CHAPTER 1

Overview

1.1. Introduction

A Shimura variety of orthogonal type arises from the Shimura datum consisting of the orthogonal group $\mathrm{SO}(L_{\mathbb{Q}})$ of a quadratic space $L_{\mathbb{Q}}$ of signature $(l-2,2)$ and the set

$$\mathbb{D} := \{N \subseteq L_{\mathbb{R}} \mid N \text{ oriented negative definite plane}\}$$

which has the structure of an Hermitian symmetric domain and can be interpreted as a conjugacy class of morphisms $\mathbb{S} = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_{\mathrm{m}} \to \mathrm{SO}(L_{\mathbb{R}})$. For any compact open subgroup $K \subseteq \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ we can form the Shimura variety

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathbb{D} \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K]$$

in which we always consider the quotient as an orbifold. It is a smooth manifold for sufficiently small K, and it always has an algebraic model M^K, in general a smooth Deligne–Mumford stack.[1]

For simplicity we assume for the moment that there is an integral lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ with cyclic discriminant group $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ of square-free order ε. Let K be the group of those integral isometries that induce the identity of the discriminant group. It goes back to Siegel [60] that in this case

$$\text{(1.1)} \quad \mathrm{vol}(\mathrm{M}^K) = 2^{-\lfloor (l-4)/2 \rfloor} \left(\prod_{i=1}^{\lfloor (l-1)/2 \rfloor} \zeta(1-2i) \right) \cdot \begin{cases} L\left(1-\frac{l}{2}, \chi\right) & 2 \mid l, \\ (-1)^{(l^2-1)/8} \prod_{p \mid \epsilon/2} \left(p^{(l-1)/2} + \left(\frac{-1}{p}\right)^{(l-1)/2} \Phi_p \right) & 2 \nmid l, \end{cases}$$

where χ is the associated quadratic character (possibly trivial), Φ_p is the Hasse invariant at p, and where the volume is understood w.r.t. to a natural volume form (highest power of the Chern class of a canonical ample automorphic line bundle). This volume is in fact an intersection number and should therefore be a rational number. Inserting the well-known formulas

$$\zeta(1-2i) = -\frac{B_{2i}}{2i}, \quad L\left(1-\frac{l}{2}, \chi\right) = -\frac{B_{l/2,\chi}}{l/2},$$

we see that the quantity (1.1) is indeed a rational number. It is also (up to a factor 2) the proportionality factor that occurs in the famous proportionality principle of Hirzebruch and Mumford [55] (cf. 2.6.22) applied to this case. For $l = 2$,

[1] defined over $\mathbb{Q}$ for $l > 2$

equation (1.1) is nothing but the classical analytic class number formula for definite binary quadratic forms. There exist formulas of similar shape for all locally symmetric spaces (see, e.g., [47] for the case of Chevalley groups). In the special case considered here M^K is, in fact, canonically defined over[2] $\mathbb{Z}[\frac{1}{2}]$. Therefore, using Arakelov theory, one obtains an analogous arithmetic 'volume element' (the highest power of the *arithmetic* Chern class of the same automorphic line bundle). To compute its Arakelov degree, which naturally is called the *arithmetic volume* of M^K, is the main objective of this book. The result is (in the special case considered here):

$$\widehat{\mathrm{vol}}(\mathrm{M}^K) \equiv \mathrm{vol}(\mathrm{M}^K) \cdot \left(\frac{l-1}{2}C + \sum_{i=1}^{\lfloor (l-1)/2 \rfloor} \left(-2\frac{\zeta'(1-2i)}{\zeta(1-2i)} - N_{2i} \right) + \begin{cases} -\dfrac{L'(1-l/2,\chi)}{L(1-l/2,\chi)} - \dfrac{1}{2}N_{l/2} - \dfrac{1}{2}\log\left|\dfrac{\varepsilon}{2}\right| & 2 \mid l, \\ \dfrac{1}{2}\sum_{p \mid \epsilon/2} \dfrac{p^{(l-1)/2} - \left(\frac{-1}{p}\right)^{(l-1)/2}\Phi_p}{p^{(l-1)/2} + \left(\frac{-1}{p}\right)^{(l-1)/2}\Phi_p} \log(p) + \dfrac{1}{2}\log(2) & 2 \nmid l, \end{cases} \right) \tag{1.2}$$

modulo rational multiples of log(2) (see B.1.3 for missing definitions).

We observe, of course, an apparent similarity between the formulas (1.1) and (1.2), and in fact, we have:

$$\mathrm{vol}(\mathrm{M}^K) = 4\tilde{\lambda}^{-1}(L_{\mathbb{Z}};0) \quad \widehat{\mathrm{vol}}(\mathrm{M}^K) \equiv 4\frac{\mathrm{d}}{\mathrm{d}s}\tilde{\lambda}^{-1}(L_{\mathbb{Z}};s)\Big|_{s=0}$$

for the function

$$\tilde{\lambda}^{-1}(L_{\mathbb{Z}};s) = 2^{-\lfloor l/2 \rfloor}(2\pi)^{s(l-1)/2}|\varepsilon|^{-s/2} \prod_{i=1}^{\lfloor (l-1)/2 \rfloor} \frac{\Gamma(s+2i)\zeta(1-2s-2i)}{\Gamma(2s+2i)\cos(\pi(s+i))} \cdot \begin{cases} 2^{s/2}\dfrac{\Gamma(s/2+l/2)L(1-s-l/2,\chi)}{\Gamma(s+l/2)\cos\big(\pi(s/2+\lfloor l/4 \rfloor)\big)} & \text{if } l \text{ is even,} \\ 2^s \displaystyle\prod_{p \mid \epsilon/2} \left(p^{(l-1)/2+s} + \left(\frac{-1}{p}\right)^{(l-1)/2} \Phi_p \right) & \text{if } l \text{ is odd,} \end{cases}$$

The function $\lambda^{-1}(L_{\mathbb{Z}};s)$ has, however, an intrinsic and much more general definition in terms of representation densities. Its study was the main subject of the paper [26] by the author. Formula (1.2) had been known only in some cases for $l \leq 4$, and was conjectured in some cases for $l = 5$.

Our proof uses only information from the "Archimedean fibre," that is, we do not need explicit computations of local intersection numbers. We generalize work of Bruinier, Burgos, and Kühn [10] which dealt with the case of Hilbert modular surfaces. Borcherds' construction of orthogonal modular forms [2], and a computation of the integral of their norm [11, 35] are used.

[2]and conjecturally over $\mathbb{Z}$

The orthogonal Shimura varieties are interesting in particular because they contain special algebraic cycles of arbitrary codimension whose arithmetic, respectively geometric volumes are conjectured to be encoded as a special value, respectively derivative of Fourier coefficients of Eisenstein series. This relation, in turn, has deep arithmetic consequences, including for example the formula of Gross and Zagier [21] and vast generalizations of it. It is also the key to the calculation of geometric and arithmetic volumes.

More precisely, these cycles are constructed as follows: For an isometry $x\colon M \hookrightarrow L$, where M is positive definite consider the subset

$$\mathbb{D}_x := \{N \in \mathbb{D} \mid N \perp x(M)\}$$

of $\mathbb{D}$. If (K-stable) lattices $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$ or more generally a K-invariant Schwartz function $\varphi \in S(M^*_{\mathbb{A}^{(\infty)}} \otimes L_{\mathbb{A}^{(\infty)}})$ is chosen, we define cycles $\mathrm{Z}(L, M, \varphi; K)$ on the Shimura variety by taking the quotient of the union of the $\mathbb{D}_x$ over all integral isometries (respectively all isometries in the support of φ in a weighted way). See Section 3.1 for the precise definition. For singular lattices M analogous cycles can be defined. Consider a model $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ of a toroidal compactification of the Shimura variety (the notation will be explained below). We consider the generating series

$$\Theta_m(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} [\mathrm{Z}(L, \langle Q \rangle, \varphi; K)] \exp(2\pi i Q \tau)$$

with values in its algebraic Chow group $\mathrm{CH}^m\left(\mathrm{M}\big({}^K_\Delta\mathbf{O}(L)\big)_{\mathbb{C}}\right) \otimes \mathbb{C}$. Assume now that $\mathrm{M}\big({}^K_\Delta\mathbf{O}(L)\big)$ is even defined over $\mathbb{Z}$ in a "reasonably canonical" way. Kudla proposes a definition of arithmetic cycles $\widehat{\mathrm{Z}}(L, M, \varphi; K, \nu)$, depending on the imaginary part ν of τ, too, and also for singular and for indefinite M, such that

$$\widehat{\Theta}_m(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} [\widehat{\mathrm{Z}}(L, \langle Q \rangle, \varphi; K, \nu)] \exp(2\pi i Q \tau)$$

should have values in a suitable Arakelov Chow group $\widehat{\mathrm{CH}}^m\left(\mathrm{M}\big({}^K_\Delta\mathbf{O}(L)\big)\right) \otimes \mathbb{C}$. He proposes specific Green's functions which have singularities at the boundary.

