CHAPTER 1

Three Wonders of Symplectic Geometry

This chapter is a mixture of a rapid introduction to symplectic topology and a review of some of its landmark achievements. We mention three wonders of symplectic topology, the first being the Eliashberg–Gromov C^0-rigidity theorem [52, 77], which is the first of several manifestations of symplectic rigidity we will encounter in this book. The second is Arnold's conjecture [4, 6] about symplectic fixed points, which served as a driving force for many modern developments in symplectic topology, among them Floer theory, to which we will return towards the end of the book. The third wonder is Hofer's geometry on the group of Hamiltonian diffeomorphisms [85, 95, 131]. Lastly, we end the chapter with several examples of symplectic manifolds and a brief discussion of J-holomorphic curves.

1.1. First wonder: C^0-rigidity

Let M^{2n} be a smooth connected $2n$-dimensional manifold without boundary, and let ω be a closed 2-form on M which satisfies the following condition:

$$(1.1) \qquad \omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}} \neq 0.$$

Then ω is called a *symplectic form*, and (M, ω) a *symplectic manifold*.

Note that ω^n is a top degree form, and hence by (1.1) a volume form, so in particular every symplectic manifold is orientable. The simplest examples include orientable surfaces equipped with area forms and their products. Here the product of two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is given by $(M_1 \times M_2, \omega_1 \oplus \omega_2)$. More sophisticated examples will be given at the end of this chapter.

A diffeomorphism f of a symplectic manifold (M, ω) is called a *symplectomorphism* if $f^* \omega = \omega$. The symplectomorphisms of (M, ω) form a group with respect to composition. We denote by $\mathrm{Symp}(M, \omega)$ the subgroup of all symplectomorphisms f with compact support: $fx = x$ for all points x outside a compact subset. By the C^k-topology, for $0 \leq k \leq \infty$, on $\mathrm{Symp}(M, \omega)$, and more generally, on the set of all diffeomorphisms of M, we mean the strong Whitney C^k-topology (see [84, Chapter 2]).

Note that symplectomorphisms are defined via their first derivatives, and hence the group $\mathrm{Symp}(M, \omega)$ is by its definition C^1-closed in the group $\mathrm{Diff}(M)$ of all compactly supported diffeomorphisms of M. However, for the same reason, a priori it is not obvious whether it is also C^0-closed.

A natural way to show that a certain class of transformations preserving a given tensor field on a manifold is C^0-closed is to characterize it by the conservation of a "C^0-robust" geometric quantity. Let us illustrate this by the following two examples:

• Let (M, g) be a closed Riemannian manifold. The group $\mathrm{Isom}(M, g)$ of all Riemannian isometries of (M, g) can be characterized by the preservation of the Riemannian distance $d(x, y)$ on M. If a sequence of diffeomorphisms f_k C^0-converges to a diffeomorphism f, we have that $d(f_k x, f_k y) \to d(fx, fy)$ for all $x, y \in M$. Thus if all f_k's are isometries, then $d(fx, fy) = d(x, y)$, and therefore f is also an isometry. We conclude that $\mathrm{Isom}(M, g)$ is C^0-closed in $\mathrm{Diff}(M)$.

• Let (M, σ) be an oriented manifold equipped with a volume form σ. The group $\mathrm{Diff}(M, \sigma)$ of all compactly supported σ-preserving diffeomorphisms of M can be characterized by the preservation of the volume $\int_U \sigma$ of open subsets $U \subset M$. It is easy to conclude from this that $\mathrm{Diff}(M, \sigma)$ is C^0-closed in $\mathrm{Diff}(M)$.

Even though no obvious candidate for such a C^0-robust quantity exists in the case of symplectomorphisms, the above phenomenon persists for symplectic manifolds [52; 77, Section 3.4.4]:

Theorem 1.1.1 (Eliashberg–Gromov rigidity theorem). *Let (M, ω) be a symplectic manifold. Then $\mathrm{Symp}(M, \omega)$ is C^0-closed in the group of all smooth compactly supported diffeomorphisms of M.*

We shall prove this result in Section 2.2 below by using methods of function theory on symplectic manifolds.

1.2. Second wonder: Arnold's conjecture

1.2.1. Mathematical model of classical mechanics. Before studying the properties of symplectic maps, a natural question to ask is whether such maps exist at all. For instance, a generic Riemannian metric on a manifold of dimension ≥ 2 admits no isometries except the identity map (see, e.g., [152, Proposition 1]). It turns out that in symplectic geometry the situation is quite different: an infinite-dimensional group of symplectomorphisms naturally arises within the mathematical model of classical mechanics. To describe this, we first need to discuss some linear algebra.

Let E^{2n} be a real vector space, equipped with an antisymmetric bilinear form $\omega \colon E \times E \to \mathbb{R}$. Define the map

$$I_\omega \colon E \to E^*, \quad \xi \mapsto i_\xi \omega = \omega(\xi, \cdot).$$

Exercise 1.2.1. Prove that the following conditions are equivalent:

(1) $\omega^n \neq 0$.
(2) I_ω is an isomorphism.

If the above equivalent conditions hold, E is called a *symplectic vector space*. For a more detailed account of symplectic vector spaces, we refer the reader to Chapter 4 in [107].

The basic example of a symplectic vector space is $\mathbb{R}^{2n}$, with coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$, equipped with the symplectic form $\omega_0 = \sum_k \mathrm{d}p_k \wedge \mathrm{d}q_k$, which we sometimes abbreviate to $\mathrm{d}p \wedge \mathrm{d}q$. According to the classical Darboux theorem, it provides a local model for the symplectic structure on an arbitrary symplectic manifold:

Theorem 1.2.2 (Darboux). *Let (M^{2n}, ω) be a symplectic manifold, and let $x \in M$. There exist local coordinates $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ near x such that in these coordinates we have $\omega = \sum_k \mathrm{d}p_k \wedge \mathrm{d}q_k$.*

Thus, symplectic manifolds have no local invariants, in contrast with Riemannian manifolds, which can be locally distinguished by their curvature tensor. We refer to these local coordinates as *a Darboux chart*. For proofs, see [107, Section 3.1] or [7, Section 43 B].

In classical mechanics, a symplectic manifold (M, ω) plays the role of the phase space of a system. A mechanical system is described by its Hamiltonian, or energy function, $H \colon M \times \mathcal{I} \to \mathbb{R}$. Here $\mathcal{I} \subset \mathbb{R}$ is a time interval which is usually assumed to contain 0. According to a basic principle of classical mechanics, the energy determines the time evolution of the system, in the following way.

