Preface

The symplectic revolution of the 1980s gave rise to the discovery of surprising rigidity phenomena involving symplectic manifolds, their subsets, and their diffeomorphisms. These phenomena have been detected with the help of a variety of novel powerful methods, including Floer theory, a version of Morse theory on the loop spaces of symplectic manifolds. A number of recent advances show that there is yet another manifestation of symplectic rigidity, taking place in function spaces associated to a symplectic manifold. These spaces exhibit unexpected properties and interesting structures, giving rise to an alternative intuition and new tools in symplectic topology, and providing a motivation to study the *function theory on symplectic manifolds*, which forms the subject of the present book.

Recall that a symplectic structure on a $2n$-dimensional manifold M is given by a closed differential 2-form ω which in appropriate local coordinates is given by $\omega = \sum_{j=1}^{n} \mathrm{d}p_j \wedge \mathrm{d}q_j$. The Poisson bracket of a pair of smooth compactly supported functions F, G on M is a canonical operation given by

$$\{F, G\} = \sum_j \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right).$$

The Poisson bracket, which is one of our main characters, plays a fundamental role in symplectic geometry and its applications. For instance, it governs Hamiltonian mechanics. The symplectic manifold M serves as the phase space of a mechanical system. The evolution (or Hamiltonian flow) $h_t \colon M \to M$ of the system is determined by its time-dependent energy $H_t \in C^\infty(M)$. Hamilton's famous equation describing the motion of the system is given, in the Heisenberg picture, by $\dot{F}_t = \{F_t, H_t\}$, where $F_t = F \circ h_t$ stands for the time evolution of an observable function F on M under the Hamiltonian flow h_t. The diffeomorphisms h_t coming from all possible energies H_t form a group $\mathrm{Ham}(M, \omega)$, called the group of Hamiltonian diffeomorphisms. For closed simply connected manifolds this group is just the identity component of the symplectomorphism group. The group Ham can be considered as an infinite-dimensional Lie group. The function space $C^\infty(M)$ is, roughly speaking, the Lie algebra of this group, and the Poisson bracket is its Lie bracket.

The structure of the function theory we are going to develop can be illustrated with the help of the following picture. Fix your favorite $t > 0$

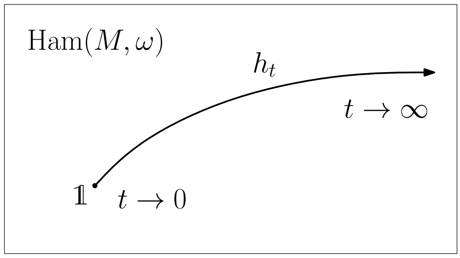

FIGURE 0.1. Two opposite regimes

and consider the natural mapping $C^\infty(M) \to \mathrm{Ham}(M)$ which takes a (time-independent) function H to the time-t map h_t of the corresponding Hamiltonian flow. In principle, this mapping enables one to translate information about Hamiltonian diffeomorphisms (which nowadays is quite a developed subject, see Chapter 4) into the language of function spaces. This naive plan works successfully in two opposite regimes, infinitesimal (when $t \to 0$) and asymptotic (when $t \to \infty$) (see Figure 0.1).

Working in the infinitesimal regime, one arrives at a surprising phenomenon of *C^0-robustness of the Poisson bracket.* Observe that the expression for the Poisson bracket involves the first derivatives of the functions F and G. Nevertheless, the functional $\Phi(F,G) := \|\{F,G\}\|$, where $\|\cdot\|$ stands for the uniform norm of a function, exhibits robustness with respect to C^0-perturbations. In particular, as we shall show in Chapter 2, Φ is lower semicontinuous in the uniform norm. Even though this result sounds analytical in nature, it turns out to be closely related to a remarkable bi-invariant geometry on the group $\mathrm{Ham}(M,\omega)$ discovered by Hofer in 1990. We shall discuss various facets of C^0-robustness of the Poisson bracket. One of them is the *Poisson bracket invariant* of a quadruple of subsets of a symplectic manifold discussed in Chapter 7. Its definition is based on an elementary looking variational problem involving the functional Φ, while its study involves a variety of methods of "hard" symplectic topology. Another facet is symplectic approximation theory, discussed in Chapter 8. Its basic objective is to find an optimal uniform approximation of a given pair of functions by a pair of (almost) Poisson commuting functions.

The asymptotic regime gives rise to the theory of *symplectic quasi-states* presented in Chapter 5. A symplectic quasi-state is a monotone functional $\zeta\colon C^\infty(M) \to \mathbb{R}$ with $\zeta(1) = 1$ which is linear on every Poisson-commutative subalgebra, but not necessarily on the whole function space. The origins of this notion go back to foundations of quantum mechanics and Aarnes' theory of topological quasi-states, an interesting branch of abstract functional analysis. In our context, nonlinear quasi-states on higher-dimensional manifolds are provided by Floer theory, the cornerstone of modern symplectic topology. Interestingly enough, symplectic quasi-states are closely related to *quasi-morphisms* on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega)$.

Roughly speaking, a quasi-morphism on a group is "a homomorphism up to a bounded error." This group-theoretical notion coming from bounded cohomology has been intensively studied in the past decade due to its various applications to geometry and dynamics. We discuss it in Chapter 3. A recent survey of quasi-states and quasi-morphisms in symplectic topology can be found in Entov's ICM-2014 talk [57].

Quasi-states serve as a useful tool for a number of problems in symplectic topology such as symplectic intersections, Hofer's geometry on groups of Hamiltonian diffeomorphisms, and Lagrangian knots. These applications are presented in Chapter 6. In addition, quasi-states provide yet another insight into robustness of the Poisson brackets, see Section 4.6.