The orthogonal Shimura varieties come equipped with a natural Hermitian automorphic line bundle $\Xi^*\overline{\mathcal{E}}$ whose metric also has singularities along the boundary. This provides us (via multiplication with a suitable power of its first Chern class and taking pushforward) with geometric (respectively arithmetic) degree maps $\deg\colon \mathrm{CH}^p(\cdots) \to \mathbb{Z}$ (respectively $\widehat{\deg}\colon \widehat{\mathrm{CH}}^p(\cdots) \to \mathbb{R}$). Assuming that an Arakelov theory can be set up to deal with all different occurring singularities, Kudla conjectures (for the geometric part this goes back to Siegel, Hirzebruch, Zagier, Kudla–Millson, Borcherds, etc.)[3]:

- (K1) Θ_m and $\widehat{\Theta}_m$ are (holomorphic, respectively nonholomorphic) Siegel modular forms of weight $l/2$ and genus m.
- (K2) $\deg(\Theta_m)$ and $\widehat{\deg}(\widehat{\Theta}_m)$ are equal to a special value (respectively the special derivative at the same point) of a *normalization* of the standard Eisenstein series of weight $l/2$ associated with the Weil representation of L [26, Section 4].

[3] If $l - r \leq m + 1$, the statement has to be modified. Here r is the Witt-rank of L.

(K3) $\Theta_{m_1}(L, \varphi_1; \tau_1) \cdot \Theta_{m_2}(L, \varphi_2; \tau_2) = \Theta_{m_1+m_2}\big(L, \varphi_1 \oplus \varphi_2, \left(\begin{smallmatrix} \tau_1 & \\ & \tau_2 \end{smallmatrix}\right)\big)$ and similarly for $\widehat{\Theta}$.

(K4) $\widehat{\Theta}_{l-1}$ can be defined with coefficients being zero-cycles on the arithmetic model, and it satisfies the properties above.

Kudla shows (see [36] for an overview) that this implies almost formally vast generalizations of the formula of Gross–Zagier [21]. In full generality the conjectures are known only for Shimura curves [44]. Section 1.3 contains a brief overview on what is known in other special cases.

(K2) is closedly related to the calculation of geometric and arithmetic volumes because the cycles in question consist themselves (for sufficiently good reduction) of models of orthogonal Shimura varieties of smaller dimension.

We are thus able to obtain partial results towards the *arithmetic part* of (K2) for *all* Shimura varieties of orthogonal type. More precisely, we prove the following:

3.5.5. Main Theorem. *Let $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ be an integral lattice in a quadratic space of signature $(l-2, 2)$. Let K be its discriminant kernel. Let D' be the product of the primes p such that $p^2 \mid D$, where D is the discriminant of $L_{\mathbb{Z}}$. We have*

(1) $\mathrm{vol}_{\mathcal{E}}\Big(\mathrm{M}\big({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}})\big)\Big) = 4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0)$,

(2) $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}\Big(\mathrm{M}\big({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}})\big)\Big) \equiv \frac{\mathrm{d}}{\mathrm{d}s}4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)|_{s=0}$ *in* $\mathbb{R}_{2D'}$.

Let $M_{\mathbb{Z}}$ be a lattice of dimension m with positive definite $Q_M \in \mathrm{Sym}^2(M_{\mathbb{Q}}^*)$. Let D'' be the product of the primes p such that $M_{\mathbb{Z}_p} \not\subseteq M_{\mathbb{Z}_p}^*$ or $M_{\mathbb{Z}_p}^*/M_{\mathbb{Z}_p}$ is not cyclic. Assume

- $l - m \geq 4$, or
- $l = 4$, $m = 1$, *and $L_{\mathbb{Q}}$ has Witt rank* 1.

Then we have for each $\kappa \in \mathbb{Z}[(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}^*]$:

(3) $\mathrm{vol}_{\mathcal{E}}\big(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)\big) = 4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)\widetilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0)$,

(4) $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}\big(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)\big) \equiv \frac{\mathrm{d}}{\mathrm{d}s}4\big(\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)\widetilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s)\big)\big|_{s=0}$ *in* $\mathbb{R}_{2DD''}$.

Here $\mathbb{R}_N$ is $\mathbb{R}$ modulo rational multiples of $\log(p)$ for $p \mid N$, and the $\tilde{\lambda}$ and $\widetilde{\mu}$ are functions in $s \in \mathbb{C}$, given by certain Euler products (3.2.12) associated with representation densities of $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$. The function $\widetilde{\mu}$ appears as the "holomorphic part" of a Fourier coefficient of the standard Eisenstein series associated with the Weil representation of L. Moreover K is the discriminant kernel and $\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}}))$ is any toroidal compactification of the orthogonal Shimura variety (see below). Finally $\overline{\mathcal{E}}$ is the integral tautological bundle on the compact dual equipped with an equivariant metric on the restriction of its complex fibre to $\mathbb{D}$; the geometric/arithmetic volumes are computed w.r.t. the associated arithmetic automorphic line bundle (Definition 2.6.4) $\Xi^*\overline{\mathcal{E}}$ on $\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}}))$.

In view of the Main Theorem it seems plausible that, if $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ is cyclic, the function $4\tilde{\lambda}^{-1}(L; s)$ is always the correct normalizing factor in (K2).[4] This is in accordance with the observations of Kudla–Rapoport–Yang [42] in the case of Shimura curves.

[4] Of course this is only a statement about its first 2 Taylor coefficients.

For a more detailed introduction to the method of proof, we refer the reader to Section 1.4. An overview on known results in the direction of the conjectures will be given in Section 1.3.

In Chapter 2 up to Section 2.5 we recall the functorial theory of

- canonical integral models of toroidal compactifications of mixed Shimura varieties of Abelian type,
- their arithmetic automorphic vector bundles,

developed in the thesis of the author [25]. This theory, for general Hodge or even Abelian type, relies on an assumption regarding the stratification of the compactification (2.5.6) which was recently proven by Madapusi [49] and previously for the orthogonal Shimura varieties—spin-version—for $l \leq 5$ by Lan [46], (the Shimura varieties are of P.E.L. type in that case). The theory is crucial even for the precise formulation of our main result mentioned above. The reader is assumed to be familiar with the theory of rational Shimura varieties as developed by Deligne [16, 17] and to some extent with Pink's thesis [58] which extends the theory to the mixed case and contains the construction of rational canonical models of toroidal compactifications.

Our models are constructed locally (i.e. over an extension of $\mathbb{Z}_{(p)}$). Input data for the theory are p-integral mixed Shimura data (abbreviated p-MSD) $\mathbf{X} = (P_\mathbf{X}, \mathbb{D}_\mathbf{X}, h_\mathbf{X})$ consisting of an affine group scheme $P_\mathbf{X}$ over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ of a certain rigid type (P) (see 2.1.6) and a set $\mathbb{D}_\mathbf{X}$ which comes equipped with a finite covering $h_\mathbf{X}\colon \mathbb{D}_\mathbf{X} \to \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_{\mathbf{X},\mathbb{C}})$ onto a certain conjugacy class, subject to some axioms, which are roughly Pink's mixed extension [58] of Deligne's axioms for a pure Shimura datum. Consider a compact open subgroup $K \subseteq P_\mathbf{X}(\mathbb{A}^{(\infty)})$ of the form $K^{(p)} \times P_\mathbf{X}(\mathbb{Z}_p)$ for a compact open subgroup $K^{(p)} \subseteq P_\mathbf{X}(\mathbb{A}^{(\infty,p)})$ (we call those admissible). For the toroidal compactification a certain rational polyhedral cone decomposition Δ of a natural conical complex $C_\mathbf{X}$ associated with $\mathbf{X}$ is needed. We call the collection ${}^K\mathbf{X}$ (respectively ${}^K_\Delta\mathbf{X}$) extended (compactified) p-integral mixed Shimura data (abbreviated p-EMSD, respectively p-ECMSD). These form categories where the morphisms ${}^K_\Delta\mathbf{X} \to {}^{K'}_{\Delta'}\mathbf{Y}$ are pairs (α, ρ) of a morphism α of Shimura data and $\rho \in P_\mathbf{Y}(\mathbb{A}^{(\infty,p)})$ satisfying compatibility conditions with the K's and Δ's. The construction of the models defines a functor M from p-ECMSD to the category of Deligne–Mumford stacks over reflex rings (above $\mathbb{Z}_{(p)}$). The functor, base-changed to $\mathbb{C}$ and restricted to p-EMSD, becomes naturally isomorphic to the one given by the construction of the analytic mixed Shimura variety. It is characterized uniquely by: Deligne's canonical model condition; Milne's extension property (integral canonicity); a stratification of the boundary into mixed Shimura varieties, together with boundary isomorphisms of the formal completions along these strata with similar completions of other (more mixed) Shimura varieties. These boundary isomorphisms, for the case of the symplectic Shimura varieties, are given by Mumford's construction [19, Appendix]. There is also a functor $\mathrm{M}^\vee$ ('compact' dual) from p-MSD to the category of schemes over reflex rings. The duals come equipped with an action of the group scheme $P_\mathbf{X}$, and we have morphisms of Artin stacks

$$\Xi_\mathbf{X}\colon\ \mathrm{M}({}^K_\Delta\mathbf{X}) \to [\mathrm{M}^\vee(\mathbf{X})/P_{\mathbf{X},\mathcal{O}_\mathbf{X}}].$$

Those constitute a pseudonatural transformation of functors, are a model of the usual construction over $\mathbb{C}$ if Δ is trivial, and are compatible with the boundary isomorphisms. This is the theory of integral automorphic vector bundles. For more information on the 'philosophy' of these objects in terms of motives see Section 1.5.