We denote $H_t(x) := H(x, t)$. Define[1] the *symplectic gradient*, or Hamiltonian vector field, of H by

$$\operatorname{sgrad} H_t = -I_\omega^{-1}(\mathrm{d}H_t),$$

where $I_\omega \colon TM \to T^*M$ is the bundle isomorphism obtained by applying the isomorphism introduced in Exercise 1.2.1 fiber-wise. That is, for any vector field η on M,

$$\omega(\eta, \operatorname{sgrad} H_t) = \mathrm{d}H_t(\eta).$$

The *Hamilton equation* is the following ordinary differential equation on M:

$$\dot{x}(t) = \operatorname{sgrad} H_t\big(x(t)\big). \tag{1.2}$$

It gives rise to a one-parameter family of diffeomorphisms $\phi_H^t \colon M \to M$, defined by $\phi_H^t(x_0) = x(t)$, where $x(\cdot)$ is the unique solution of (1.2) with initial condition $x(0) = x_0$. (On noncompact manifolds, some extra assumptions on the behavior of H_t at infinity are required in order to guarantee that the solution $x(t)$ exists for all $t \in \mathcal{I}$.) The family $\{\phi_H^t\}$ is called the *Hamiltonian flow* of H, and each diffeomorphism in the family is called a *Hamiltonian diffeomorphism*.

Proposition 1.2.3. *The flow ϕ_H^t preserves ω.*

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[1]Different authors may use different signs in the definitions of certain notions playing an important role in this book. This includes, in particular, Hamiltonian vector fields and Poisson brackets.

PROOF. The proof is a straightforward computation using Cartan's formula (we use the notation $\mathcal{L}_X$ for the Lie derivative with respect to the vector field X):

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_H^t)^*\omega = (\phi_H^t)^*\mathcal{L}_{\mathrm{sgrad}\,H_t}\omega = (\phi_H^t)^*(\mathrm{d}i_{\mathrm{sgrad}\,H_t}\omega + i_{\mathrm{sgrad}\,H_t}\underbrace{\mathrm{d}\omega}_{=0})$$

$$= (\phi_H^t)^*(-\mathrm{d}^2 H_t) = 0.$$

Hence, $(\phi_H^t)^*\omega = (\phi_H^0)^*\omega = \omega$ for all $t \in \mathcal{I}$. □

Therefore, any Hamiltonian diffeomorphism is a symplectomorphism of (M, ω).

Example 1.2.4. Consider $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_0 = \mathrm{d}p \wedge \mathrm{d}q$. We interpret q as the position vector, and p as the momentum. In this case the Hamilton equation (1.2) takes the form

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}; \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}. \end{cases}$$

Consider the particular case of the motion of a particle with mass m in $\mathbb{R}^n(q)$, in the presence of a potential $U(q)$ which depends only on the position. The force in this case is $F = -\partial U/\partial q$. The velocity of the particle is $\dot{q}$ and its momentum is defined by $p = m\dot{q}$. The Hamiltonian of the particle is its total energy $H(p, q) = K(p) + U(q)$, where

$$K = \frac{1}{2}m\dot{q}^2 = \frac{p^2}{2m}$$

is the *kinetic energy*. Thus,

$$H(p, q) = \frac{p^2}{2m} + U(q).$$

Hence, Hamilton's equations are

$$\begin{cases} \dot{q} = \dfrac{p}{m}; \\ \dot{p} = -\dfrac{\partial U}{\partial q} = F. \end{cases}$$

Combining these two equations, we obtain *Newton's second law*: $m\ddot{q} = F$. The passage from solving Newton's equation, a second order ODE in the configuration space $\mathbb{R}^n$, to Hamilton's equation, a first order ODE in the phase space $\mathbb{R}^{2n}$, brings classical mechanics into the framework of the theory of dynamical systems.

One of the first important results in classical mechanics was Liouville's theorem, which states that the time evolution of a mechanical system under Hamilton's equation preserves volume in the phase space. Since the natural

volume form on M is, up to a constant factor, ω^n, this theorem follows from Proposition 1.2.3 above.

The discussion above leads us to the following question: *which dynamical features distinguish symplectomorphisms from general volume preserving maps*? One of the answers is given in the next section.

1.2.2. Fixed points of Hamiltonian diffeomorphisms. Let $f \colon M \to M$ be a smooth map. Denote by $\mathrm{Fix}(f)$ its set of fixed points. We say that $x \in \mathrm{Fix}(f)$ is a nondegenerate fixed point if 1 is not an eigenvalue of the differential f_{*x}, or, equivalently, if graph $f \subset M \times M$ intersects the diagonal $\Delta = \{(y, y) : y \in M\}$ transversally at (x, x). We write $\# X$ for the cardinality of a set X.

Conjecture 1.2.5 (Arnold [4, 6]). *Let (M, ω) be a closed symplectic manifold. Let $f \colon M \to M$ be a Hamiltonian diffeomorphism with only nondegenerate fixed points. Then*

$$\# \mathrm{Fix}(f) \geq \dim_{\mathbb{R}} H_*(M; \mathbb{R}) = \sum_k \dim_{\mathbb{R}} H_k(M; \mathbb{R}).$$

Starting from pioneering works by Conley and Zehnder [48] and Floer [67], Arnold's conjecture became a central subject in symplectic topology. It is currently established for all symplectic manifolds, though the amount of subtle technicalities increases drastically when one deals with the general case, and there is no consensus among the experts on whether all of them are completely fixed or not, yet. Fortunately, for a reasonably broad and interesting class of symplectic manifolds the proof of Arnold's conjecture is transparent. In Chapter 11 we will outline an argument based on the powerful machinery of Floer homology.

The link between fixed points of Hamiltonian diffeomorphisms and Morse theory can be clearly seen in the following toy example: Let f_t be the Hamiltonian flow of a *time-independent* Hamiltonian F. Assume further that F is a Morse function, that is, for every critical point of F, $x_0 \in \mathrm{Crit}(F)$, $\det\big(\partial^2 F / \partial x^2(x_0)\big) \neq 0$. Then for any sufficiently small t, f_t has nondegenerate fixed points only. Observe that $\mathrm{sgrad}\, F(x_0) = 0$ for $x_0 \in \mathrm{Crit}(F)$, and hence $f_t x_0 = x_0$. This means that every critical point of F is a fixed point of f_t. In fact, it follows from a theorem by Yorke [162] that for small values of t the converse statement is true: Every fixed point of f_t is a critical point of F. Thus, by the Morse inequality

$$\# \mathrm{Fix}(f_t) = \# \mathrm{Crit}(F) \geq \dim H_*(M; \mathbb{R}).$$

Remark 1.2.6. Incidentally, Arnold's conjecture illustrates the fact that, in general, not every symplectomorphism is Hamiltonian, even if it is isotopic to the identity through symplectomorphisms. Consider, for instance, the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ equipped with the standard area form. Here

$$\dim H_*(\mathbb{T}^2; \mathbb{R}) = \dim H_0(\mathbb{T}^2; \mathbb{R}) + \dim H_1(\mathbb{T}^2; \mathbb{R}) + \dim H_2(\mathbb{T}^2; \mathbb{R})$$
$$= 1 + 2 + 1 = 4.$$

Take a family of (area-preserving!) translations $f_v \colon x \mapsto x + v$, $v \in \mathbb{R}^2$. If $v \neq \mathbb{Z}^2$, the translation f_v has no fixed points, and hence it is not a Hamiltonian diffeomorphism. The last statement can be proved without referring to Arnold's conjecture, see, e.g., [132, Chapter 14] for an elementary proof.

