CHAPTER 6

Levi Decomposition. Identification of the Radical and Levi Factor

The Levi theorem 2.12 in its full generality states that for an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic zero it is possible to construct a semidirect sum decomposition

$$\mathfrak{g} = \mathfrak{p} \niplus \mathfrak{r}. \tag{6.1}$$

Here $\mathfrak{r} \equiv \mathrm{R}(\mathfrak{g})$ is the *radical* of $\mathfrak{g}$, i.e., its maximal solvable ideal, and $\mathfrak{p}$ is a semisimple Lie algebra. The radical is unique. The complement $\mathfrak{p}$, i.e., the *Levi factor*, is isomorphic to the factor algebra $\mathfrak{g}/\mathfrak{r}$ and is unique up to automorphisms of $\mathfrak{g}$.

In other words, given a basis of $\mathfrak{g}$, say $(x_1, \dots, x_n)$, it is always possible to find a new basis

$$\{s_1, s_2, \dots, s_\sigma, r_1, r_2, \dots, r_\rho\}, \quad \sigma + \rho = n, \tag{6.2}$$

such that $\mathfrak{r} = \mathrm{span}\{r_1, \dots, r_\rho\}$, $\mathfrak{p} = \mathrm{span}\{s_1, \dots, s_\sigma\}$, and the commutation relations are such that

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}, \quad [\mathfrak{r}, \mathfrak{r}] \subsetneq \mathfrak{r}, \quad [\mathfrak{p}, \mathfrak{r}] \subseteq \mathfrak{r}. \tag{6.3}$$

The question that we address here is: how does one find a convenient change of basis that realizes the Levi decomposition? Notice that a Levi decomposition can be performed for both decomposable and indecomposable Lie algebras. From the point of view of identifying a Lie algebra $\mathfrak{g}$, it is usually preferable to first perform a direct decomposition into indecomposable components and then construct a Levi decomposition for each component.

Since the factor algebra $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{p}$ is semisimple and hence *perfect*, i.e., its derived algebra $D(\mathfrak{p})$ satisfies $D(\mathfrak{p}) = \mathfrak{p}$, we have

$$D(\mathfrak{g}) + \mathfrak{r} = \mathfrak{g}. \tag{6.4}$$

From (6.4) we obtain the isomorphism

$$D(\mathfrak{g})/[D(\mathfrak{g}) \cap \mathfrak{r}] \cong \mathfrak{g}/\mathfrak{r} \tag{6.5}$$

and hence $\dim(D(\mathfrak{g})/[D(\mathfrak{g}) \cap \mathfrak{r}]) = n - \rho = \sigma$.

The problem of obtaining a Levi decomposition is one of linear algebra. We shall address it in two ways.

6.1. Original algorithm

First, we present an algorithm as given in [102]. Its essence is a reduction of the general case to the case with an Abelian radical. We proceed in four steps.

Step 1. Find the radical $\mathfrak{r} = \mathrm{R}(\mathfrak{g})$. This is a simple task of linear algebra, since we can use the property [59, 139]

$$\mathrm{R}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid K(x, y) = 0,\ \forall y \in D(\mathfrak{g})\}, \tag{6.6}$$

where $K(x, y)$ is the Killing form (2.31)

$$K(x, y) = \mathrm{Tr}\big(\mathrm{ad}(x)\,\mathrm{ad}(y)\big).$$

If $\mathfrak{g} = \mathrm{R}(\mathfrak{g})$, then $\mathfrak{g}$ is solvable and $\mathfrak{p} = 0$ in (6.1). If $\mathrm{R}(\mathfrak{g}) = 0$, then $\mathfrak{g} = \mathfrak{p}$ is semisimple. In both cases the Levi decomposition is trivial.

Step 2. If $0 \neq \mathfrak{p} \neq \mathfrak{g}$, we calculate the derived series (2.5) of $\mathfrak{g}$:

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \dots, \mathfrak{g}^{(k+1)} = \big[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}\big], \dots$$

till we arrive, after a finite number of steps, at a perfect Lie algebra

$$\mathfrak{g}^{(k+1)} = \mathfrak{g}^{(k)}, \quad \mathfrak{g}^{(k)} \neq \mathfrak{g}^{(k-1)}. \tag{6.7}$$

If we know the Levi decomposition of $\mathfrak{g}^{(k)}$, i.e.,

$$\mathfrak{g}^{(k)} = \mathfrak{p} \niplus R\big(\mathfrak{g}^{(k)}\big) \tag{6.8}$$

then we obtain the Levi decomposition of $\mathfrak{g}$ by extending the basis $(r_1, \dots, r_\tau)$ of $R\big(\mathfrak{g}^{(k)}\big)$ to a basis of $\mathrm{R}(\mathfrak{g})$: $(r_1, \dots, r_\tau, r_{\tau+1}, \dots, r_\rho)$.

Step 3. From now on we assume that $\mathfrak{g}$ is perfect which also implies that the radical of $\mathfrak{g}$ is nilpotent (since $D(\mathfrak{g}) \subseteq \mathfrak{p} \niplus NR(\mathfrak{g})$). We consider two cases separately, namely that of an Abelian and a non-Abelian radical, respectively. If $\mathrm{R}(\mathfrak{g})$ is Abelian, the problem is a simple task of linear algebra, solved below in Step 4. Let us first reduce the case of a non-Abelian radical to that of an Abelian one. We have

$$\mathfrak{g} = D(\mathfrak{g}), \quad D\big(\mathrm{R}(\mathfrak{g})\big) \neq 0. \tag{6.9}$$

We note that $D\big(\mathrm{R}(\mathfrak{g})\big)$ is an ideal in $\mathfrak{g}$.