In particular this enables us to do the following construction: Let $\mathbf{X}$ be, for simplicity, a pure Shimura datum with reflex field $\mathbb{Q}$, and let $\mathcal{E}$ be a $P_{\mathbf{X}}$-equivariant vector bundle on $\mathrm{M}^{\vee}(\mathbf{X})$ equipped with a $P_{\mathbf{X},\mathbb{R}}$-equivariant Hermitian metric $h_{\mathcal{E}}$ on its restriction to the image of the Borel embedding $h_{\mathbf{X}}(\mathbb{D}_{\mathbf{X}}) \hookrightarrow \mathrm{M}^{\vee}(\mathbf{X})(\mathbb{C})$. The above morphism of stacks and its 2-isomorphism over $\mathbb{C}$ with the analytic period construction allows us to obtain a *well-defined* Arakelov vector bundle $(\Xi^*\mathcal{E}, \Xi^*h_{\mathcal{E}})$. The metric $\Xi^*h_{\mathcal{E}}$, however, has (rather mild) singularities along the boundaries of toroidal compactifications (see below). Without the existence of this canonical pullback, it would not even make sense to speak of arithmetic volumes.

The remaining sections of Chapter 2 are completely revised with respect to [25]: In Section 2.6 we define integral Hermitian automorphic vector bundles and the notions of arithmetic and geometric volume. Furthermore we set up an Arakelov theory which has enough properties to deal with singularities of the natural Hermitian metrics on "fully decomposed" automorphic vector bundles. This uses work of Burgos, Kramer, and Kühn [14, 15] but the arithmetic Chow groups are defined using a technically simpler method.

In Sections 2.7 and 2.8 a precise general q-expansion principle is derived from the abstract properties of Section 2.5.

Chapter 3 is concerned entirely with the theory of orthogonal Shimura varieties: In Section 3.1 the structure of the models of orthogonal Shimura varieties is investigated and special cycles are defined.

In Section 3.2 we use the general q-expansion principle to prove that Borcherds products with their natural norm yield *integral* sections of an appropriate integral Hermitian line bundle. Among other things the product expansions of Borcherds' are adelized and their Galois properties investigated.

In Section 3.3 we prove that the bundle of vector valued modular forms for the Weil representation (which appear as input forms in the construction of Borcherds products) has a rational structure. Then we use this to construct input forms with special properties which will be needed later.

In Section 3.4 we relate Borcherds' theory and Arakelov geometry on the orthogonal Shimura varieties. Its main result is an (averaged) arithmetic Siegel–Weil formula which will be crucial for the proof of the Main Theorem.

In Section 3.5 the Main Theorem is proven. An overview on the proof may be found in Section 1.4.

In Appendix A additional material on quadratic forms, on "lacunarity of modular forms," and on semilinear representations is provided which will be needed in the proofs in Section 3.5. Appendix B contains a continuation of the calculation in [26] of the function $\lambda(L_{\mathbb{Z}}; s)$ for the special case of lattices with square-free discriminant.

Finally, it is a pleasure to thank the Department of Mathematics and Statistics at McGill University and especially Henri Darmon, Jayce Getz and Eyal Goren, for providing a very inspiring working environment during the preparation of this book.

1.2. A brief introduction to Siegel–Weil theory

1.2.1. Consider two lattices $L_{\mathbb{Z}} \cong \mathbb{Z}^l$, and $M_{\mathbb{Z}} \cong \mathbb{Z}^m$ with integral and positive definite quadratic forms Q_L, and Q_M. It is a classical problem, to which already Gauss, Euler and in particular Siegel devoted themselves, to determine the representation number, that is, the number of elements in the set of isometric embeddings

$$\mathrm{I}(L_{\mathbb{Z}}, M_{\mathbb{Z}}) = \{\alpha \colon M_{\mathbb{Z}} \hookrightarrow L_{\mathbb{Z}} \mid \alpha \text{ is an isometry}\}.$$

It includes (for $m = 1$) questions like: "In how many ways can an integer be represented as a sum of l squares?".

If $M_{\mathbb{Z}} = \mathbb{Z}^m$ then Q_M is given by an element in $\mathrm{Sym}^2\big((\mathbb{Z}^m)^*\big)$. We write $\langle Q \rangle$ for the lattice $\mathbb{Z}^m$ with quadratic form given by Q. The *generating series*, the *theta series* of $L_{\mathbb{Z}}$,

$$\Theta_m(L_{\mathbb{Z}}; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} \#\, \mathrm{I}(L_{\mathbb{Z}}, \langle Q \rangle) \exp(2\pi i Q \cdot \tau), \tag{1.3}$$

(here τ is an element in Siegel's upper half space $\mathbb{H}_m \subset (\mathbb{C}^m \otimes \mathbb{C}^m)^s$, the subset of elements with positive definite imaginary part) is a *Siegel modular form* of weight $l/2$ for a certain congruence subgroup of Sp'_{2m} (the symplectic or metaplectic group, according to the parity of l). For example $\Theta_1(\langle 1 \rangle; \tau)$ is just the classical theta function.

Under some conditions on the dimensions, a certain weighted sum of these theta functions over all *classes* $L_{\mathbb{Z}}^{(i)}$ in the *genus* $L_{\hat{\mathbb{Z}}}$ is an Eisenstein series (cf. [26, Section 4] for details):

1.2.2. Theorem (Siegel–Weil). *If $l > 2m + 2$, we have*

$$\sum_i c_i \Theta_m(L_{\mathbb{Z}}^{(i)}; \tau) = E_m(\Phi; \tau, s_0).$$

The additional parameter s_0 indicates that this Eisenstein series is in fact the (holomorphic) special value of a nonholomorphic Eisenstein series $E_m(\Phi; \tau, s)$ at $s = s_0 := (l - m + 1)/2$. The Fourier coefficients of the series are given by a product formula

$$\mu(L_{\mathbb{Z}}, \langle Q \rangle; s, y) = \mu_\infty(L, \langle Q \rangle; s, y) \prod_p \mu_p(L_{\mathbb{Z}_p}, \langle Q \rangle; s). \tag{1.4}$$

Here y is the imaginary part of τ—its appearance indicates that this series is nonholomorphic for general s. At $s = 0$ the μ_p are just the p-adic volumes of the 'spheres' $I(L, \langle Q \rangle)(\mathbb{Z}_p)$, classically called *representation densities*. For almost all p, the functions μ_p are very simple polynomials in p^{-s} (see, e.g., [26, Theorem 8.1]). Otherwise they may be computed by determining sufficiently many representation numbers of the congruences modulo p^n.

Essentially, the Siegel–Weil formula (1.2.2) is valid, if and only if $l > m + 1$, but if $l \leq 2m + 2$, the value of the Eisenstein series has to be defined via analytic continuation in s and the theta function has sometimes to be complemented by indefinite coefficients.