1.3. Third wonder: Hofer's metric

1.3.1. The group of Hamiltonian diffeomorphisms. Let (M, ω) be a symplectic manifold. Having in mind an allusion to classical mechanics (see Section 1.2.1 above), by a Hamiltonian on M we mean a smooth time-dependent function $H \colon M \times \mathcal{I} \to \mathbb{R}$, where $\mathcal{I} \subset \mathbb{R}$ is an interval containing 0. We denote, as before, $H_t(\cdot) = H(\cdot, t)$. A time-independent Hamiltonian is called *autonomous*, in which case it is considered as a function on M. The Hamiltonian flow of H is denoted by φ_H^t or h_t. Observe that after adding to H a function of t we get a new Hamiltonian $H'(x, t) = H(x, t) + a(t)$, for which $\operatorname{sgrad} H' = \operatorname{sgrad} H$, and hence H and H' generate the same dynamics. This ambiguity can be resolved if one deals with normalized Hamiltonians. We call a Hamiltonian $H(x, t)$ *normalized* if

- $\int_M H_t \omega^n = 0$ for all $t \in \mathcal{I}$ in the case when the manifold M is closed;
- $\bigcup_{t \in \mathcal{I}} \operatorname{supp} H_t \subset K$ for some compact $K \subset M$ when M is open.

We denote the space of all normalized *time-independent* Hamiltonians by $\mathcal{H}$.

The maps h_t forming the Hamiltonian flow of a normalized Hamiltonian are called *Hamiltonian diffeomorphisms*. By Proposition 1.2.3, they preserve the symplectic form ω. Denote by $\operatorname{Ham}(M, \omega)$ the set of all Hamiltonian diffeomorphisms. When ω or M are clear from the context, we will sometimes abbreviate this to $\operatorname{Ham}(M)$, or simply Ham.

In classical geometry one often deals with groups of all transformations preserving a given geometric structure on a manifold, such as isometries of a Riemannian metric, volume preserving diffeomorphisms, or symplectomorphisms. Since Hamiltonian diffeomorphisms are not defined via preservation of any structure, it is not immediately clear that they form a group. However, as the name of this section implies, this is indeed the case.

Proposition 1.3.1. *The set* $\operatorname{Ham}(M, \omega)$ *is a group with respect to composition.*

We break the proof of this important proposition into several steps.

Exercise 1.3.2. Let $\phi \colon M \to M$ be a symplectomorphism, and let $F \colon M \to \mathbb{R}$ be a smooth function. Prove that

$$\operatorname{sgrad}(F \circ \phi^{-1}) = \phi_* \operatorname{sgrad} F.$$

Proposition 1.3.3 (Product formula). *Let $\{f_t\}, \{g_t\}, 0 \leq t \leq 1$, be the Hamiltonian flows generated by normalized Hamiltonians F_t, G_t, respectively. Then the flow $h_t := f_t g_t$ is generated by the normalized Hamiltonian $H_t := F_t + G_t \circ f_t^{-1}$.*

PROOF. We know that

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t x = \operatorname{sgrad} F_t(f_t x), \quad \frac{\mathrm{d}}{\mathrm{d}t} g_t x = \operatorname{sgrad} G_t(g_t x).$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} h_t x &= \operatorname{sgrad} F_t(f_t g_t x) + f_{t*} \operatorname{sgrad} G_t(g_t x) \\
&= \operatorname{sgrad} F_t(f_t g_t x) + \operatorname{sgrad}(G_t \circ f_t^{-1})(f_t g_t x) \\
&= \operatorname{sgrad}(F_t + G_t \circ f_t^{-1})(f_t g_t x) = \operatorname{sgrad} H_t(h_t x). \qquad \square
\end{aligned}$$

Corollary 1.3.4. *Suppose that $\{f_t\}$ is the Hamiltonian flow generated by F_t. Then $\{f_t^{-1}\}$ is the Hamiltonian flow generated by $-F_t \circ f_t$.*

Every Hamiltonian diffeomorphism can be obtained as the time-one map h_1 of the Hamiltonian flow h_t generated by a normalized Hamiltonian $H: M \times [0,1] \to \mathbb{R}$. This readily follows from the next proposition.

Proposition 1.3.5 (Change of time in Hamiltonian flows). *Suppose that the Hamiltonian $H_t: M \times \mathbb{R} \to \mathbb{R}$ generates the flow $\{h_t\}$. Given any smooth function $a: \mathbb{R} \to \mathbb{R}$, the flow $\{h_{a(t)}\}$ is generated by the Hamiltonian $a'(t)H(x, a(t))$. In particular, one has the following* time rescaling *property: for a fixed $s \in \mathbb{R}$, the flow $\{h_{st}\}$ is generated by the Hamiltonian sH_{st}.*

The proof is left as an exercise. Using Proposition 1.3.3 and Corollary 1.3.4 we can now prove that $\operatorname{Ham}(M, \omega)$ is a group.

PROOF OF PROPOSITION 1.3.1. Indeed, if $f, g \in \operatorname{Ham}(M, \omega)$ are the time-one maps of the normalized Hamiltonians F, G, respectively, then by Proposition 1.3.3, fg is the time-one map of the normalized Hamiltonian $F_t + G_t \circ f_t^{-1}$. Additionally, if f is the time-one map of F, then f^{-1} is the time-one map of the normalized Hamiltonian $-F_t \circ f_t$. Therefore, $\operatorname{Ham}(M, \omega) \subset \operatorname{Symp}(M, \omega)$ is a subgroup. $\qquad \square$

By definition, if $f \in \operatorname{Ham}(M, \omega)$, there exists a path $\{f_t\} \subset \operatorname{Ham}(M, \omega)$ such that $f_0 = \mathbb{1}$ and $f_1 = f$. Consider now an arbitrary path $\{f_t\} \subset \operatorname{Ham}(M, \omega)$ which starts at $\mathbb{1}$. Although each f_t is a Hamiltonian diffeomorphism, a priori there is no reason why $\{f_t\}$ itself should be the Hamiltonian flow of some Hamiltonian. However, this turns out to be true, due to a theorem by Banyaga [13, Proposition II.3.3; 107, Proposition 10.17].