Let us choose a basis for $\mathfrak{g}$ in the form

$$(e_1, \dots, e_\mu, r_1, \dots, r_\nu, x_1, \dots, x_\sigma), \qquad \mu + \nu = \rho \tag{6.10}$$

where $(e_1, \dots, e_\mu)$ is a basis for $D\big(\mathrm{R}(\mathfrak{g})\big)$, $(r_1, \dots, r_\nu)$ for a complement of $D\big(\mathrm{R}(\mathfrak{g})\big)$ in $\mathrm{R}(\mathfrak{g})$, and $(x_1, \dots, x_\sigma)$ for a complement of $\mathrm{R}(\mathfrak{g})$ in $\mathfrak{g}$. Next, we construct the factor algebra $\bar{\mathfrak{g}} = \mathfrak{g}/D\big(\mathrm{R}(\mathfrak{g})\big)$ satisfying $\bar{\mathfrak{g}} = D(\bar{\mathfrak{g}})$ and $D\big(\mathrm{R}(\bar{\mathfrak{g}})\big) = 0$. We have $\dim \bar{\mathfrak{g}} = n - \mu < n$ and the commutation relations for $\bar{\mathfrak{g}}$ are obtained by setting $e_1 = \dots = e_\mu = 0$ in the commutation relations for $\mathfrak{g}$. Using the method described below (for Abelian radicals), we obtain the Levi decomposition

$$\bar{\mathfrak{g}} = \bar{\mathfrak{p}} \niplus \mathrm{R}(\bar{\mathfrak{g}}) \tag{6.11}$$

of $\bar{\mathfrak{g}}$. From the residue classes in $\bar{\mathfrak{p}}$ we construct their representative elements in $\mathfrak{g}$ and denote by $\mathfrak{p}_1$ the vector space spanned by them. We obtain a proper subalgebra of $\mathfrak{g}$, namely

$$\mathfrak{g}_1 = D\big(\mathrm{R}(\mathfrak{g})\big) + \mathfrak{p}_1, \tag{6.12}$$

satisfying $\dim \mathfrak{g}_1 = n - \nu < n$. If $D\big(\mathrm{R}(\mathfrak{g})\big)$ is Abelian, we find a Levi decomposition of $\mathfrak{g}_1$; if not, we repeat Steps 2 and 3 until, after a finite number of steps, we arrive at an algebra $\mathfrak{g}_k$ with an Abelian radical. For it we obtain

$$\mathfrak{g}_k = \mathfrak{p} \niplus \mathrm{R}(\mathfrak{g}_k). \tag{6.13}$$

The Levi decomposition of $\mathfrak{g}$ is then (6.1) with $\mathfrak{p}$ as in (6.13).

Step 4. Finally, let us consider the case of an Abelian radical. We have $\mu = 0$ in (6.10), and the commutation relations for $\mathfrak{g}$ are

$$[x_i, x_k] = c_{ik}{}^l x_l + f_{ik}{}^q r_q, \quad [r_p, r_q] = 0, \quad [x_i, r_p] = h_{ip}{}^q r_q, \qquad 1 \le i, k, l \le \sigma, \quad 1 \le p, q \le \rho. \tag{6.14}$$

Summation over repeated indices over the appropriate ranges is understood throughout.

We now replace the basis elements x_i by

$$s_i = x_i + b_i^p r_p \tag{6.15}$$

and require that the s_i form a Lie algebra:

$$[s_i, s_k] = c_{ik}{}^l s_l. \tag{6.16}$$

Equations (6.14) – (6.16) imply that the unknown coefficients b_i^p must satisfy a system of inhomogeneous linear equations

$$c_{ik}{}^l b_l^q - h_{ip}{}^q b_k^p + h_{kp}{}^q b_i^p = f_{ik}{}^q, \quad 1 \le p, q \le \rho, \quad 1 \le i, k \le \sigma. \tag{6.17}$$

The system (6.17) involves $\rho\sigma$ unknowns b_i^p and $\frac{1}{2}\sigma(\sigma-1)\rho/2$ equations. It follows from the Levi theorem that the system is compatible and that a solution b_i^p can always be found. It is actually possible to solve (6.17) explicitly and analytically, making use of the second order Casimir operator of the factor algebra $\mathfrak{p} = \mathfrak{g}/\mathfrak{r}$ [139]. From the computational point of view it is usually preferable to solve (6.17) directly in each specific case.

6.2. Modified algorithm

A streamlined alternative formulation of the algorithm combines Steps 3 and 4 into one. Its basic idea is a construction of a proper subalgebra of the Lie algebra $\mathfrak{g}$ which contains $\mathfrak{p}$, without constructing the factor algebra $\mathfrak{g}/D\big(\mathrm{R}(\mathfrak{g})\big)$. Repeating the procedure finitely many times one obtains a subalgebra of $\mathfrak{g}$ which has vanishing radical, i.e., coincides with $\mathfrak{p}$. From the computational point of view it is equivalent to the original algorithm.

We observe that $D(\mathfrak{r}) \equiv \mathfrak{r}^2$ is a characteristic ideal of $\mathfrak{g}$ and consequently the subalgebra $\tilde{\mathfrak{g}} = \mathfrak{p} \dotplus D(\mathfrak{r})$ of $\mathfrak{g}$ is a Levi decomposable algebra with the Levi factor $\mathfrak{p}$ and radical $\mathfrak{r}^2$. We shall proceed to construct its basis.

Let us suppose that we have a basis of $\mathfrak{g}$ of the form (6.10). Thus we have the following particular Lie brackets

$$[x_i, x_j] = c_{ij}{}^k x_k + d_{ij}{}^p r_p + f_{ij}{}^l e_l, \tag{6.18}$$

$$[x_i, r_p] = g_{ip}{}^q r_q + h_{ip}{}^m e_m. \tag{6.19}$$

Summation over repeated indices $k = 1, \ldots, \sigma$, $l, m = 1, \ldots, \mu$ and $p, q = 1, \ldots, \nu$ applies throughout.

A basis of $\tilde{\mathfrak{g}} = \mathfrak{p} \dotplus D(\mathfrak{r})$ can be without loss of generality chosen in the form

$$\{e_1, \ldots, e_\mu, \hat{x}_1, \ldots, \hat{x}_\sigma\} \tag{6.20}$$

where $\hat{x}_k \in \operatorname{span}\{x_k, r_1, \ldots, r_\nu\}$, i.e.,

$$\hat{x}_j = x_j + b_j^p r_p. \tag{6.21}$$

The span of the set (6.20) is by construction complementary to span$\{r_1, \dots, r_\nu\}$. Therefore it forms a basis of $\mathfrak{p} \dotplus D(\mathfrak{r})$ if and only if it closes under the Lie bracket. That is always true for commutators of the type $[\hat{x}_i, e_l]$ since $D(\mathfrak{r})$ is an ideal in $\mathfrak{g}$. It remains to satisfy

$$[\hat{x}_i, \hat{x}_j] = c_{ij}{}^k \hat{x}_k + \widehat{f_{ij}}^l e_l \tag{6.22}$$

for some constants $\widehat{f_{ij}}^l$. Notice that the structure constants $c_{jk}{}^k$ are the same in (6.18) and (6.22) because they are the structure constants of the semisimple factor algebra $\mathfrak{g}/\mathfrak{r}$ in the same basis $x_j \bmod \mathfrak{r} = \hat{x}_j \bmod \mathfrak{r}$, $j = 1, \dots, \sigma$.