The mere fact that the representation numbers (in an average over classes) should be given by a product over local volumes or densities can be explained easily in the adelic language:

1.2.3. Let $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ now be an integral lattice in an arbitrary quadratic space (not necessarily definite) and $M_{\mathbb{Q}}$ a positive definite quadratic space. Assume $l - m \geq 3$, for simplicity, for the rest of the discussion. On the adelic points $\mathrm{SO}(L_{\mathbb{A}})$ of the special orthogonal group of $L_{\mathbb{Q}}$, there is a canonical measure μ. It is a product over local measures μ_ν on the various $\mathrm{SO}(L_{\mathbb{Q}_\nu})$, constructed by any algebraic volume form defined over $\mathbb{Q}$ [65]. The product μ is independent of the choice of this form. The volume of $\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}})$ which turns out to be finite, is called the **Tamagawa number**, and we have

1.2.4. Theorem ([65]). *For $l \geq 3$*

$$\mathrm{vol}\big(\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}})\big) = 2.$$

From this the fact that the representation numbers (in an average over classes) should be given by a product over local volumes already follows, as we will explain now (in a slightly broader context):

1.2.5. Let $\varphi \in S(L_{\mathbb{A}^{(\infty)}} \otimes M^*_{\mathbb{A}^{(\infty)}})$ be a Schwartz-Bruhat function (i.e., locally constant with compact support). Let $K = \prod_p K_p$ be a compact open subgroup of $\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ which stabilizes φ. For example, K could be the stabilizer of the lattice $L_{\widehat{\mathbb{Z}}}$ and φ the characteristic function of $L_{\widehat{\mathbb{Z}}}$. Let K_∞ be a maximal compact subgroup of $\mathrm{SO}(L_{\mathbb{R}})$. (If L is definite, this will be equal to $\mathrm{SO}(L_{\mathbb{R}})$.)

From 1.2.4 we may infer that the volume of the real analytic orbifold

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash(\mathrm{SO}(L_{\mathbb{A}})/K_\infty K)],$$

induced by the quotient of μ_∞ and some measure on K_∞, is:

$$2\prod_\nu \mathrm{vol}_\nu^{-1}(K_\nu). \tag{1.5}$$

We have a finite disjoint decomposition

$$\mathrm{I}(L,M)(\mathbb{A}^{(\infty)}) \cap \mathrm{supp}(\varphi) = \bigcup_i K\alpha_i$$

If this set is nonempty, we have by Hasse's principle an $\alpha' \in \mathrm{I}(L,M)(\mathbb{Q})$ and hence $g_i \in \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ with $g_i^{-1}\alpha' = \alpha_i$. There is a lattice $L_{\mathbb{Z}}^{(i)} \subset L_{\mathbb{Q}}$ satisfying $L_{\widehat{\mathbb{Z}}}^{(i)} = g_i^{-1}L_{\widehat{\mathbb{Z}}}$. We denote by abuse of notation by $\alpha_i^\perp$ the lattice $\mathrm{im}(\alpha')^\perp \cap L_{\mathbb{Z}}^{(i)}$. We have $\alpha_i^\perp \otimes \widehat{\mathbb{Z}} \cong \mathrm{im}(\alpha_i)^\perp$. However, only the genus of $\alpha_i^\perp$ is well-defined, but we will use the notation only for objects which depend only on this genus.

Consider the symmetric space[5]

$$\mathbb{D}(L) = \{\text{maximal negative definite subspaces of } L_{\mathbb{R}}\} = \mathrm{SO}(L_{\mathbb{R}})/K_\infty.$$

We have an embedding $\mathbb{D}(\alpha_i^\perp) \times \mathrm{SO}\big((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}}\big) \hookrightarrow \mathbb{D}(L) \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$, given by the natural inclusion of $\mathbb{D}(\alpha_i^\perp) \hookrightarrow \mathbb{D}(L)$ and multiplication of the adelic part by g_i from the right.

We form the *special cycle* $\mathrm{Z}(L,M,\varphi;K)$, the following formal sum (with real coefficients):

$$\sum_i \varphi(\alpha_i)\Big[\mathrm{SO}((\alpha_i^\perp)_{\mathbb{Q}})\backslash\mathbb{D}(\alpha_i^\perp) \times \mathrm{SO}\big((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}}\big)\Big/\Big({}^{g_i}K \cap \mathrm{SO}\big((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}}\big)\Big)\Big]$$

---

[5] *Caution*: This definition differs from the later definition of $\mathbb{D}_{\mathbf{O}(L)}$ in case that L has signature $(l-2, 2)$

which we consider, by means of the embeddings above, as a formal sum of real analytic suborbifolds of $[\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]$. It does not depend on the choices made above.

The *canonical* measures [26, 2.4] on $\mathrm{SO}(L_{\mathbb{Q}})$, $\mathrm{SO}(\alpha_i^{\perp})$ and $\mathrm{I}(L,M)$ over any $\mathbb{Q}_\nu$ are related by an orbit equation [26, 5.6]—an equation of the shape:

$$\text{‘volume of space’} = \sum_{\text{orbits}} \frac{\text{‘volume of group’}}{\text{‘volume of stabilizer’}},$$

similar to the corresponding formula for actions of finite groups on sets.

From this and (1.5) above

$$\frac{\mathrm{vol}(Z(L,M,\varphi;K))}{\mathrm{vol}(\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L)\times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)} = \frac{\mathrm{vol}(K_\infty)}{\mathrm{vol}(K'_\infty)} \int_{\mathrm{I}(L,M)(\mathbb{A}^{(\infty)})} \varphi(\alpha)\mu(\alpha) \tag{1.6}$$

follows immediately. K'_∞ is any maximal compact subgroup of any of the $\mathrm{SO}(\alpha_{i,\mathbb{R}}^{\perp})$. We define $\mu_\infty(L,M)$ to be the quantity $\mathrm{vol}(K_\infty)/\mathrm{vol}(K'_\infty)$ (computed w.r.t. the canonical measures). If L is definite, it is equal to:

$$\mathrm{vol}(\mathrm{I}(L,M)(\mathbb{R})) = \prod_{k=l-m+1}^{l} 2\frac{\pi^{k/2}}{\Gamma(k/2)}.$$

Observe that

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)] = \bigcup_j [(\mathrm{SO}(L_{\mathbb{Q}}) \cap {}^{g_j}K)\backslash\mathbb{D}(L)],$$

with respect to a set $\{g_j\}_j$ of representatives of $\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K$, i.e., of the *classes of* $\mathrm{SO}(L_{\mathbb{Q}})$ with respect to the compact open group K. (If K is the stabilizer of a lattice $L_{\widehat{\mathbb{Z}}}$, this coincides with the classical notion of classes in the genus $L_{\widehat{\mathbb{Z}}}$.) Similarly, we have

$$\mathrm{Z}(L,M,\varphi;K) = \sum_{i,k} \varphi(\alpha_{ik})\big[\big(\mathrm{SO}\big((\alpha_i^{\perp})_{\mathbb{Q}}\big) \cap K^{g_{ik}}\big)\backslash\mathbb{D}(\alpha_i^{\perp})\big], \tag{1.7}$$

where $\{g_{ik}\}_k$ is a set of representatives of the classes of $\mathrm{SO}\big((\alpha_i^{\perp})_{\mathbb{Q}}\big)$ w.r.t. ${}^{g_i}K \cap \mathrm{SO}\big((\alpha_i^{\perp})_{\mathbb{A}^{(\infty)}}\big)$.

Let now K be the stabilizer of $L_{\widehat{\mathbb{Z}}}$ and φ the characteristic function. We have the following easy

1.2.6. Lemma. *There is a bijection*

$$\left\{\begin{matrix} \textit{class } L_{\mathbb{Z}}^{(j)} \textit{ in the genus } L_{\widehat{\mathbb{Z}}}, \\ \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\textit{-orbit } \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha \textit{ in } \mathrm{I}(L^{(j)},M)(\mathbb{Z}) \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \mathrm{SO}(L_{\widehat{\mathbb{Z}}})\textit{-orbit } \mathrm{SO}(L_{\widehat{\mathbb{Z}}})\alpha \textit{ in } \mathrm{I}(L,M)(\widehat{\mathbb{Z}}), \\ \textit{class in } \mathrm{SO}(\alpha_{\mathbb{Q}}^{\perp})\backslash \mathrm{SO}(\alpha_{\mathbb{A}^{(\infty)}}^{\perp})/K \cap \mathrm{SO}(\alpha_{\mathbb{A}^{(\infty)}}^{\perp}) \end{matrix}\right\}.$$

We have, of course, a similar statement for any K.