Theorem 1.3.6. *Every smooth path in $\operatorname{Ham}(M, \omega)$ based at the identity is the Hamiltonian flow generated by some normalized Hamiltonian.*

An elementary particular case of this statement is valid in the following more general context: Let (M, ω) be a symplectic manifold (open or closed). Recall that $\mathrm{Symp}(M, \omega)$ stands for the group of all compactly supported symplectomorphisms of (M, ω). Denote by $\mathrm{Symp}_0(M, \omega)$ its identity component. Since $\mathrm{Ham}(M, \omega)$ is by definition path connected, we have the inclusions

$$\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}_0(M, \omega) \subset \mathrm{Symp}(M, \omega).$$

In general, $\mathrm{Ham}(M, \omega) \neq \mathrm{Symp}_0(M, \omega)$. However, in some cases these two groups coincide:

Proposition 1.3.7. *Suppose that the first cohomology with compact support of M vanishes, that is $H_c^1(M; \mathbb{R}) = 0$. Then $\mathrm{Ham}(M, \omega) = \mathrm{Symp}_0(M, \omega)$.*

PROOF. Let $f \in \mathrm{Symp}_0(M, \omega)$. Then there is a path $\{f_t\} \subset \mathrm{Symp}_0(M, \omega)$ such that $f_0 = \mathbb{1}$ and $f_1 = f$. Define the time-dependent vector field ξ_t by

$$\xi_t(f_t x) = \frac{\mathrm{d}}{\mathrm{d}t} f_t x.$$

Then

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f_t^* \omega = f_t^* \mathcal{L}_{\xi_t} \omega.$$

Since $\mathrm{d}\omega = 0$, by Cartan's formula we obtain $\mathrm{d}i_{\xi_t}\omega = \mathcal{L}_{\xi_t}\omega = 0$, that is, for every t the form $i_{\xi_t}\omega$ is closed. Since $H_c^1(M; \mathbb{R}) = 0$, these forms are exact: $i_{\xi_t}\omega = -\mathrm{d}F_t$, where $F_t \colon M \to \mathbb{R}$ has compact support. Therefore $\xi_t = \mathrm{sgrad}\, F_t$, and the path $\{f_t\}$ is the Hamiltonian flow of the time-dependent Hamiltonian F_t. $\square$

Although in general the inclusion $\mathrm{Ham} \subset \mathrm{Symp}_0$ is strict (see Remark 1.2.6 above), the difference between the two groups is 'not too big'. The next statement, which was an important open problem known as the flux conjecture, was finally settled by Ono [122]:

Theorem 1.3.8. *For every closed symplectic manifold (M, ω)*

$$\mathrm{Symp}_0(M, \omega)/\mathrm{Ham}(M, \omega) = H^1(M; \mathbb{R})/\Gamma,$$

where $\Gamma \subset H^1(M; \mathbb{R})$ is a discrete subgroup.

For instance, for the torus $(\mathbb{T}^2, \mathrm{d}p \wedge \mathrm{d}q)$ the group Γ is simply the integral lattice $H^1(\mathbb{T}^2; \mathbb{Z}) \subset H^1(\mathbb{T}^2; \mathbb{R})$, see, e.g., [132].

Exercise 1.3.9. The group $\mathrm{Ham}(M, \omega)$ acts on M by its definition. Prove that when M is connected this action is transitive. *Hint*: First prove the result for the open ball $B^{2n}(r) \subset \mathbb{R}^{2n}$. For the general case, use the previous one and Darboux's Theorem 1.2.2 to prove that for any $x \in M$, the orbit $\{f(x) : f \in \mathrm{Ham}(M, \omega)\}$ is open. Deduce the result.

1.3.2. Ham as a Lie group. A useful viewpoint to adopt is to consider $\mathrm{Ham}(M,\omega)$ as an infinite-dimensional Lie group. Its Lie algebra, which is by definition the tangent space at the identity $T_{\mathbb{1}}\,\mathrm{Ham}$, consists of all (time-independent) Hamiltonian vector fields, and therefore can be identified with the space $\mathcal{H}$ of all normalized autonomous Hamiltonians on M. Further, we identify the tangent space to Ham at an arbitrary point g with the Lie algebra via right translations:

$$T_g\,\mathrm{Ham} = (R_g)_* T_{\mathbb{1}}\,\mathrm{Ham},$$

where

$$R_g\colon f \mapsto fg.$$

For example, consider a path $\{f_t\}$ generated by a normalized Hamiltonian F_t. Let us express the tangent vector

$$\xi := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} f_t \in T_{f_s}\,\mathrm{Ham}$$

in terms of the Hamiltonian. By our identification, ξ corresponds to

$$\left(R_{f_s}^{-1}\right)_* \xi = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} R_{f_s}^{-1} f_t = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} f_t f_s^{-1} = \mathrm{sgrad}\,F_t \circ (f_t f_s^{-1})|_{t=s} = \mathrm{sgrad}\,F_s.$$

That is, $(\mathrm{d}/\mathrm{d}t)|_{t=s} f_t$ corresponds to $F_s \in \mathcal{H}$. Note that here s is fixed, and F_s is thought of as an autonomous Hamiltonian.

Next, let us find the adjoint action of Ham on its Lie algebra. Let $f \in \mathrm{Ham}$, $G \in \mathcal{H}$ and let $\{g_t\}$ be the Hamiltonian flow of G. By definition,

$$\mathrm{Ad}_f(G) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f g_t f^{-1} = \mathrm{sgrad}(G \circ f^{-1})$$

by Exercise 1.3.2. By a slight abuse of notation we write $\mathrm{Ad}_f(G) = G \circ f^{-1}$.

Finally, let us calculate the Lie bracket on the Lie algebra $\mathcal{H}$. Take $F, G \in \mathcal{H}$, and let $\{f_t\}$ be the Hamiltonian flow of F. Again, by definition,

$$\{F, G\} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}_{f_t} G = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} G \circ f_t^{-1} = -\,\mathrm{d}G(\mathrm{sgrad}\,F)$$
$$= \omega(\mathrm{sgrad}\,G, \mathrm{sgrad}\,F).$$

This Lie bracket is called the *Poisson bracket* on $\mathcal{H}$. It can be rewritten as

$$\{F, G\} = \mathrm{d}F(\mathrm{sgrad}\,G).$$

The Poisson bracket was defined above for normalized Hamiltonians $F, G \in \mathcal{H}$, but the definition can be extended to any smooth functions $F, G \in C^\infty(M)$ by the same formula:

$$\{F, G\} = \omega(\mathrm{sgrad}\,G, \mathrm{sgrad}\,F) = \mathrm{d}F(\mathrm{sgrad}\,G).$$

Exercise 1.3.10. Show that in $(\mathbb{R}^{2n}, \mathrm{d}p \wedge \mathrm{d}q)$, the Poisson bracket takes the form

$$\{F, G\} = \sum_j \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right).$$

Two functions $F, G \in C^\infty(M)$ are said to be *Poisson commuting* if their Poisson bracket vanishes identically, $\{F, G\} \equiv 0$. Note that if F and G Poisson commute, then F is constant along trajectories of the Hamiltonian flow of G (and vice versa).

Exercise 1.3.11. Let F, G be Poisson commuting functions, and denote their Hamiltonian flows by f_t and g_t, respectively. Prove that the flows commute:
$$f_t g_s = g_s f_t, \quad \forall s, t.$$
Prove that the flow $f_t g_t$ is generated by the Hamiltonian $F + G$.

Exercise 1.3.12. Prove that for any $F, G \in C^\infty(M)$
$$\mathrm{d}F \wedge \mathrm{d}G \wedge \omega^{n-1} = -\frac{1}{n}\{F, G\}\omega^n.$$
Deduce that the Poisson bracket of any two Hamiltonian functions has zero mean.