Substituting (6.21) into (6.22) and dropping any term proportional to e_l we obtain the following set of equations, to be satisfied for all $1 \le i < j \le \sigma$

$$d_{ij}{}^q r_q + b_j^p g_{ip}{}^q r_q - b_i^p g_{jp}{}^q r_q = c_{ij}{}^k b_k^q r_q, \tag{6.23}$$

i.e., a set of $\frac{1}{2}\sigma(\sigma-1)\nu$ linear inhomogeneous equations

$$g_{jp}{}^q b_i^p - g_{ip}{}^q b_j^p + c_{ij}{}^k b_k^q = d_{ij}{}^q \tag{6.24}$$

for $\sigma\nu$ unknowns b_i^p. Due to the Levi theorem the set of equations (6.24) always has a solution. Thus, once we find any particular solution of (6.24) we have a basis of $\tilde{\mathfrak{g}} = \mathfrak{p} \dotplus D(\mathfrak{r})$. We repeat the procedure until we arrive at $k \in \mathbb{N}$ such that

$$\mathfrak{r}^{(k)} = 0.$$

Notice that it is advantageous to suppose that $\mathfrak{g}$ is perfect but the streamlined version of the procedure does not depend on it.

Remark 6.1. An analogous procedure was given in [35] where the lower central series of the nilpotent radical $\mathfrak{r}$ of a perfect Lie algebra $\mathfrak{g}$ was used instead of the derived series employed here. Consequently, the procedure has to be repeated more times but the arising systems of equations (6.24) are smaller and thus easier to solve. However, a comparable computational simplification can be achieved in our algorithm by simply using a basis of the nilpotent radical $\mathfrak{r}$ which respects the lower central series, as in (9.19) in Section 9.3 below.

6.3. Examples

In order to illuminate the algorithms outlined above we conclude this chapter by two examples.

Example 6.2. Let us consider the finite dimensional part of the algebra of infinitesimal point symmetries of the heat equation [86]. It is spanned by the following vector fields in $\mathbb{R}^3$ with coordinates t, x, u

$$\begin{aligned} &Y_1 = 4t^2\partial_t + 4xt\partial_x - (2t + x^2)u\partial_u, && Y_2 = 4t\partial_t + 2x\partial_x, \\ &Y_3 = \partial_t, && Y_4 = -2t\partial_x + xu\partial_u, \\ &Y_5 = u\partial_u, && Y_6 = \partial_x. \end{aligned} \tag{6.25}$$

Evaluation of the commutators gives the following Lie brackets of an abstract Lie algebra (with y_i replacing the vector fields Y_i)

$$\begin{aligned} &[y_1, y_2] = -4y_1, && [y_1, y_3] = -2y_2 + 2y_5, && [y_1, y_6] = 2y_4, \\ &[y_2, y_3] = -4y_3, && [y_2, y_4] = 2y_4, && [y_2, y_6] = -2y_6, \\ &[y_3, y_4] = -2y_6, && [y_4, y_6] = -y_5. \end{aligned} \tag{6.26}$$

In Steps 1 and 2 we find that the Lie algebra (6.26) is perfect and its radical is spanned by y_4, y_5, y_6. The radical (equal to the nilradical) is the Heisenberg algebra $\mathfrak{h}(1) \equiv \mathfrak{n}_{3,1}$. The derived algebra of the radical is spanned by y_5. The basis of the form (6.10) can be chosen as

$$e_1 = y_5, \quad r_1 = y_4, \quad r_2 = y_6, \quad x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3 \tag{6.27}$$

with the Lie brackets

$$\begin{aligned} &[r_1, r_2] = -e_1, \quad [r_1, x_2] = -2r_1, \quad [r_1, x_3] = 2r_2, \quad [r_2, x_1] = -2r_1, \\ &[r_2, x_2] = 2r_2, \quad [x_1, x_2] = -4x_1, \quad [x_1, x_3] = 2e_1 - 2x_2, \quad [x_2, x_3] = -4x_3. \end{aligned} \tag{6.28}$$

Step 3 of the procedure now requires the construction of a 5-dimensional Lie algebra $\mathfrak{g}/\operatorname{span}\{e_1\}$ with an Abelian radical. Its Lie brackets are

$$\begin{aligned} &[\tilde{r}_1, \tilde{x}_2] = -2\tilde{r}_1, \quad [\tilde{r}_1, \tilde{x}_3] = 2\tilde{r}_2, \quad [\tilde{r}_2, \tilde{x}_1] = -2\tilde{r}_1, \quad [\tilde{r}_2, \tilde{x}_2] = 2\tilde{r}_2, \\ &[\tilde{x}_1, \tilde{x}_2] = -4\tilde{x}_1, \quad [\tilde{x}_1, \tilde{x}_3] = -2\tilde{x}_2, \quad [\tilde{x}_2, \tilde{x}_3] = -4\tilde{x}_3 \end{aligned} \tag{6.29}$$

where tildes were used to distinguish residue classes in $\mathfrak{g}/\operatorname{span}\{e_1\}$ from the corresponding elements in $\mathfrak{g}$. Its (Abelian) radical is spanned by $\tilde{r}_1$ and $\tilde{r}_2$.

In Step 4 the Levi factor of $\mathfrak{g}/\operatorname{span}\{e_1\}$ is obtained by an explicit solution of (6.17) or by inspection; it can be chosen to be the span of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$.

Thus, we have $\mathfrak{p} \dotplus D\big(R(\mathfrak{g})\big) = \operatorname{span}\{e_1, x_1, x_2, x_3\}$. The same conclusion is immediate in our modified algorithm since e_1, x_1, x_2, x_3 span a subalgebra of $\mathfrak{g}$ by an inspection of (6.28).