Besides applications to some mainstream problems in symplectic topology, function theory on symplectic manifolds opens up a prospect of using "hard" symplectic methods in quantum mechanics. Mathematical quantization and, mostnotably, the quantum-classical correspondence principle provide a tool which enables one to translate basic notions of classical mechanics into quantum language. In general, a meaningful translation of symplectic rigidity phenomena involving subsets and diffeomorphisms faces serious analytical and conceptual difficulties. However, such a translation becomes possible if one shifts the focus from subsets and morphisms of manifolds to function spaces. We present some first steps in this direction in Chapter 9.

The book is a fusion of a research monograph on function theory on symplectic manifolds and an introductory survey of symplectic topology. On the introductory side, the first chapter discusses some basic symplectic constructions and fundamental phenomena, including the Eliashberg–Gromov C^0-rigidity theorem, Arnold's symplectic fixed point conjecture, and Hofer's metric, while in the last three chapters the reader will find an informal crash course on Floer theory. Even though our intention was to make the book as self-contained as possible, the reader is encouraged to consult earlier symplectic literature, such as the classical monographs [107, 108] by McDuff and Salamon. We also refer the reader to the manuscript by Oh [121] on Floer theory. The reader is assumed to have familiarity with basic differential and algebraic topology.

Most of the results presented in the book are based on a number of joint papers by L.P. with Michael Entov. L.P. expresses his gratitude to Michael for long years of pleasant collaboration. Furthermore, some central results of the book are joint with Lev Buhovsky (Poisson bracket invariants and symplectic approximation), Yakov Eliashberg (Lagrangian knots), and Frol Zapolsky (Poisson bracket inequality and rigidity of partitions of unity). L.P. cordially thanks all of them.

Parts of the material have been taught by L.P. in graduate courses at University of Chicago and Tel Aviv University, in a lecture series at UCLA, and (with the assistance of D.R.) in a mini-course at University of Melbourne. We thank these institutions for such an invaluable opportunity. We are indebted to Strom Borman, Lev Buhovsky, François Charette, Adi Dickstein, Michael Entov, Morimichi Kawasaki, Asaf Kislev, Karina Samvelyan, Egor Schelukhin, and Frol Zapolsky for very useful comments on the manuscript, as well as to Dorit Aharonov, Peter Albers, Paul Biran, Paul Busch, Danny Calegari, Yakov Eliashberg, Helmut Hofer, Vincente Humilière, Gil Kalai, Vadim Kaloshin, Yael Karshon, Guy Kindler, Dusa McDuff, Yong-Geun Oh, Yaron Ostrover, Iosif Polterovich, Victor Polterovich, Dietmar Salamon, Felix Schlenk, Paul Seidel, Sobhan Seyfaddini, Ivan Smith, Misha Sodin, Shmuel Weinberger, Amie Wilkinson, and Jinxin Xue for illuminating discussions on various topics of the book. Additionally, we thank the anonymous referees for very helpful critical remarks.

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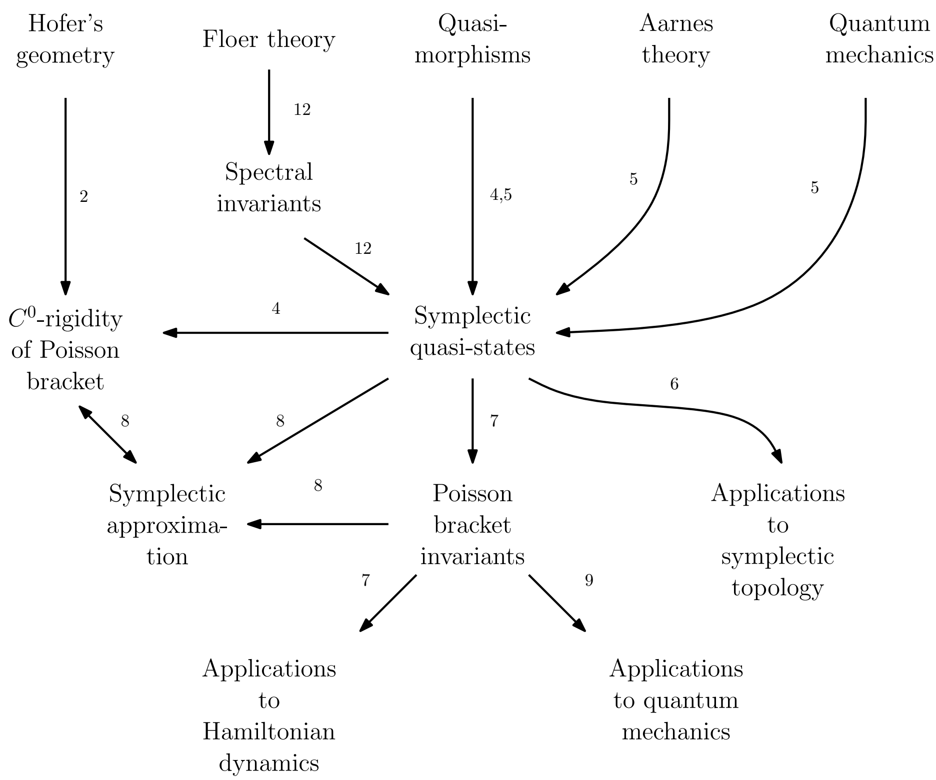

FIGURE 0.2. Subject road map. Numbers next to arrows indicate relevant chapters.