We denote the cycle in this case by $\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}})$ and it is, according to the lemma and (1.7), equal to:

$$\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}) = \sum_j \sum_{\mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha \subset \mathrm{I}(L^{(j)},M)(\mathbb{Z})} \big[\big(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\big)\backslash\mathbb{D}(L)\big].$$

1.2.7. Now, if the form Q_L is *positive definite*, the quotient of volumes (1.6) has an interpretation as a global representation number. For this observe that in this case

$$\mathrm{vol}\big(\mathrm{SO}(L_{\mathbb{Z}})\backslash\mathbb{D}(L)\big) = \frac{1}{\#\,\mathrm{SO}(L_{\mathbb{Z}})}$$

and similarly

$$\mathrm{vol}\big((\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)}))\backslash\mathbb{D}(\alpha^{\perp})\big) = \frac{1}{\#\big(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\big)}.$$

Furthermore, we have by the set theoretical orbit equation,

$$\frac{\#\,\mathrm{I}(L^{(j)}, M)(\mathbb{Z})}{\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})} = \sum_{\mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha \in \mathrm{I}(L^{(j)}, M)(\mathbb{Z})} \frac{1}{\#\big(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\big)}.$$

Hence we get

$$\frac{\mathrm{vol}\big(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}})\big)}{\mathrm{vol}(\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)} = \frac{\sum_j \#\,\mathrm{I}(L_{\mathbb{Z}}^{(j)}, M)(\mathbb{Z})/\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})}{\sum_j 1/\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})}$$

which is precisely a weighted sum over the representation numbers. Combined with (1.6), we get *Siegel's formula*. The deep part, of course, is hidden in Theorem 1.2.4.

1.2.8. If the quadratic form on L is *indefinite*, say of signature (p, q), then these representation numbers do not make sense because there are always infinitely many isometries. However, equation (1.6) tells us, what the correct analogue in the indefinite case is: the quotient of volumes

$$\frac{\mathrm{vol}\big(\mathrm{Z}(L, M, \varphi; K)\big)}{\mathrm{vol}([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)])}.$$

For every cohomology theory H (in a very broad sense) one might in addition consider the *classes* $[Z(L, \langle Q\rangle, \varphi; K)]^H$ of these cycles and define their generating theta series:

$$\Theta_m^H(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} [\mathrm{Z}(L, \langle Q\rangle, \varphi; K)]^H \cup e_q^{m-r(Q)} \exp(2\pi i Q \cdot \tau),$$

where e_q is a certain Euler class, and $r(Q)$ is the rank of Q. One always expects modularity of this function and a relation to Eisenstein series.

Kudla and Millson [37, 38] have shown (generalizing work of Hirzebruch and Zagier [24]) that the generating series

$$\Theta_m^{\mathrm{B}}(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^m)^*)} [\mathrm{Z}(L, \langle Q\rangle, \varphi; K)]^{\mathrm{B}} \cup e_q^{m-r(Q)} \exp(2\pi i Q \cdot \tau),$$

with values in the Betti cohomology groups

$$H^{(p-m)q}([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)], \mathbb{C})$$

is a modular form. Under certain conditions on l, m and the Witt rank of L, its 'degree' is the special value of an Eisenstein series:

$$\langle \Theta_m^{\mathrm{B}}(L, \varphi; \tau), e_q^{l-m}\rangle = \mathrm{vol}_{e_q}([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]) E_m(\Phi; \tau, s_0).$$

The latter equation follows essentially again from the Siegel–Weil formula (in its full generality) or the Tamagawa number result, respectively. If $L_{\mathbb{Q}}$ is anisotropic, the locally symmetric space is compact and the pairing on the left is the degree of the product in cohomology (Poincaré duality pairing). If $L_{\mathbb{Q}}$ is isotropic, the locally symmetric space is noncompact but the expression still makes sense, because the natural forms defining e_q^{l-m} are integrable on the special cycles. For details see [36].

The $\Theta_m^H(L,\varphi;\tau)$'s are also always expected to satisfy a product relation like:

$$\Theta^H_{m_1}(L,\varphi_1;\tau_1)\cup\Theta^H_{m_2}(L,\varphi_2;\tau_2)=\Theta^H_{m_1+m_2}\left(L,\varphi_1\otimes\varphi_2;\begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix}\right). \tag{1.8}$$

1.2.9. The above theory is particularly interesting if the signature is $(l-2,2)$. As mentioned in the beginning, the locally symmetric orbifold

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L)\times(\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]$$

has, in this case, a complex structure and is associated with an algebraic Deligne–Mumford stack $\mathrm{M}\big({}^K\mathbf{O}(L)\big)$, a Shimura variety of orthogonal type. In particular, the special cycles may be considered as *algebraic* cycles on $\mathrm{M}({}^K\mathbf{O}(L))$ and we have the following theta function:

$$\Theta^{\mathrm{CH}}_m(L,\varphi;\tau)=\sum_{Q\in\mathrm{Sym}^2((\mathbb{Z}^m)^*)}[\mathrm{Z}(L,\langle Q\rangle,\varphi;K)]^{\mathrm{CH}}\cup\mathrm{c}_1(\Xi^*\mathcal{E})^{m-r(Q)}\exp(2\pi iQ\cdot\tau),$$

with values in $\mathrm{CH}^m\big(\mathrm{M}({}^K_\Delta\mathbf{O}(L))_{\mathbb{C}}\big)\otimes\mathbb{C}$, where $\mathrm{M}\big({}^K_\Delta\mathbf{O}(L)\big)$ is a toroidal compactification of $\mathrm{M}\big({}^K\mathbf{O}(L)\big)$.

$\Xi^*\mathcal{E}$ is a certain ample (automorphic) line bundle on $\mathrm{M}\big({}^K_\Delta\mathbf{O}(L)\big)$. It is equipped with a Hermitian metric $\Xi^*h_{\mathcal{E}}$ (singular along the boundary of the compactification), whose associated Chern form is (roughly) e_2 above (see 3.2.5). The series is therefore a 'lift' of Θ^{B}_m with respect to the cycle class map.

The only known fact, however, in the direction of modularity *in arbitrary dimensions* is the following theorem:

1.2.10. Theorem. *For $m=1,2$, $\Theta^{\mathrm{CH}}_m(L,\varphi;\tau)$ is a modular form of weight $l/2$.*

This was proven by Borcherds [3] for $m=1$, and for $m=2$ independently by Martin Raum and by Bruinier [8] (using results of Wei Zhang).

The theta functions Θ^{B}_m, in this case, do satisfy the relation (1.8) [32]. An analogue of this for Θ^{CH}_m is not known in general.

1.3. Known results

1.3.1. In this section, some of the recent developments in the direction of Kudla's conjectures will be presented with no aim whatsoever towards completeness.

For lattices of small dimensions the associated Shimura varieties are of P.E.L. type and were already subject to a variety of classical work of Gross, Heegner, Hilbert, Hirzebruch, Riemann, Shimura, Siegel, Zagier and many others. The cases are listed in Table 1.1.

Modularity of $\Theta^{\widehat{\mathrm{CH}}}_m$ is widely unknown, especially for higher-dimensional varieties with nonempty boundary which requires the use of extended Arakelov theories like, e.g., defined in Section 2.6. Modularity was obtained so far only for the cases II and III above—for II, by work of Kudla, Rapoport and Yang [42,43] culminating in their recent book *Modular forms and special cycles on Shimura curves* [44]. They obtained modularity of $\Theta_1^{\widehat{\mathrm{CH}}}$ and $\Theta_2^{\widehat{\mathrm{CH}}}$, as well as their relations to the corresponding special derivatives of Eisenstein series of genus 1, respectively 2 and of weight $\frac{3}{2}$. Also the formula (K3) was established, yielding inner product formulas. This completed earlier work started by Kudla in the 90s [31, 33, 34] and [40]. We mention also related work of Liu for higher-dimensional unitary Shimura varieties [48].

To obtain results (at least) about the equality of heights with the 'nonholomorphic' part of the special derivatives of Eisenstein series, there are in principle two approaches:

(1) The *first* approach is by comparison of direct calculations of the finite intersection numbers of the cycles $\widehat{\mathrm{Z}}(L, M, \varphi; K, y)$ and of the special derivative of the Eisenstein series, respectively. These lines have been followed predominantly in the above mentioned work. In these cases, the equality of the 'nonholomorphic part' of the special derivative with the integral of the corresponding Kudla-Millson Green's functions has also been verified.

Evidence in higher dimensions had been provided so far only by work of Kudla and Rapoport, [39] for Hilbert–Blumenthal varieties (V), and [41] for Siegel modular varieties (VII, VIII). These approaches rely heavily on explicit use of the underlying moduli problem. In particular, the special cycles are defined algebraically via a submoduli problem involving additional special endomorphisms.

(2) The *second* approach which is used in the main part of this book, is an inductive method, generalized from Burgos, Bruinier and Kühn [10], who investigated a special case of (V) above. However, for cycles of codimension $n > 1$, it seems to be restricted to the case of indices Q_M, where Q_M has 'good shape' at all primes p considered, at least such that $M_{\mathbb{Z}_p}^*/M_{\mathbb{Z}_p}$ is at most cyclic (i.e., essentially the codimension one case). Otherwise it seems to require at least as much knowledge about bad reduction as a direct computation of finite intersection numbers in the first approach requires.