Next, we construct the Levi decomposition of $\mathfrak{p} \dotplus D\big(R(\mathfrak{g})\big) = \operatorname{span}\{e_1, x_1, x_2, x_3\}$ with the nonvanishing Lie brackets

$$[x_1, x_2] = -4x_1, \quad [x_1, x_3] = 2e_1 - 2x_2, \quad [x_2, x_3] = -4x_3. \tag{6.30}$$

This step coincides in both versions of the algorithm. The solution of the set of linear equations (6.17) or, equivalently, (6.24) is

$$b_1^1 = 0, \quad b_2^1 = -1, \quad b_3^1 = 0$$

and leads to the desired basis for the Levi factor in the form

$$x_1 = y_1, \quad x_2 - e_1 = y_2 - y_5, \quad x_3 = y_3. \tag{6.31}$$

Notice that in this case the Levi factor of the subalgebra $\operatorname{span}\{e_1, x_1, x_2, x_3\}$ coincides with its derived algebra and is therefore unique. However, as a Levi factor of the whole algebra $\mathfrak{g}$ it is not unique since a different choice of the preimage $\mathfrak{p}_1$ of $\bar{\mathfrak{p}}$ was possible in Step 3. As stated in Theorem 2.13, the Levi factor $\mathfrak{p}$ is determined only up to inner automorphisms of $\mathfrak{g}$.

We conclude that the Levi decomposition of the "heat algebra" $\mathfrak{g}$ (6.25) is

$$\mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{R}) \ni \mathfrak{h}(1) \tag{6.32}$$

where the Levi factor $\mathfrak{sl}(2, \mathbb{R})$ has the basis

$$\begin{aligned} Y_1 &= 4t^2\partial_t + 4xt\partial_x - (2t + x^2)u\partial_u, \\ Y_2 - Y_5 &= 4t\partial_t + 2x\partial_x - u\partial_u, \quad Y_3 = \partial_t \end{aligned} \tag{6.33}$$

and the radical is the Heisenberg algebra spanned by Y_4, Y_5, Y_6.

As we have just seen explicitly, the difference between the two versions of the algorithm is conceptual rather than computational. Therefore, in our second example we shall employ only the second, streamlined version of it.

Example 6.3. Let us consider an 8-dimensional Lie algebra $\mathfrak{g}$ with the Lie brackets

$$\begin{aligned} &[y_1, y_2] = 2y_2, && [y_1, y_3] = -2y_3 + 2y_6, && [y_1, y_4] = 2y_4, \\ &[y_1, y_5] = 2y_5, && [y_1, y_7] = 2y_7, && [y_2, y_3] = y_1 + y_5 + y_8, \\ &[y_2, y_6] = y_5, && [y_2, y_8] = 2y_4, && [y_3, y_5] = -y_4 + y_6, \\ &[y_3, y_8] = y_6, && [y_4, y_8] = 2y_4, && [y_5, y_6] = y_4, \\ &[y_5, y_8] = y_5, && [y_6, y_8] = y_6, && [y_7, y_8] = 2y_7. \end{aligned} \tag{6.34}$$

Its radical is spanned by y_4, y_5, y_6, y_7, y_8 and the nilradical by y_4, y_5, y_6, y_7. In fact, the radical $\{y_4, \ldots, y_8\}$ is isomorphic to the algebra $\mathfrak{s}_{5,22,a=1,b=2}$ of Section 18.3 and the nilradical is decomposable into $\mathfrak{n}_{3,1} \oplus \mathfrak{n}_{1,1}$ where $\mathfrak{n}_{3,1} \equiv \mathfrak{h}(1)$ is the Heisenberg algebra.

We start by Step 2. The derived series is

$$\begin{aligned} \mathfrak{g}^{(1)} &= D(\mathfrak{g}) = \operatorname{span}\{y_1 + y_8, y_2, \ldots, y_7\}, \\ \mathfrak{g}^{(2)} &= \mathfrak{g}^{(3)} = \operatorname{span}\{y_1 + y_8, y_2, y_3, y_4, y_5, y_6\}. \end{aligned} \tag{6.35}$$

Thus, the Levi factor of $\mathfrak{g}$ is found once the Levi factor of $\mathfrak{g}^{(2)}$ is constructed using the algorithm of Section 6.2. The nilpotent non-Abelian radical of $\mathfrak{g}^{(2)}$ is spanned by y_4, y_5, y_6 with a single nonvanishing Lie bracket

$$[y_5, y_6] = y_4.$$

We choose the basis as in (6.10)

$$e_1 = y_4, \quad r_1 = y_5, \quad r_2 = y_6, \quad x_1 = y_1 + y_8, \quad x_2 = y_2, \quad x_3 = y_3. \tag{6.36}$$

The Lie brackets of $\tilde{\mathfrak{g}} = \mathfrak{g}^{(2)}$ in this basis become

$$\begin{aligned} &[r_1, r_2] = e_1, && [r_1, x_1] = -r_1, && [r_1, x_3] = e_1 - r_2, \\ &[r_2, x_1] = r_2, && [r_2, x_2] = -r_1, && [x_1, x_2] = -2e_1 + 2x_2, \\ &[x_1, x_3] = r_2 - 2x_3, && [x_2, x_3] = r_1 + x_1. \end{aligned} \tag{6.37}$$

In order to find $\mathfrak{p} \dotplus D\big(R(\tilde{\mathfrak{g}})\big)$ we have to perform the change of basis (6.21). The conditions (6.24) reduce to the equations

$$b_2^2 = 0, \quad b_3^1 = 0, \quad b_2^1 + b_1^2 = 0, \quad b_3^2 - b_1^1 = 1.$$

Thus, a particular solution of (6.24) is

$$b_1^1 = 1, \qquad b_1^2 = 0, \qquad b_2^p = b_3^p = 0, \quad p = 1, 2,$$

which corresponds to the change of basis

$$\hat{x}_1 = x_1 + r_1 = y_1 + y_5 + y_8, \quad \hat{x}_2 = x_2 = y_2, \quad \hat{x}_3 = x_3 = y_3. \tag{6.38}$$

The vectors $e_1, \hat{x}_1, \hat{x}_2, \hat{x}_3$ form a basis of $\mathfrak{p} \dotplus D\big(R(\tilde{\mathfrak{g}})\big)$. In this basis we have the Lie brackets

$$[\hat{x}_1, \hat{x}_2] = -2e_1 + 2\hat{x}_2, \qquad [\hat{x}_1, \hat{x}_3] = e_1 - 2\hat{x}_3, \qquad [\hat{x}_2, \hat{x}_3] = \hat{x}_1. \tag{6.39}$$