Exercise 1.3.13. Consider the unit sphere
$$S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$$
with the induced Euclidean area form. Use Exercise 1.3.12 to show that the Poisson brackets of the coordinate functions x_1, x_2, x_3 are given by the relation $\{x_1, x_2\} = -x_3$ and its cyclic permutations.

Exercise 1.3.14. Let (M, ω) and (N, σ) be two symplectic manifolds. Prove that a diffeomorphism $f \colon M \to N$ is a symplectomorphism if and only if it preserves the Poisson bracket, that is, for any $\phi, \psi \in C^\infty(N)$ one has
$$\{f^*\phi, f^*\psi\} = f^*\{\phi, \psi\}.$$

1.3.3. Hofer's metric on $\mathrm{Ham}(M, \omega)$. Recall that a *Finsler metric* on a manifold is a smooth assignment of a norm on each tangent space. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is equipped with a norm $\|\cdot\|$ which is invariant under the adjoint action. Then we can extend $\|\cdot\|$ to $T_g G$ for all $g \in G$, by right or left translations, and by the invariance of $\|\cdot\|$ both yield the same result. In this way G becomes a Finsler manifold, which enables us to introduce in the standard way various geometric notions on G. First, define the length of a curve:
$$\mathrm{length}\{g_t\} = \int_0^1 \|\dot{g}_t\| \, \mathrm{d}t.$$
Further, for $\phi, \psi \in G$ put
$$d(\phi, \psi) = \inf\{\mathrm{length}\{f_t\} : f_0 = \phi, f_1 = \psi\}.$$
The function d on $G \times G$ satisfies the axioms of a pseudo-metric, namely
- $d(\phi, \psi) \geq 0$;
- $d(\phi, \psi) = d(\psi, \phi)$;
- $d(\phi, \psi) + d(\psi, \theta) \geq d(\phi, \theta)$;

and the additional axiom of *bi-invariance*:

- $d(\theta\phi, \theta\psi) = d(\phi\theta, \psi\theta) = d(\phi, \psi)$.

Note that the only property d is a priori lacking in order to be a genuine metric is nondegeneracy:

- $d(\phi, \psi) > 0$ for all $\phi \neq \psi$.

Let us return to the group $\mathrm{Ham}(M, \omega)$. Here the Lie algebra is $\mathcal{H}$, and we have several choices of invariant norms on it, such as the L_p norms for $1 \leq p < \infty$ or the uniform norm $\|F\| = \max|F|$.

Theorem 1.3.15. *For the uniform norm, the pseudo-metric d is a genuine metric.*

This metric was introduced in 1990 by Hofer (see [85]), who proved its nondegeneracy for the linear symplectic space $\mathbb{R}^{2n}$. Later on, this result was extended to some more complicated symplectic manifolds in [131], and finally proved in full generality in [95]. All the proofs involves methods of "hard" symplectic topology. We present a proof for closed symplectic manifolds in Chapter 4 (see Corollary 4.7.2). We shall refer to the metric d associated to the uniform norm as the Hofer metric. Throughout the book we reserve the notation $\|\cdot\|$ for the uniform norm and d for the Hofer metric. Remarkably, Hofer's metric is essentially the unique nondegenerate bi-invariant Finsler metric on the group Ham [35]:

Theorem 1.3.16 (Buhovsky–Ostrover). *Let (M, ω) be a closed symplectic manifold. Let $\|\cdot\|'$ be a Ham-invariant norm on $\mathcal{H}$ which is continuous in the C^∞-topology, and let d' be the associated bi-invariant pseudo-metric on Ham. Then d' is nondegenerate if and only if the norm $\|\cdot\|'$ is equivalent to the uniform norm: there exist constants $C, c > 0$ so that*

$$c\|\cdot\| \leq \|\cdot\|' \leq C\|\cdot\|.$$

The existence of a bi-invariant Finsler metric on $\mathrm{Ham}(M, \omega)$ becomes especially surprising when confronted with the following two results. The first is a fundamental algebraic property of the group of Hamiltonian diffeomorphisms [13]. Recall that a group is called *simple* if it contains no nontrivial normal subgroups.

Theorem 1.3.17 (Banyaga). *Let (M, ω) be a closed symplectic manifold. Then $\mathrm{Ham}(M, \omega)$ is a simple group.*

The second result is a general fact about bi-invariant Finsler metrics on simple *finite-dimensional* Lie groups:

Proposition 1.3.18. *Let G be a simple finite-dimensional Lie group with a bi-invariant Finsler metric. Then G is compact.*

PROOF. It is a classical fact [109, Corollary 21.5] that if G admits a bi-invariant *Riemannian* metric and its Lie algebra $\mathfrak{g}$ has trivial center, then its Ricci curvature is strictly positive and bounded away from zero, and

hence the group is necessarily compact. Let us construct a bi-invariant Riemannian metric on G. Consider the unit ball $K \subset \mathfrak{g}$ with respect to the given Finsler metric. Let E be the unique ellipsoid of maximal volume contained in K (E is called the John ellipsoid, see [16, Section V.2]). Then E is the unit ball of a Euclidean norm on $\mathfrak{g}$. Moreover, by the uniqueness of the John ellipsoid, E is invariant under the adjoint action of G on $\mathfrak{g}$, and hence defines a bi-invariant Riemannian metric on G.

It remains to show that the Lie algebra $\mathfrak{g}$ has trivial center. Indeed, denote its center by $\mathcal{Z}$, and let $H = \exp \mathcal{Z}$ be its image under the exponential map. We note that for any $X \in \mathcal{Z}$ and $Y \in \mathfrak{g}$, X and Y commute, and hence

$$(1.3) \qquad \exp(X)\exp(Y) = \exp(X + Y) = \exp(Y)\exp(X).$$

Therefore, if we let $U \subset G$ be a neighborhood of the identity which lies in the image of the exponential map, then (1.3) implies that $HU = UH$, or in other words, U normalizes H. But since G is connected, it is generated by U, and hence H is a normal subgroup of G. Since G is simple, H is either trivial or all of G. But the latter cannot hold, since it would imply that G is abelian, again contradicting its simplicity. Therefore, H is trivial, and hence $\mathcal{Z}$ is trivial as well. This completes the proof. $\square$

The existence of Hofer's metric on Ham is thus a purely infinite-dimensional phenomenon. Let us mention also that for a compact manifold M, every degenerate pseudo-metric on $\mathrm{Ham}(M, \omega)$ vanishes identically. This follows from Theorem 1.3.17 and the following elementary observation:

Proposition 1.3.19. *Let G be a simple group, and let d be a bi-invariant pseudo-metric on G. If there exists $g \in G$ such that $d(\mathbb{1}, g) > 0$, then d is a metric.*

PROOF. Define
$$K = \{h \in G : d(\mathbb{1}, h) = 0\}.$$
Using the triangle inequality, K is easily seen to be a subgroup of G, and by the bi-invariance of d, it is a normal subgroup. By the simplicity of G, either $K = G$ or $K = \mathbb{1}$, and since $g \notin K$, the latter holds. That is, for any $h \neq \mathbb{1}$, $d(\mathbb{1}, h) > 0$. $\square$

1.3.4. A small scale in symplectic topology. There is a natural 'small scale' for sets in symplectic topology. The following notion is motivated by [85].