This method has, however, the advantage of giving results in arbitrary dimension, even for non-P.E.L. type Shimura varieties, and with boundary, too—cases which only recently seem to come within reach of the first method. It uses modular forms on these Shimura varieties, constructed by Borcherds [2] by purely analytic means, using ideas from physics. They have a divisor consisting precisely of the codimension one cycles $\mathrm{Z}(L, \langle q\rangle, \varphi; K)$ and have integral Fourier coefficients. A computation of the integral of their norm is needed. This was accomplished before independently by Kudla [35] and by Bruinier and Kühn [11].

Table 1.1

	signature	**Witt rk.**	**classical name**
I	(0,2)	0	Heegner points (compact)
II	(1,2)	0	Shimura curves (compact)
III	(1,2)	1	modular curve (moduli space of elliptic curves)
IV	(2,2)	0	Shimura surfaces (compact)
V	(2,2)	1	Hilbert–Blumenthal varieties
VI	(2,2)	2	product of modular curves
VII	(3,2)	1	twisted Siegel modular threefolds
VIII	(3,2)	2	Siegel modular threefold (moduli space of Abelian surfaces)

In this book, using the second approach, we compute the respective arithmetic volumes for *all* Shimura varieties of orthogonal type and *all* cycles $\mathrm{Z}(L, M, \varphi; K)$ on them, but only up to contributions (multiples of $\log(p)$) from primes p, where the above requirement of 'good shape' is violated and up to contributions from $p = 2$, due to the still incomplete theory of good reduction of integral models of Shimura varieties of non-P.E.L. type. In case $m = 1$ (codimension 1) the heights of the special cycles can be computed for *all* $M = \langle q \rangle$, $q > 0$ only up to contributions from bad reduction of the surrounding Shimura variety.

For the modular curve, Yang [66] verified the modularity of $\Theta_1^{\widehat{\mathrm{CH}}}$ and the identity of $\langle \Theta_1^{\widehat{\mathrm{CH}}}, \hat{\mathrm{c}}_1(\Xi^* \overline{\mathcal{E}}) \rangle$ with the special derivative of an Eisenstein series, using Chow groups of an extended Arakelov theory as in [14, 15] which, however, for the case needed here (arithmetic surfaces) had already been constructed long before by Kühn [45] and by Bost [4], independently. It should be mentioned that the equality of $\deg(\Theta_1^{\mathrm{B}})$ with the special *value* of the same Eisenstein series in this case is more difficult because the modular curve is, in a sense, an extremal case. One has to introduce also negative, nonholomorphic Fourier coefficients. The positive ones here are given by the class numbers of binary quadratic forms (the $\mathrm{Z}(L_{\mathbb{Z}}, \langle q \rangle_{\mathbb{Z}})$ consist of special *points* in this case). This special value of the Eisenstein series which is accordingly also nonholomorphic, is Zagier's famous Eisenstein series [67] of weight $\frac{3}{2}$. The other conjectures have not been verified so far in this special case, but Bruinier and Yang succeeded in obtaining the formula of Gross and Zagier and generalizations directly, also using Borcherds products [12].

1.4. An overview on the proof

1.4.1. Let $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ be a lattice of signature $(l-2, 2)$ and let D be the discriminant of $L_{\mathbb{Z}}$. Let $K \subseteq \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ be its discriminant kernel. Let $\mathrm{M}({}^K\mathbf{O}(L_{\mathbb{Z}}))$ be a global canonical model of the Shimura variety

$$[\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathbb{D}_{\mathbf{O}(L)} \times \mathrm{SO}(\mathbb{A}^{(\infty)})/K].$$

We want to prove the assertion

$$\widehat{\mathrm{vol}}\Big(\mathrm{M}\big({}^K\mathbf{O}(L_{\mathbb{Z}})\big)\Big) \equiv 4 \frac{\mathrm{d}}{\mathrm{d}s} \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) \bigg|_{s=0} \tag{1.9}$$

(up to some contributions of the form $\mathbb{Q}\log(p)$, but with control over these primes!). To illuminate the main ideas we ignore here completely the issue of compactification. This is legitimate insofar as the arithmetic volume will be shown to be independent of the choice of toroidal compsactification. The idea is to prove (1.9) first up to rational multiples of $\log(p)$, where p is either 2 or such that the Shimura variety has bad reduction there, i.e., for $p \mid 2D$. This will be done by induction on l, the dimension of L, using *codimension 1* cycles. We are immediately reduced (using (1.11) below) to show for each prime $p \nmid 2D$ the statement

$$\widehat{\mathrm{vol}}\Big(\mathrm{M}\big({}^K\mathbf{O}(L_{\mathbb{Z}})\big)\Big) \equiv \mathrm{vol}\Big(\mathrm{M}\big({}^K\mathbf{O}(L_{\mathbb{Z}})\big)\Big) \cdot \frac{\mathrm{d}\log \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)}{\mathrm{d}s} \bigg|_{s=0} \tag{1.10}$$

up to rational multiples of $\log(q)$ for all primes $q \neq p$. Here we gain the flexibility of taking any K of the form $\mathrm{SO}(L_{\mathbb{Z}_p})K^{(p)}$, where $K^{(p)} \subset \mathrm{SO}(L_{\mathbb{A}^{(\infty,p)}})$ is arbitrary, and to work with the local canonical model of the Shimura variety over $\mathbb{Z}_{(p)}$. In certain cases (e.g., if $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ is cyclic), we will be able to show (1.9) a posteriori merely up to $\mathbb{Q}\log(2)$, where $\mathrm{M}\big({}^{K}\mathbf{O}(L_{\mathbb{Z}})\big)$ has to be interpreted as a model induced by an embedding into a nonsingular $\mathrm{M}\big({}^{K'}\mathbf{O}(L'_{\mathbb{Z}})\big)$ for bigger L'. It seems that, at least if $L^*_{\mathbb{Z}}/L_{\mathbb{Z}}$ is cyclic *of square-free order*, such a model is still *canonical* at all $p \neq 2$ in a way similiar to our notion of canonical for p of good reduction.

We will reprove along the way also the statement:

$$\mathrm{vol}\Big(\mathrm{M}\big({}^{K}\mathbf{O}(L_{\mathbb{Z}})\big)\Big) = 4\tilde{\lambda}^{-1}(L_{\mathbb{Z}};0) \tag{1.11}$$

where K is the discriminant kernel.

1.4.2. Let $q \in \mathbb{Q}$, $q > 0$ be given such that $|q|_p = 1$. Let κ be (the characteristic function of) a coset in $L^*_{\widehat{\mathbb{Z}}}/L_{\widehat{\mathbb{Z}}}$ and assume that K fixes κ, e.g., let it be contained in the discriminant kernel. First we observe that, by definition, the arithmetic/geometric volume of the special cycle $\mathrm{Z}(L, \langle q\rangle, \kappa; K)$ (here understood to be the Zariski closure of those defined in the introduction, see 3.1.7 for the precise definition) is

$$\mathrm{vol}\big(\mathrm{Z}(L, \langle q\rangle, \kappa; K)\big) = \sum_{K\alpha\subset \mathrm{I}(L,\langle q\rangle)(\mathbb{A}^{(\infty)})\cap\kappa} \mathrm{vol}\Big(\mathrm{M}\big({}^{K_\alpha}\mathbf{O}(\alpha^\perp_{\mathbb{Z}})\big)\Big) \tag{1.12}$$

$$\widehat{\mathrm{vol}}\big(\mathrm{Z}(L, \langle q\rangle, \kappa; K)\big) = \sum_{K\alpha\subset \mathrm{I}(L,\langle q\rangle)(\mathbb{A}^{(\infty)})\cap\kappa} \widehat{\mathrm{vol}}\Big(\mathrm{M}\big({}^{K_\alpha}\mathbf{O}(\alpha^\perp_{\mathbb{Z}})\big)\Big) \tag{1.13}$$

ignoring the problem of compactifications here for simplicity. The sums run over K-orbits in the adelic sphere $\mathrm{I}(L, \langle q\rangle)(\mathbb{A}^{(\infty)})$ of radius q, more precisely, over those lying in κ. In other words, the decomposition of the special cycles into sub-Shimura varieties is governed by these orbits. We have furthermore:

$$\lambda^{-1}(L_{\mathbb{Z}};s)\mu(L_{\mathbb{Z}}, \langle q\rangle, \kappa; s) = \sum_{K\alpha\subset \mathrm{I}(L,\langle q\rangle)(\mathbb{A}^{(\infty)})\cap\kappa} \lambda^{-1}(\alpha^\perp; s) \tag{1.14}$$

by [26, Theorem 5.10]. Here $\mu(L_{\mathbb{Z}}, \langle q\rangle, \kappa; s)$ is (part of) the qth Fourier coefficient of the Eisenstein series of genus 1 associated with the Weil representation of L, cf. [26, 10.2, Equation (18)].