The radical of $\mathfrak{p} \dotplus D\big(R(\tilde{\mathfrak{g}})\big)$ is spanned by e_1 which coincides with the center of $\mathfrak{p} \dotplus D\big(R(\tilde{\mathfrak{g}})\big)$. Thus, the Lie algebra (6.39) is decomposable into a direct sum of a simple algebra $\mathfrak{sl}(2, \mathbb{F})$ and a central component spanned by e_1, and the algorithm of the previous chapter can be used. Alternatively, we use the change of basis (6.21)

$$\bar{x}_1 = \hat{x}_1 + \hat{b}_1^1 e_1, \quad \bar{x}_2 = \hat{x}_2 + \hat{b}_2^1 e_1, \quad \bar{x}_3 = \hat{x}_3 + \hat{b}_3^1 e_1$$

once again, arriving at the conditions (6.24) expressed as

$$\hat{b}_1^1 = 0, \quad \hat{b}_2^1 = -1, \quad 2\hat{b}_3^1 = -1.$$

Thus, we have constructed a basis $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ of the Levi factor $\mathfrak{p}$ of $\mathfrak{g}$ in the form

$$\bar{x}_1 = \hat{x}_1 = y_1 + y_5 + y_8, \quad \bar{x}_2 = \hat{x}_2 - e_1 = y_2 - y_4, \quad \bar{x}_3 = \hat{x}_3 - \tfrac{1}{2}e_1 = y_3 - \tfrac{1}{2}y_4.$$

To sum up, the Levi factor of the algebra (6.34) is

$$\mathfrak{p} = \operatorname{span}\left\{y_1 + y_5 + y_8, y_2 - y_4, y_3 - \tfrac{1}{2}y_4\right\} \tag{6.40}$$

and is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$.

Let us recall that the Levi factor (6.40) is generically far from unique. In our example, another choice for it is spanned by

$$\tilde{x}_1 = y_1 + y_8, \quad \tilde{x}_2 = y_2 - y_4, \quad \tilde{x}_3 = y_3 - y_6 \tag{6.41}$$

and, in fact, the choice (6.41) is more convenient because the Lie brackets of the algebra (6.34) are more compact when written in terms of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. Different choices of Levi factors arise through different choices of the particular solutions of the systems of linear equations involved.

CHAPTER 18

Five-Dimensional Lie Algebras

18.1. Nilpotent five-dimensional Lie algebras

- $\mathfrak{n}_{5,1}$

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	0
e_3			0	0	e_1
e_4				0	e_2

US = [2, 5]
CS = [5, 2, 0]
DS = [5, 2, 0]
$\dim \mathfrak{Der} = 13$

Casimir invariants:

$$e_1, \quad e_2, \quad e_2e_3 - e_1e_4$$

- $\mathfrak{n}_{5,2}$

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	0
e_3			0	e_2	e_1
e_4				0	e_3

US = [2, 3, 5]
CS = [5, 3, 2, 0]
DS = [5, 3, 0]
$\dim \mathfrak{Der} = 10$

Casimir invariants:

$$e_1, \quad e_2, \quad e_3^2 + 2e_2e_5 - 2e_1e_4$$

- $\mathfrak{n}_{5,3}$

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	e_1	0
e_3			0	0	e_1
e_4				0	0

US = [1, 5]
CS = [5, 1, 0]
DS = [5, 1, 0]
$\dim \mathfrak{Der} = 15$

Casimir invariants:

$$e_1$$

- $\mathfrak{n}_{5,4}$

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	e_1
e_3			0	e_1	0
e_4				0	e_2

US = [1, 3, 5]
CS = [5, 2, 1, 0]
DS = [5, 2, 0]
$\dim \mathfrak{Der} = 10$

Casimir invariants:

$$e_1$$

- $\mathfrak{n}_{5,5}$

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	e_1
e_3			0	0	e_2
e_4				0	e_3

US $= [1, 2, 3, 5]$
CS $= [5, 3, 2, 1, 0]$
DS $= [5, 3, 0]$
$\dim \mathfrak{Der} = 9$

Casimir invariants:

$$e_1, \quad 2e_1e_3 - e_2^2, \quad e_2^3 + 3e_4e_1^2 - 3e_2e_3e_1$$

- $\mathfrak{n}_{5,6}$

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	0
e_2		0	0	0	e_1
e_3			0	e_1	e_2
e_4				0	e_3

US $= [1, 2, 3, 5]$
CS $= [5, 3, 2, 1, 0]$
DS $= [5, 3, 0]$
$\dim \mathfrak{Der} = 8$

Casimir invariants:

$$e_1$$

18.2. Solvable five-dimensional Lie algebras with the nilradical $4\mathfrak{n}_{1,1}$

	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2		0	0	0
e_3			0	0

- $\mathfrak{s}_{5,1}$

	e_1	e_2	e_3	e_4
e_5	0	e_1	e_2	e_4

US $= [1, 2, 3]$
CS $= [5, 3, 2, 1]$
DS $= [5, 3, 0]$

Casimir invariants:

$$e_1, \quad e_2^2 - 2e_1e_3, \quad e_4 \exp\left(-\frac{e_2}{e_1}\right)$$

- $\mathfrak{s}_{5,2}$

	e_1	e_2	e_3	e_4
e_5	0	e_1	e_3	$e_3 + e_4$

US $= [1, 2]$
CS $= [5, 3, 2]$
DS $= [5, 3, 0]$

Casimir invariants:

$$e_1, \quad \frac{e_1e_4 - e_2e_3}{e_3}, \quad e_3 \exp\left(-\frac{e_2}{e_1}\right)$$

- $\mathfrak{s}_{5,3}$

	e_1	e_2	e_3	e_4
e_5	0	e_1	e_3	ae_4

US $= [1, 2]$
CS $= [5, 3, 2]$
DS $= [5, 3, 0]$

where $0 < |a| \leq 1$. If $|a| = 1$ then $\arg(a) \leq \pi$.

Casimir invariants:

$$e_1, \quad \frac{e_3^a}{e_4}, \quad e_3 \exp\left(-\frac{e_2}{e_1}\right)$$

- $\mathfrak{s}_{5,4}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,3}$)

	e_1	e_2	e_3	e_4
e_5	0	e_1	$\alpha e_3 - e_4$	$e_3 + \alpha e_4$

US $= [1, 2]$
CS $= [5, 3, 2]$
DS $= [5, 3, 0]$

where $0 \leq \alpha$.