Definition 1.3.20. A set $X \subset M$ is called *displaceable* if there exists a Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(M, \omega)$ such that

$$\varphi(X) \cap \mathrm{Closure}(X) = \varnothing.$$

Example 1.3.21. Let us illustrate this notion in the following basic example. Consider the sphere S^2 equipped with an area form ω of total area 1. A simple closed smooth curve on S^2 is called *an equator* if it divides the sphere into two discs of equal areas.

Exercise 1.3.22. Show that every equator of S^2 is nondisplaceable.

Exercise 1.3.23. Show that the group $\mathrm{Ham}(S^2)$ acts transitively on the space of equators. *Hint*: Let L_0, L_1 be two equators. Choose an orientation preserving diffeomorphism ϕ with $\phi(L_0) = L_1$. Modify if necessary ϕ so that $\phi^*\omega$ coincides with ω on $T_{L_0}S^2$. A version of Moser's argument [107, Section 3.2; 114] yields a diffeomorphism ψ of S^2 so that ψ point-wise preserves L_0 and $\psi^*\phi^*\omega = \omega$. Thus, $\phi\psi$ is a symplectomorphism of S^2 taking L_0 to L_1. It remains to note that the group of symplectomorphisms $\mathrm{Symp}(S^2)$ of S^2 coincides with $\mathrm{Ham}(S^2)$. Indeed, Moser's argument shows that the group $\mathrm{Diff}_+(S^2)$ of orientation preserving diffeomorphisms of S^2 acts transitively on the space of symplectic forms on S^2 which are compatible with the orientation and have total area 1. The latter space is contractible (why?), and hence $\mathrm{Diff}_+(S^2)$ retracts to the stabilizer of the standard symplectic form, that is, to $\mathrm{Symp}(S^2)$. But the group $\mathrm{Diff}_+(S^2)$ is known to be path connected, see Exercise 1.5.3 below. Thus $\mathrm{Symp}(S^2) = \mathrm{Symp}_0(S^2)$. Finally, apply Proposition 1.3.7 to conclude that $\mathrm{Symp}_0(S^2) = \mathrm{Ham}(S^2)$.

On the other hand, let $B \subset S^2$ be a closed disc with smooth boundary of area $< \frac{1}{2}$. We claim that B is displaceable. Indeed, fix the standard model of the sphere,

$$S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3, \quad \omega = \frac{1}{4\pi}\Omega,$$

where Ω is the standard spherical area form. It readily follows from Exercise 1.3.23 that there exists $\phi \in \mathrm{Ham}(S^2)$ such that $\phi(B)$ lies in the upper hemisphere $\{x_3 > 0\}$ (prove this!). The latter is displaceable by the rotation by 180 degrees around the x_1-axis. Therefore, the disc B is displaceable.

Example 1.3.24. The product of equators in $(S^2 \times S^2, \omega \oplus \omega)$ is not displaceable. This can be shown with the help of Lagrangian Floer homology (see [117]). In Section 6.1.1 we extend this result to certain singular subsets of $S^2 \times S^2$ and prove it by using tools of function theory on symplectic manifolds.

Following Hofer, we define an important invariant of a displaceable subset $U \subset M$, *Hofer's displacement energy $e_H(U)$*, as $\inf d(\mathbb{1}, h)$, where the infimum is taken over all $h \in \mathrm{Ham}(M, \omega)$ displacing U. Remarkably, Hofer's displacement energy of every open displaceable subset is strictly positive. We shall prove this for closed symplectic manifolds in Chapter 4 (see Corollary 4.7.2). We follow the convention that $e_H(U) = +\infty$ if U is nondisplaceable.

1.4. The universal cover $\widetilde{\mathbf{Ham}}(M, \omega)$

In the present book we often work not on the group $\mathrm{Ham}(M, \omega)$ itself, but on its universal cover, denoted by $\widetilde{\mathrm{Ham}}(M, \omega)$ (partly for technical, and partly for conceptual reasons). It is convenient to think of an element of

$\widetilde{\text{Ham}}$ as a pair $(\phi, [\alpha])$, where $\phi \in \text{Ham}$ is a Hamiltonian diffeomorphism, $\alpha = \{\alpha_t\}_{t \in [0,1]}$ is a smooth path of Hamiltonian diffeomorphisms with $\alpha_0 = \mathbb{1}$ and $\alpha_1 = \phi$, and $[\alpha]$ stands for the homotopy class of α with fixed endpoints. The universal cover $\widetilde{\text{Ham}}$ is a group with respect to the composition

$$(\phi, [\alpha]) \circ (\psi, [\beta]) = (\phi\psi, [\{\alpha_t\beta_t\}]).$$

The identity element of $\widetilde{\text{Ham}}$, which is still denoted by $\mathbb{1}$, is given by $(\mathbb{1}, [\alpha])$, where α is the constant path.

Exercise 1.4.1. Show that $[\{\alpha_t\beta_t\}] = [\gamma]$, where γ is a concatenation of the paths α_{2t}, $t \in [0, \frac{1}{2}]$ and $\alpha_1\beta_{2t-1}$, $t \in [\frac{1}{2}, 1]$.

Exercise 1.4.2. Let $\{f_t\}_{t \in [0,1]}$ be the Hamiltonian flow generated by a 1-periodic Hamiltonian $F_{t+1} = F_t$. Denote $\tilde{f} = [\{f_t\}_{t \in [0,1]}] \in \widetilde{\text{Ham}}(M, \omega)$. Prove that the element $(\tilde{f})^k$ can be represented by the flow of the Hamiltonian kF_{kt}. *Hint*: Use Proposition 1.3.5 and Exercise 1.4.1.

The fundamental group $\pi_1(\text{Ham})$ is a subgroup of $\widetilde{\text{Ham}}$. Here every homotopy class of loops $\{\gamma_t\}$ with $\gamma_0 = \gamma_1 = \mathbb{1}$ is identified with the element $(\mathbb{1}, [\{\gamma_t\}])$ of $\widetilde{\text{Ham}}$.