1.4.3. Taking the *value* of (1.14) at $s = 0$ and inserting (1.11), we get for K, respectively K_α being the respective discriminant kernels:

$$\begin{aligned}\mathrm{vol}\Big(\mathrm{M}\big({}^{K}\mathbf{O}(L_{\mathbb{Z}})\big)\Big)\widetilde{\mu}(L_{\mathbb{Z}}, \langle q\rangle, \kappa; 0) &= \sum_{\alpha K\subset \mathrm{I}(L,\langle q\rangle)(\mathbb{A}^{(\infty)})\cap\kappa} \mathrm{vol}\Big(\mathrm{M}\big({}^{K_\alpha}\mathbf{O}(\alpha^\perp_{\mathbb{Z}})\big)\Big) \\ &= \mathrm{vol}\big(\mathrm{Z}(L, \langle q\rangle, \kappa; K)\big).\end{aligned} \tag{1.15}$$

This is just a version of the Siegel–Weil formula for the indefinite case and can be seen *directly* in several ways. For example, using Borcherds theory, see 1.4.5 below, or using the original version of the Siegel–Weil formula and Kudla–Millson theory. Hence equation (1.11) is true for $\mathrm{M}({}^{K}\mathbf{O}(L_{\mathbb{Z}}))$ if and only if it is true for all $\mathrm{M}({}^{K_\alpha}\mathbf{O}(\alpha^\perp_{\mathbb{Z}}))$. The result follows by induction.

1.4.4. Taking the *derivative* of (1.14)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\lambda^{-1}(L_{\mathbb{Z}};s)\mu(L_{\mathbb{Z}}, \langle q\rangle, \kappa; s) + \lambda^{-1}(L_{\mathbb{Z}};s)\frac{\mathrm{d}}{\mathrm{d}s}\mu(L_{\mathbb{Z}}, \langle q\rangle, \kappa; s) \\ &\qquad = \sum_{\alpha K\subset \mathrm{I}(L,\langle q\rangle)(\mathbb{A}^{(\infty)})\cap\kappa} \frac{\mathrm{d}}{\mathrm{d}s}\lambda^{-1}(\alpha^\perp_{\mathbb{Z}}; s)\end{aligned}$$

at $s = 0$ (which remains true [26, Theorem 10.5] up to $\mathbb{Q}\log(q)$ for primes $q \neq p$, if we replace λ by $\tilde{\lambda}$ and μ by $\widetilde{\mu}$) and inserting (1.10), we get:

$$\begin{aligned}(1.16)\quad &\widehat{\mathrm{vol}}\Big(\mathrm{M}\big({}^K\mathbf{O}(L_{\mathbb{Z}})\big)\Big)\widetilde{\mu}(L_{\mathbb{Z}}, \langle q\rangle, \kappa; 0)\\ &\qquad + \mathrm{vol}\Big(\mathrm{M}\big({}^K\mathbf{O}(L_{\mathbb{Z}})\big)\Big)\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mu}(L_{\mathbb{Z}}, \langle q\rangle, \kappa; s)\Big|_{s=0}\\ &= \sum_{K\alpha\subset \mathrm{I}(L,\langle q\rangle)(\mathbb{A}^{(\infty)})\cap\kappa} \widehat{\mathrm{vol}}\Big(\mathrm{M}\big({}^{K_\alpha}\mathbf{O}(\alpha_{\mathbb{Z}}^{\perp})\big)\Big) = \widehat{\mathrm{vol}}\big(\mathrm{Z}(L, \langle q\rangle, \kappa; K)\big).\end{aligned}$$

As soon as we are able to show this relation *directly*, too, we get that (1.10) holds for $\mathrm{M}\big({}^K\mathbf{O}(L_{\mathbb{Z}})\big)$ if and only if it holds for all $\mathrm{M}\big({}^{K_\alpha}\mathbf{O}(\alpha_{\mathbb{Z}}^{\perp})\big)$. Again, the result follows by induction.

To show (1.16) is the main step of the proof. In fact a certain average of it (cf. Theorem 3.4.2) will be sufficient. In its proof Borcherds' theory enters.

1.4.5. Recall from [2] that a Borcherds product is a meromorphic modular form F on $\mathrm{M}\big({}^K\mathbf{O}(L_{\mathbb{Z}})\big)_{\mathbb{C}}$ having singularities precisely at the (codimension $m = 1$) special cycles. It is a multiplicative lift of an integral vector valued modular form f of weight $1 - l/2$, holomorphic in $\mathbb{H}$, and meromorphic in the cusp $i\infty$ for the Weil representation of $\mathrm{Sp}_2'(\mathbb{Z})$ restricted to $\mathbb{C}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}]$. Here $\mathrm{Sp}_2'(\mathbb{Z})$ is $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$ if l is even and $\mathrm{Mp}_2(\mathbb{Z})$ if l is odd. Such an f has a Fourier expansion

$$f = \sum_{k\in\mathbb{Q}_{\gg -\infty}} a_k q^k,$$

where only a_k's with k having bounded denominator actually occur and $a_k \in \mathbb{Z}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}]$. The divisor of F (on the uncompactified Shimura variety) is given by $\sum_{k<0}\mathrm{Z}(L_{\mathbb{Z}}, \langle -k\rangle, a_k; K)$ and its weight is equal to $a_0(0)$. Assume for the moment (for simplicity) that an f can be found such that $a_k = 0$ for $k < 0$ except $a_{-q} = \kappa$ for some $q > 0$.

Using the corresponding meromorphic function F, Theorem 3.4.2 precisely gives relation (1.15), by an argument involving Serre duality already used by Borcherds [3] to show modularity of the Chow-group-valued theta function. However, we obtain relation (1.16), too, by a calculation in Arakelov theory, using a calculation of the integral of the Hermitian norm of F. This however requires the solution of additional problems:

(1) Need to prove that F is an *integral and 'monic'* modular form. This follows from an application of the q-expansion principle to Borcherds' famous product expansion of F. This implies that also its *arithmetic* divisor (away from the boundary) is precisely $\sum_{k<0}\mathrm{Z}(L_{\mathbb{Z}}, \langle -k\rangle, a_k; K)$, i.e., that no vertical components occur.

(2) Need to show that the boundary divisor of F does not contribute. This requires rather hard estimates on the occurring Green's functions.

Another issue concerns the induction process itself. Since many of the low-dimensional orthogonal Shimura varieties are compact, the here described method (q-expansion principle) cannot be used for them. Hence we start with the modular curve (the variety associated with a lattice of dimension 3 and Witt rank 1) and then for lattices of dimension ≥ 4 ensure that all components in the divisor of F have good reduction at p and are not compact. This yields the result for all non-compact orthogonal Shimura varieties. The compact case is obtained by exploiting the opposite implication of 1.4.4. Consequently, we need to design modular forms f prescribing their negative Fourier coefficients to some extent. This involves showing that the space of input forms f has a rational structure, and Serre duality like arguments to reduce to showing that modular forms with sparsely occurring Fourier coefficients (lacunary modular forms) vanish. This idea of designing appropriate f appears in principle already in [10].

In the same way, exploiting the opposite implication of 1.4.4, we finally also get equation (1.9) only up to contributions of the form $\mathbb{Q}\log(2)$ for an ample class of orthogonal Shimura varieties.

1.5. The philosophy of Shimura varieties

1.5.1. Even to define an intrinsic notion of arithmetic volume for an orthogonal Shimura varieties in higher dimensions, *canonical* integral models are necessary. To apply Arakelov theoretical methods in a reasonable way, the models have to be compactified, too. The arithmetic special cycles $\widehat{\mathrm{Z}}$ (good reduction case) on them are built from models of this kind themselves. Furthermore the Hermitian line bundle $\Xi^*(\mathcal{E}, h)$, involved in the definition of the arithmetic theta function and used to compute the 'degree' of this function, has to be defined as a 'canonical integral model' of an automorphic line bundle. In addition, to be able to work with Borcherds products as sections of them, one needs a 'q-expansion principle' to establish its integrality. The best and broadest context for all of these considerations is a fully functorial theory of canonical integral models of toroidal compactifications of mixed Shimura varieties, of the standard principal bundle on them, and of their 'compact' dual.

Consider a p-integral mixed Shimura datum $\mathbf{X} = (P_{\mathbf{X}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$, where: $P_{\mathbf{X}}$ is a group scheme $P_{\mathbf{X}}$ over $\mathbb{Z}_{(p)}$ of a certain rigid type which we will call type (P) (see Section 2.1); $\mathbb{D}_{\mathbf{X}}$ is a generalized Hermitian symmetric space (a principal $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-space, where $U_{\mathbf{X}}$ is a certain subgroup of the unipotent radical of $P_{\mathbf{X}}$); $h_{\mathbf{X}} : \mathbb{D} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{X},\mathbb{C}})$ is an equivariant morphism. The data is subject to the condition roughly that $(P_{\mathbf{X},\mathbb{Q}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$ satisfy Pink's axioms for a mixed Shimura datum [58].