Casimir invariants:

$$e_1, \quad (e_3^2 + e_4^2) \exp\left(-2\alpha \frac{e_2}{e_1}\right), \quad \frac{e_2}{e_1} + \arctan \frac{e_3}{e_4}$$

- $\mathfrak{s}_{5,5}$

	e_1	e_2	e_3	e_4
e_5	e_1	$e_1 + e_2$	$e_2 + e_3$	$e_3 + e_4$

US $= [0]$
CS $= [5, 4]$
DS $= [5, 4, 0]$

Casimir invariants:

$$\frac{2e_1e_3 - e_2^2}{e_1^2}, \quad \frac{3e_1^2e_4 - 3e_1e_2e_3 + e_2^3}{e_1^3}, \quad e_1 \exp\left(-\frac{e_2}{e_1}\right)$$

- $\mathfrak{s}_{5,6}$

	e_1	e_2	e_3	e_4
e_5	e_1	$e_1 + e_2$	ae_3	$e_3 + ae_4$

US $= [0]$
CS $= [5, 4]$
DS $= [5, 4, 0]$

where $0 < |a| \leq 1$. If $|a| = 1$ then $\arg(a) \leq \pi$.

Casimir invariants:

$$\frac{e_1e_4 - e_2e_3}{e_1e_3}, \quad \frac{e_1^a}{e_3}, \quad e_1 \exp\left(-\frac{e_2}{e_1}\right)$$

- $\mathfrak{s}_{5,7}$

	e_1	e_2	e_3	e_4
e_5	e_1	$e_1 + e_2$	$e_2 + e_3$	ae_4

US $= [0]$
CS $= [5, 4]$
DS $= [5, 4, 0]$

where $a \neq 0$.

Casimir invariants:

$$\frac{2e_1e_3 - e_2^2}{e_1^2}, \quad \frac{e_1^a}{e_4}, \quad e_1 \exp\left(-\frac{e_2}{e_1}\right)$$

- $\mathfrak{s}_{5,8}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,6}$)

	e_1	e_2	e_3	e_4
e_5	$\alpha e_1 - e_2$	$e_1 + \alpha e_2$	$e_1 + \alpha e_3 - e_4$	$e_2 + e_3 + \alpha e_4$

US $= [0]$
CS $= [5, 4]$
DS $= [5, 4, 0]$

where $0 \leq \alpha$.

Casimir invariants:

$$\frac{e_1e_4 - e_2e_3}{e_1^2 + e_2^2}, \quad \left(e_1^2 + e_2^2\right) \exp\left(-2\alpha \frac{e_1e_3 + e_2e_4}{e_1^2 + e_2^2}\right), \quad \frac{e_1e_3 + e_4e_2}{e_1^2 + e_2^2} + \arctan \frac{e_1}{e_2}$$

- $\mathfrak{s}_{5,9}$

	e_1	e_2	e_3	e_4
e_5	e_1	ae_2	be_3	ce_4

US = [0]
CS = [5, 4]
DS = [5, 4, 0]

where the values of the parameters a, b, c are

$$0 < |c| \leq |b| \leq |a| \leq 1.$$

If one or more equalities hold further restrictions on the parameters are necessary in order to avoid redundancies, as discussed in Section 10.4.

Casimir invariants:

$$\frac{e_1^a}{e_2}, \quad \frac{e_1^b}{e_3}, \quad \frac{e_1^c}{e_4}$$

- $\mathfrak{s}_{5,10}$

	e_1	e_2	e_3	e_4
e_5	e_1	$e_1 + e_2$	ae_3	be_4

US = [0]
CS = [5, 4]
DS = [5, 4, 0]

where $0 < |b| \leq |a|$, if $|b| = |a|$ then $\arg(a) \leq \arg(b)$.

Casimir invariants:

$$\frac{e_1^a}{e_3}, \quad \frac{e_1^b}{e_4}, \quad e_1 \exp\left(-\frac{e_2}{e_1}\right)$$

- $\mathfrak{s}_{5,11}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,9}$)

	e_1	e_2	e_3	e_4
e_5	αe_1	βe_2	$\gamma e_3 - e_4$	$e_3 + \gamma e_4$

US = [0]
CS = [5, 4]
DS = [5, 4, 0]

where $\alpha > 0$, $\beta \neq 0$, $\alpha \geq \beta \geq -\alpha$.

Casimir invariants:

$$\frac{e_1^\beta}{e_2^\alpha}, \quad \frac{(e_3^2 + e_4^2)^\beta}{e_2^{2\gamma}}, \quad e_1 \exp\left(\alpha \arctan \frac{e_3}{e_4}\right)$$

- $\mathfrak{s}_{5,12}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,10}$)

	e_1	e_2	e_3	e_4
e_5	e_1	$e_1 + e_2$	$\alpha e_3 - \beta e_4$	$\beta e_3 + \alpha e_4$

US = [0]
CS = [5, 4]
DS = [5, 4, 0]

where $\beta > 0$.

Casimir invariants:

$$\frac{e_3^2 + e_4^2}{e_1^{2\alpha}}, \quad e_1 \exp\left(-\frac{e_2}{e_1}\right), \quad \beta \frac{e_2}{e_1} + \arctan\left(\frac{e_3}{e_4}\right)$$

- $\mathfrak{s}_{5,13}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,9}$)

	e_1	e_2	e_3	e_4
e_5	$\alpha e_1 - e_2$	$e_1 + \alpha e_2$	$\beta e_3 - \gamma e_4$	$\gamma e_3 + \beta e_4$

US $= [0]$
CS $= [5, 4]$
DS $= [5, 4, 0]$

where $0 < \gamma \leq 1$, $0 \leq \alpha$. If $\alpha = 0$ then $0 \leq \beta$. If $\gamma = 1$ then $|\alpha| \leq |\beta|$.
Casimir invariants:

$$\frac{(e_1^2 + e_2^2)^\beta}{(e_3^2 + e_4^2)^\alpha}, \quad \arctan\frac{e_3}{e_4} - \gamma\arctan\frac{e_1}{e_2}, \quad (e_1^2 + e_2^2)\exp\left(2\alpha\arctan\frac{e_1}{e_2}\right).$$