Exercise 1.4.3. Show that the center of $\widetilde{\text{Ham}}$ coincides with $\pi_1(\text{Ham})$. Thus, we have a central extension

$$1 \to \pi_1(\text{Ham}) \to \widetilde{\text{Ham}} \to \text{Ham} \to 1.$$

Every Hamiltonian flow $\{f_t\}$, $t \in [0, T]$, with $f_0 = \mathbb{1}$ can be uniquely lifted to a path $\{\tilde{f}_t\}$ on $\widetilde{\text{Ham}}$, where

$$\tilde{f}_t = (f_t, [\{f_{st}\}_{s \in [0,1]}]).$$

Hofer's metric d on Ham lifts to a bi-invariant pseudo-metric $\tilde{d}$ on $\widetilde{\text{Ham}}$ as follows: Given an element $\tilde{\phi} = (\phi, [\alpha]) \in \widetilde{\text{Ham}}$, define

$$\tilde{d}(\mathbb{1}, \tilde{\phi}) = \inf \text{length}\,\beta,$$

where the infimum is taken over all paths $\beta = \{\beta_t\}_{t \in [0,1]}$ with $\beta_0 = \mathbb{1}$, $\beta_1 = \phi$, and $[\beta] = [\alpha]$. By definition, $\tilde{d}(\tilde{a}, \tilde{b}) = \tilde{d}(\mathbb{1}, \tilde{a}^{-1}\tilde{b})$. For many interesting symplectic manifolds one can show that $\tilde{d}$ is nondegenerate; however, this is still unknown in the general case: a priori, $\tilde{d}(\mathbb{1}, \tilde{\phi})$ might vanish for $\tilde{\phi} \in \pi_1(\text{Ham})$. We refer to [105, 106] for a discussion.

1.5. More examples: Kähler manifolds

An important example of symplectic manifolds is given by the Kähler manifolds, that is manifolds equipped with compatible symplectic and complex structures.

Let us first recall some linear algebra. Let E be a complex vector space equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$. Write

$$\langle u, v \rangle = g(u, v) - \mathrm{i}\omega(u, v).$$

Then $g(\cdot, \cdot)$ is a real inner product, ω an antisymmetric bilinear form, and the two are related by the formula

$$g(u, v) = \omega(u, \mathrm{i}v).$$

For example, in $\mathbb{R}^{2n}(p, q) \simeq \mathbb{C}^n(p + \mathrm{i}q)$ with the standard Hermitian inner product

$$\langle z, w \rangle = \sum_{k=1}^{n} z_k \overline{w}_k$$

we get the standard symplectic form $\omega_0 = \sum_k \mathrm{d}p_k \wedge \mathrm{d}q_k$.

Proposition 1.5.1. *The* 2-*form* ω *is nondegenerate.*

PROOF. Let $\xi \neq 0$. We must find η such that $\omega(\xi, \eta) \neq 0$. Take $\eta = \mathrm{i}\xi$. Then

$$\omega(\xi, \eta) = \omega(\xi, \mathrm{i}\xi) = g(\xi, \xi) > 0. \qquad \square$$

Definition 1.5.2. Let M^{2n} be a smooth manifold. An *almost complex structure* on M is a smooth field of automorphisms

$$J_x \colon T_x M \to T_x M$$

such that

$$J_x^2 = -\mathbb{1}.$$

An almost complex structure J is called *integrable* (or simply a *complex structure*) if M has the structure of a complex manifold (i.e., a smooth manifold with holomorphic transition maps $\mathbb{C}^n \to \mathbb{C}^n$) so that J is the multiplication by $\mathrm{i} = \sqrt{-1}$.

The *Newlander–Nirenberg theorem* [115] states that the almost complex structure J is integrable if and only if the *Nijenhuis tensor* $N_J \colon TM \otimes TM \to TM$,

$$N_J(\xi, \eta) := [\xi, \eta] + J[J\xi, \eta] + J[\xi, J\eta] - [J\xi, J\eta],$$

vanishes. Here ξ, η are local extensions of the vectors ξ and η to vector fields on M, and $[\cdot, \cdot]$ stands for the commutator. Observe that $N_J(\xi, \xi) = N_J(\xi, J\xi) = 0$. If $\dim M = 2$ and $\xi \neq 0$, the tangent space to M is generated by ξ and $J\xi$ and therefore N_J vanishes identically. We conclude that *every almost complex structure on a surface is integrable.* Furthermore, if $M = S^2$, the uniformization theorem of complex analysis states that all complex structures on M are diffeomorphic. Therefore the sphere S^2 carries a canonical (up to a diffeomorphism) almost complex structure.

Exercise 1.5.3. Prove Smale's theorem [150]: the group $\mathrm{Diff}_+(S^2)$ of orientation preserving diffeomorphisms of S^2 retracts to $SO(3)$. *Hint*: Denote by $\mathcal{J}$ the space of all almost complex structures on S^2 compatible with the orientation. By the above discussion, $\mathrm{Diff}_+(S^2)$ acts transitively on $\mathcal{J}$. Since $\mathcal{J}$ is a contractible space (why?), $\mathrm{Diff}_+(S^2)$ retracts to the stabilizer of the standard complex structure $j \in \mathcal{J}$, that is to the group of Möbius transformations $\mathrm{PSL}(2, \mathbb{C})$. The latter retracts to $SO(3)$.

Let M be a manifold equipped with a symplectic form ω and an almost complex structure J. We say that ω and J are *compatible* if $\omega(\cdot, J\cdot)$ defines a Riemannian metric.

Definition 1.5.4. A Kähler manifold is a manifold M^{2n} equipped with a symplectic form ω and a compatible complex structure J.

A basic example of a Kähler manifold is the complex projective space $\mathbb{C}P^n$ equipped with the Fubini–Study metric, described in Section 1.7 below. Furthermore, any complex submanifold of a Kähler manifold is necessarily Kähler with respect to the induced symplectic and complex structures (prove this!). In particular, every complex projective manifold is Kähler.

1.6. J-holomorphic curves

The theory of J-holomorphic curves, introduced by Gromov in his famous paper [76], is an important tool in studying symplectic manifolds. In this section we briefly introduce J-holomorphic curves, which will be used many times throughout this book. For a comprehensive treatment of the subject we refer the reader to the book [108], as well as the lecture notes [160].

Recall from Section 1.5 that an almost complex structure J on M is said to be *compatible* with ω if the bilinear form $\omega(\cdot, J\cdot)$ is a Riemannian metric. We refer the reader to [107] for preliminaries on compatible almost complex structures.

Exercise 1.6.1. Let J be a compatible almost complex structure on M, and let g be the corresponding Riemannian metric. Show that for every smooth function H on M its gradient is given by $\nabla H = -J \operatorname{sgrad} H$.

Given an almost complex structure J on M, the tangent bundle TM becomes a complex vector bundle and hence one can define its first Chern class $c_1 = c_1(TM, J)$ (see [30] or [110] for details about the first Chern class). The space of all compatible almost complex structures is known to be contractible, and in particular connected. Therefore, c_1 does not depend on the specific choice of a compatible J. In what follows we refer to $c_1(TM, J)$ as to the first Chern class of M and focus on its restriction to spherical classes only. In other words c_1 will stand for the functional $\pi_2(M) \to \mathbb{Z}$.