When $\alpha \to 0$ the first and third invariant become dependent and one of them should be replaced by

$$(e_3^2 + e_4^2)\exp\left(2\beta\arctan\frac{e_1}{e_2}\right).$$

18.3. Solvable five-dimensional Lie algebras with the nilradical $\mathfrak{n}_{3,1} \oplus \mathfrak{n}_{1,1}$

	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2		0	e_1	0
e_3			0	0

- $\mathfrak{s}_{5,14}$

	e_1	e_2	e_3	e_4
e_5	0	0	e_2	e_4

US $= [1, 2, 3]$
CS $= [5, 3, 2, 1]$
DS $= [5, 3, 0]$

Casimir invariant:

$$e_1$$

- $\mathfrak{s}_{5,15}$

	e_1	e_2	e_3	e_4
e_5	0	e_2	$-e_3$	e_1

US $= [1, 2]$
CS $= [5, 3]$
DS $= [5, 3, 1, 0]$

Casimir invariant:

$$e_1$$

- $\mathfrak{s}_{5,16}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,15}$)

	e_1	e_2	e_3	e_4
e_5	0	$-e_3$	e_2	e_1

US $= [1, 2]$
CS $= [5, 3]$
DS $= [5, 3, 1, 0]$

Casimir invariant:

$$e_1$$

- $\mathfrak{s}_{5,17}$

	e_1	e_2	e_3	e_4
e_5	0	e_2	$-e_3$	ae_4

US = [1]
CS = [5, 4]
DS = [5, 4, 1, 0]

where the values of the parameter a are
over the field $\mathbb{C}$: $0 \le \mathrm{Re}(a)$, if $\mathrm{Re}(a) = 0$ then $0 < \mathrm{Im}(a)$;
over the field $\mathbb{C}$: $0 < a$.
Casimir invariant:

$$e_1$$

- $\mathfrak{s}_{5,18}$

	e_1	e_2	e_3	e_4
e_5	0	$-e_2$	$e_3 + e_4$	e_4

US = [1]
CS = [5, 4]
DS = [5, 4, 1, 0]

Casimir invariant:

$$e_1$$

- $\mathfrak{s}_{5,19}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,17}$)

	e_1	e_2	e_3	e_4
e_5	0	$-e_3$	e_2	αe_4

US = [1]
CS = [5, 4]
DS = [5, 4, 1, 0]

where $\alpha > 0$.
Casimir invariant:

$$e_1$$

- $\mathfrak{s}_{5,20}$

	e_1	e_2	e_3	e_4
e_5	e_1	e_2	e_4	0

US = [1]
CS = [5, 3, 2]
DS = [5, 3, 0]

Casimir invariant:

$$e_4$$

- $\mathfrak{s}_{5,21}$

	e_1	e_2	e_3	e_4
e_5	$2e_1$	$e_2 + e_3$	$e_3 + e_4$	e_4

US = [0]
CS = [5, 4]
DS = [5, 4, 1, 0]

Casimir invariant:

$$\frac{e_4^2}{e_1}$$

- $\mathfrak{s}_{5,22}$

	e_1	e_2	e_3	e_4
e_5	$(a+1)e_1$	e_2	ae_3	be_4

US = [0]
CS = [5, 4]
DS = [5, 4, 1, 0]

where the values of the parameters a, b are
over the field $\mathbb{C}$: $0 < |a| \le 1$, $b \ne 0$, if $|a| = 1$ then $\arg(a) < \pi$;

over the field $\mathbb{R}$: $-1 < a \leq 1$, $a, b \neq 0$.
Casimir invariant:

$$\frac{e_4^{a+1}}{e_1^b}$$

- $\mathfrak{s}_{5,23}$

	e_1	e_2	e_3	e_4
e_5	$(a+1)e_1$	ae_2	$e_3 + e_4$	e_4

US = [0]
CS = [5, 4]
DS = [5, 4, 1, 0]

where $a \neq 0, -1$.
Casimir invariant:

$$\frac{e_4^{a+1}}{e_1}$$

- $\mathfrak{s}_{5,24}$

	e_1	e_2	e_3	e_4
e_5	$2e_1$	$e_2 + e_3$	e_3	ae_4

US = [0]
CS = [5, 4]
DS = [5, 4, 1, 0]

where $a \neq 0$.
Casimir invariant:

$$\frac{e_1^a}{e_4^2}$$

- $\mathfrak{s}_{5,25}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,22}$)

	e_1	e_2	e_3	e_4
e_5	$2\alpha e_1$	$\alpha e_2 - e_3$	$e_2 + \alpha e_3$	βe_4

US = [0]
CS = [5, 4]
DS = [5, 4, 1, 0]

where $\alpha \neq 0$, $\beta > 0$.
Casimir invariant:

$$\frac{e_4^{2\alpha}}{e_1^{\beta}}$$

- $\mathfrak{s}_{5,26}$

	e_1	e_2	e_3	e_4
e_5	$(a+1)e_1$	e_2	ae_3	$e_1 + (a+1)e_4$

US = [0]
CS = [5, 4]
DS = [5, 4, 1, 0]

where the values of the parameter a are
over the field $\mathbb{C}$: $0 < |a| \leq 1$, if $|a| = 1$ then $\arg(a) < \pi$;
over the field $\mathbb{R}$: $-1 < a \leq 1$, $a \neq 0$.
Casimir invariant:

$$e_1 \exp\left(-(a+1)\frac{e_4}{e_1}\right)$$

- $\mathfrak{s}_{5,27}$

	e_1	e_2	e_3	e_4
e_5	$2e_1$	$e_2 + e_3$	e_3	$e_1 + 2e_4$

US = [0]
CS = [5, 4]
DS = [5, 4, 1, 0]

Casimir invariant:

$$e_1 \exp\left(-2\frac{e_4}{e_1}\right)$$

- $\mathfrak{s}_{5,28}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,26}$)

	e_1	e_2	e_3	e_4
e_5	$2\alpha e_1$	$\alpha e_2 + e_3$	$-e_2 + \alpha e_3$	$e_1 + 2\alpha e_4$

US = [0]
CS = [5, 4]
DS = [5, 4, 1, 0]

where $\alpha > 0$.

Casimir invariant:

$$e_1 \exp\left(-2\alpha\frac{e_4}{e_1}\right)$$

- $\mathfrak{s}_{5,29}$

	e_1	e_2	e_3	e_4
e_5	e_1	0	$e_3 + e_4$	e_4

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

Casimir invariant:

$$\frac{e_4}{e_1}$$

- $\mathfrak{s}_{5,30}$

	e_1	e_2	e_3	e_4
e_5	e_1	e_2	0	ae_4

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

where $a \neq 0$.

Casimir invariant:

$$\frac{e_1^a}{e_4}$$

- $\mathfrak{s}_{5,31}$

	e_1	e_2	e_3	e_4
e_5	e_1	e_2	0	$e_1 + e_4$

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

Casimir invariant:

$$e_1 \exp\left(-\frac{e_4}{e_1}\right)$$

- $\mathfrak{s}_{5,32}$

	e_1	e_2	e_3	e_4
e_5	e_1	0	$e_3 + e_4$	$e_1 + e_4$

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

Casimir invariant:

$$e_1 \exp\left(-\frac{e_4}{e_1}\right)$$

18.4. Solvable five-dimensional Lie algebras with the nilradical $\mathfrak{n}_{4,1}$

	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2		0	0	e_1
e_3			0	e_2

- $\mathfrak{s}_{5,33}$

	e_1	e_2	e_3	e_4
e_5	0	$-e_2$	$-2e_3$	e_4

US = [1]
CS = [5, 4]
DS = [5, 4, 2, 0]

Casimir invariant

$$e_1$$

- $\mathfrak{s}_{5,34}$

	e_1	e_2	e_3	e_4
e_5	$3e_1$	$2e_2$	e_3	$e_3 + e_4$

US = [0]
CS = [5, 4]
DS = [5, 4, 2, 0]

Casimir invariant:

$$\frac{(2e_1e_3 - e_2^2)^3}{e_1^4}$$

- $\mathfrak{s}_{5,35}$

	e_1	e_2	e_3	e_4
e_5	$(a + 2)e_1$	$(a + 1)e_2$	ae_3	e_4

US = [0]
CS = [5, 4]
DS = [5, 4, 2, 0]

where $a \neq 0, -2$.
Casimir invariant:

$$\frac{(2e_1e_3 - e_2^2)^{2+a}}{e_1^{2(1+a)}}$$

- $\mathfrak{s}_{5,36}$

	e_1	e_2	e_3	e_4
e_5	$2e_1$	e_2	0	e_4

US = [0]
CS = [5, 3]
DS = [5, 3, 1, 0]

Casimir invariant:

$$\frac{2e_1e_3 - e_2^2}{e_1}$$

- $\mathfrak{s}_{5,37}$

	e_1	e_2	e_3	e_4
e_5	e_1	e_2	e_3	0

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

Casimir invariant:

$$\frac{2e_1e_3 - e_2^2}{e_1^2}$$

- $\mathfrak{s}_{5,38}$

	e_1	e_2	e_3	e_4
e_5	e_1	e_2	$\epsilon e_1 + e_3$	0

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

where the value of the parameter ϵ is
over the field $\mathbb{C}$: $\epsilon = 1$;
over the field $\mathbb{R}$: $\epsilon = \pm 1$.
Casimir invariant:

$$e_1^{-2\epsilon} \exp\left(\frac{2e_1e_3 - e_2^2}{e_1^2}\right)$$

18.5. Solvable five dimensional Lie algebras with the nilradical $3\mathfrak{n}_{1,1}$

	e_1	e_2	e_3
e_1	0	0	0
e_2		0	0

- $\mathfrak{s}_{5,39}$

	e_1	e_2	e_3	e_4
e_4	0	e_2	0	0
e_5	0	0	e_3	e_1

US = [1]
CS = [5, 3, 2]
DS = [5, 3, 0]

Casimir invariant:

$$e_1$$

- $\mathfrak{s}_{5,40}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,39}$)

	e_1	e_2	e_3	e_4
e_4	0	e_2	e_3	0
e_5	0	$-e_3$	e_2	e_1

US = [1]
CS = [5, 3, 2]
DS = [5, 3, 0]

Casimir invariant:

$$e_1$$

- $\mathfrak{s}_{5,41}$

	e_1	e_2	e_3	e_4
e_4	e_1	0	ae_3	0
e_5	0	e_2	be_3	0

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

where the values of the parameters a, b are

$$0 < |b| \leq |a| \leq 1.$$

If one or both equalities hold further restrictions on the parameters are necessary in order to avoid redundancies, as discussed in Section 10.4.
Casimir invariant:

$$\frac{e_1^a e_2^b}{e_3}$$

- $\mathfrak{s}_{5,42}$

	e_1	e_2	e_3	e_4
e_4	ae_1	e_2	e_3	0
e_5	e_1	0	e_2	0

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

Casimir invariant:

$$\frac{e_2^a}{e_1}\exp\left(\frac{e_3}{e_2}\right)$$

- $\mathfrak{s}_{5,43}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,41}$)

	e_1	e_2	e_3	e_4
e_4	αe_1	e_2	e_3	0
e_5	βe_1	$-e_3$	e_2	0

US = [0]
CS = [5, 3]
DS = [5, 3, 0]

where $\alpha^2 + \beta^2 \neq 0$.

Casimir invariant:

$$\frac{e_1^2}{(e_2^2+e_3^2)^\alpha}\exp\left(2\beta\arctan\frac{e_2}{e_3}\right)$$

18.6. Solvable five-dimensional Lie algebras with the nilradical $\mathfrak{n}_{3,1}$

	e_1	e_2	e_3
e_1	0	0	0
e_2		0	e_1

- $\mathfrak{s}_{5,44}$

	e_1	e_2	e_3	e_4
e_4	e_1	e_2	0	0
e_5	0	e_2	$-e_3$	0

US = [0]
CS = [5, 3]
DS = [5, 3, 1, 0]

Casimir invariant:

$$\frac{e_1e_5+e_2e_3}{e_1}$$

- $\mathfrak{s}_{5,45}$ (over the field $\mathbb{C}$: isomorphic to $\mathfrak{s}_{5,44}$)

	e_1	e_2	e_3	e_4
e_4	$2e_1$	e_2	e_3	0
e_5	0	e_3	$-e_2$	0

US = [0]
CS = [5, 3]
DS = [5, 3, 1, 0]

Casimir invariant:

$$\frac{2e_1e_5+e_2^2+e_3^2}{e_1}$$

18.7. Five-dimensional Levi decomposable Lie algebra

- $\mathfrak{sl}(2,\mathbb{F}) \niplus 2\mathfrak{n}_{1,1}$

	e_1	e_2	e_3	e_4	e_5
e_1	0	$2e_1$	$-e_2$	e_5	0
e_2		0	$2e_3$	e_4	$-e_5$
e_3			0	0	e_4
e_4				0	0

US = [0]
CS = [5]
DS = [5]

Casimir invariant:

$$e_1e_4^2 - e_2e_4e_5 - e_3e_5^2$$