In general, almost complex manifolds do not carry holomorphic functions. It was a great insight of Gromov [76] that on symplectic manifolds

(M, ω) equipped with a compatible almost complex structure one can develop a theory of *curves* which to a large extent resembles classical algebraic geometry of curves in complex manifolds. Let (Σ, j) be a real surface equipped with an almost complex (and hence, complex, see Section 1.5) structure j. We call (Σ, j) a *Riemann surface*. A map $u: (\Sigma, j) \to (M, J)$ is called *J-holomorphic* if $u_* \circ j = J \circ u_*$. Geometrically, if u is J-holomorphic and $u_{*z} \neq 0$ for some $z \in \Sigma$, the tangent space $T_{u(z)}u(\Sigma) \subset T_{u(z)}M$ is invariant under J.

Definition 1.6.2. A J-holomorphic map from a Riemann surface to M is called a J-holomorphic (or pseudoholomorphic) curve.[2]

Exercise 1.6.3. Let $\mathbb{C}$ be the complex line equipped with the complex coordinate $x + iy$. Prove that a map $u: (\mathbb{C}, i) \to (M, J)$ is J-holomorphic if and only if

$$(1.4) \qquad \frac{\partial u}{\partial x} + J \frac{\partial u}{\partial y} = 0.$$

Note that in the case when $(M, J) = (\mathbb{C}, i)$ this becomes the usual Cauchy–Riemann equation.

More generally, any J-holomorphic curve $u: (\Sigma, j) \to (M, J)$ satisfies (1.4) in local complex coordinates $z = x + iy$ on Σ.

Exercise 1.6.4. Prove that for any nonconstant J-holomorphic curve $u: \Sigma \to M$

$$\int_\Sigma u^*\omega > 0.$$

Hint: Use the local form (1.4) to show that the integrand $u^*\omega$ is positive near a regular point $z \in \Sigma$ and is nonnegative everywhere.

1.7. Marsden–Weinstein reduction

Let us present a fundamental construction called the Marsden–Weinstein reduction [100], which in particular yields the above-mentioned Fubini–Study symplectic form on $\mathbb{C}P^n$. Let H be a time-independent Hamiltonian function on a symplectic manifold (M, ω) generating a Hamiltonian flow $\{h_t\}$.

Exercise 1.7.1 (Energy conservation law). Prove that $H \circ h_t = H$.

Therefore, M is foliated by the level sets of H, which are invariant under the flow $\{h_t\}$.

Exercise 1.7.2. Fix an energy level $\Sigma = \{H = c\}$, and take any point $x \in \Sigma$ such that $\operatorname{sgrad} H(x) \neq 0$. Show that the vector $\operatorname{sgrad} H(x)$ generates the ω-orthogonal subspace

$$T_x\Sigma^\omega = \{\xi \in T_x M : \omega(\xi, \eta) = 0 \; \forall \eta \in T_x\Sigma\}.$$

---

[2]The terminology *curve* comes from the fact that a Riemann surface is a one-dimensional complex manifold.

Assume now in addition that the flow h_t is T-periodic, i.e., $h_T = \mathbb{1}$ for some $T > 0$. In other words, h_t defines a Hamiltonian S^1-action on M. Suppose that Σ is a regular level set of H on which the action is free. Then the quotient manifold $N^{2n-2} = \Sigma/S^1$ carries a natural symplectic form σ, which is uniquely determined by the fact that $\pi^*\sigma = \omega|_\Sigma$, where $\pi \colon \Sigma \to N$ is the natural projection. Explicitly, let $x \in N$, and let $\xi, \eta \in T_x N$. Choose $\tilde{x} \in \pi^{-1}(x)$ and $\tilde{\xi}, \tilde{\eta} \in T_{\tilde{x}}\Sigma$ such that

$$\pi_*\tilde{\xi} = \xi, \quad \pi_*\tilde{\eta} = \eta.$$

Put

$$\sigma(\xi, \eta) = \omega(\tilde{\xi}, \tilde{\eta}).$$

We claim that the form σ is well defined. Observe that the vectors $\tilde{\xi}, \tilde{\eta}$ are defined up to addition of vectors from $\ker \pi_*$. The line $\ker \pi_*$ is tangent to the fiber $\pi^{-1}(x)$, and hence it is generated by $\operatorname{sgrad} H$. By Exercise 1.7.2, $\operatorname{sgrad} H$ lies in the kernel of $\omega|_\Sigma$. Therefore $\omega(\tilde{\xi}, \tilde{\eta})$ does not depend on the specific choice of the lifts $\tilde{\xi}$ and $\tilde{\eta}$. Furthermore, any two points $\tilde{x}, \tilde{y} \in \pi^{-1}(x)$ are related by $\tilde{y} = h_t\tilde{x}$ for some t. Since h_t is a symplectomorphism, one readily checks that $\sigma(\xi, \eta)$ does not depend on the lift $\tilde{x} \in \pi^{-1}(x)$. The claim follows.

We leave it as an exercise to show that the 2-form σ is closed and non-degenerate, and hence is a symplectic form on N. We refer to (N, σ) as the *reduced* symplectic manifold.

Example 1.7.3 (Harmonic oscillator). Consider $\mathbb{R}^{2n} \simeq \mathbb{C}^n(p + \mathrm{i}q)$, and take

$$H(p, q) = \pi \sum_{k=1}^{n} (p_k^2 + q_k^2).$$

Then the level set

$$\Sigma := \{H = 1\}$$

is the sphere S^{2n-1} of radius $1/\sqrt{\pi}$ centered at the origin. Hamilton's equations

$$\begin{cases} \dot{p}_k &= -2\pi q_k, \\ \dot{q}_k &= 2\pi p_k, \end{cases}$$

can be rewritten in the complex coordinates as

$$\dot{z}_k = 2\pi \mathrm{i} z_k.$$

Therefore, the Hamiltonian flow h_t defines the standard circle action

$$z \mapsto \mathrm{e}^{2\pi \mathrm{i}t} z$$

on Σ, and the reduced manifold is the complex projective space $\mathbb{C}P^{n-1}$. The resulting symplectic form σ on $\mathbb{C}P^{n-1}$ is called the Fubini–Study form. One readily checks that the complex structure induced from $\mathbb{C}^n$ is compatible with σ, which makes $\mathbb{C}P^{n-1}$ a Kähler manifold.

Exercise 1.7.4. Show that for the projective line $\mathbb{C}P^1 \subset \mathbb{C}P^{n-1}$

$$\int_{\mathbb{C}P^1} \sigma = 1.$$

Exercise 1.7.5. Identify

$$S^2 = \{|z|^2 + x^2 = 1\} \subset \mathbb{C} \times \mathbb{R}.$$

Define $\Phi\colon (\mathbb{C}P^1, \sigma) \to (S^2, 1/(4\pi)\Omega)$ by

$$[z_1 : z_2] \mapsto \left(\frac{2z_1\bar{z}_2}{|z_1|^2 + |z_2|^2}, \frac{|z_2|^2 - |z_1|^2}{|z_1|^2 + |z_2|^2} \right).$$

(Here σ is the Fubini–Study form on $\mathbb{C}P^1$ and Ω the induced Euclidean area form on S^2). Prove that Φ is a symplectomorphism.