To understand why analytic locally symmetric varieties (or orbifolds) of the form

$$[P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)]$$

should have *canonical* algebraic models defined over number fields (or even rings of integers) at all, and where this structure is supposed to come from, one should bear in mind the following philosophy:

Let $L_{\mathbb{Z}_{(p)}}$ be free $\mathbb{Z}_{(p)}$-module of finite rank with a faithful representation (closed embedding) $\rho\colon P_{\mathbf{X}} \to \mathrm{GL}(L_{\mathbb{Z}_{(p)}})$, fixing some weight filtration $\{W_i\}$, $W_i \subset L_{\mathbb{Z}_{(p)}}$ and polarization. There is always a finite set of tensors $v_1, \dots, v_n$, $v_i \in L_{\mathbb{Z}_{(p)}}^{\otimes}$ such that the image of ρ (in the stabilizer of the weight filtration in the similitude group of the polarization form) is precisely the stabilizer of these tensors. The complex manifold $\mathbb{D}_{\mathbf{X}}$ can be seen as an *open* $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-orbit in the parameter space of (polarized) mixed Hodge structures (w.r.t. the filtration $\{W_i\}$) on $L_{\mathbb{C}}$, having the property that all v_i lie in $(L^{\otimes})^{(0,0)}$. Furthermore there is a category (groupoid) of families of mixed Hodge structures on arbitrary local systems over a base analytic space B. It is convenient to take local systems of $\mathbb{Q}$-vector spaces and equip the families with a K-level structure (for a compact open $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$). This groupoid is denoted by

$$\left[\,\boldsymbol{B}\text{-}{}^{\boldsymbol{K}}\mathbf{X}\text{-}\boldsymbol{L}\text{-}\mathbf{loc\text{-}mhs}\,\right].$$

In fact, these groupoids (with varying B) form a category fibered in groupoids which is an analytic orbifold represented by the quotient

$$[P_{\mathbf{X}}(\mathbb{Q})\backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)]$$

i.e., the analytic mixed Shimura variety associated with $\mathbf{X}$.

If the group $P_{\mathbf{X}}$ and $\mathbb{D}_{\mathbf{X}}$ form a p-integral Shimura datum (2.2.3) one expects that over any base scheme S over $\mathcal{O}$ (a reflex ring of $\mathbf{X}$) there is a category (groupoid) of mixed motives

$$\left[\,\boldsymbol{S}\text{-}\mathbf{X}\text{-}\boldsymbol{L}\text{-}\mathbf{mot}\,\right]$$

which should (very roughly) be seen as the category of those polarized mixed motives M of fixed weight filtration type with morphisms $v_i'\colon \mathbb{Z}(0) \to M^{\otimes}$ which have the following property: etale locally, there is an isomophism (respecting weight filtration and polarization) of some realization (H^{et} or H^{dR}, say) with L that maps $H(v_i')$ to v_i for every i. This will be made precise for certain Shimura data and certain associated standard representations—corresponding to 1-motives—in Section 2.7. It can be made precise for all ‘P.E.L. situations’ (pure weight 1 and all v_i endomorphisms) and we refer to [30] or [46] for this. For Hodge-type Shimura data, the truth of the Hodge conjecture would as well allow to pose a moduli problem involving the existence of certain algebraic cycles.

Furthermore, one expects (functorial) maps

$$\left[\,\boldsymbol{S}\text{-}{}^{\boldsymbol{K}}\mathbf{X}\text{-}\boldsymbol{L}\text{-}\mathbf{mot}\,\right] \to \left[\,\boldsymbol{S}^{\mathbf{an}}\text{-}{}^{\boldsymbol{K}}\mathbf{X}\text{-}\boldsymbol{L}\text{-}\mathbf{loc\text{-}mhs}\,\right], \tag{1.17}$$

if S is of finite type over $\mathbb{C}$, which are *equivalences* for $S = \mathrm{Spec}(\mathbb{C})$. Here, on the left hand side, we consider now motives up to $\mathbb{Z}_{(p)}$-isogeny with a $K^{(p)}$-level structure (on the etale realization in $\mathbb{A}^{(\infty,p)}$-vector spaces), too. (Assume that K is admissible, i.e., of the form $P_{\mathbf{X}}(\mathbb{Z}_p) \times K^{(p)}$, in particular, hyperspecial.)

$\left[\,\boldsymbol{S}\text{-}{}^{\boldsymbol{K}}\mathbf{X}\text{-}\boldsymbol{L}\text{-}\mathbf{mot}\,\right]$ should be (represented by) an *algebraic* smooth Deligne–Mumford stack $\mathrm{M}({}^{K}\mathbf{X})$ over $\mathrm{Spec}(\mathcal{O})$, which would be a model of the analytic Shimura variety because (1.17) is an equivalence for $S = \mathrm{Spec}(\mathbb{C})$.

It is also important to look at the categories of motives, like above, equipped with a trivialization of H^{et} (with values in $\mathbb{A}^{(\infty,p)}$ vector spaces, say), H^{dR}, and, in the analytic setting, of H_{B}—in each case *respecting* the $P_{\mathbf{X}}$-structure (given by the tensors, polarization and weight filtration). These groupoids should be represented by . . .

- in the etale case:

$$\mathrm{M}^p(\mathbf{X}) := \varprojlim_{K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)}) \text{ admissible}} \mathrm{M}({}^{K}\mathbf{X}), \tag{1.18}$$

- in the de Rham case:

$$\mathrm{P}({}^{K(1)}\mathbf{X}), \tag{1.19}$$

  which is a right $P_{\mathbf{X}}$-torsor on $\mathrm{M}({}^{K(1)}\mathbf{X})$, called the standard principal bundle,

- in the Betti case:

$$\mathbb{D}_{\mathbf{X}} \tag{1.20}$$

itself, as mentioned above.

Analytic comparison isomorphisms should give embeddings $\mathbb{D}_{\mathbf{X}} \hookrightarrow (\mathrm{M}^p(\mathbf{X})_{\mathbb{C}})^{\mathrm{an}}$ and $\mathbb{D}_{\mathbf{X}} \hookrightarrow (\mathrm{P}({}^{K(1)}\mathbf{X})_{\mathbb{C}})^{\mathrm{an}}$. An equivariant embedding $\mathbb{D}_{\mathbf{X}} \hookrightarrow (\mathrm{P}({}^{K(1)}\mathbf{X})_{\mathbb{C}})^{\mathrm{an}}$ trivializes $\mathrm{P}({}^{K(1)}\mathbf{X})_{\mathbb{C}}^{\mathrm{an}}$ locally up to translation by $P_{\mathbf{X}}(\mathbb{Z})$. The image of a point under this trivialization followed by ρ would be precisely a *period matrix* of the corresponding motive. The standard principal bundle therefore is sometimes also called 'period torsor' because it encodes (when the above philosophy can be realized) relations between periods.

The main point, which makes it possible to approach the theory of these models without having an appropriate theory of mixed motives, is that all objects $\mathrm{M}({}^K\mathbf{X})$, $\mathrm{P}({}^K\mathbf{X})$, $\mathbb{D}_{\mathbf{X}}$, etc. should be *independent of the representation* ρ. Moreover, it *is* possible to characterize models intrinsically, which we call *canonical*. These should always represent the corresponding moduli problem, if an appropriate one in terms of motives can be posed. The intrinsic characterization of the models is as follows:

(1) $\mathbb{D}_{\mathbf{X}}$ is seen as a certain conjugacy class of morphisms in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{X},\mathbb{C}})$ (defined over $\mathbb{R}$ modulo $U_{\mathbf{X}}(\mathbb{C})$, a part of the unipotent radical). If a representation ρ is chosen, composition with it yields morphisms $\mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(L_{\mathbb{C}})$ which are splittings for the corresponding mixed Hodge structures. In particular, this determines already an intrinsic complex analytic structure (via Borel embedding) of the Shimura variety.

(2) The characterization of the projective limit of the rational models $\mathrm{M}({}^K\mathbf{X})_E$ is reduced, requiring functoriality in Shimura data, to the case where $P_{\mathbf{X}}$ is a torus and accordingly the analytic Shimura variety is 0-dimensional. The characterization in that case is in terms of class field theory and is motivated by the theory of complex multiplication of Abelian varieties. This marvelous idea is due to Deligne [16, 17] and was extended to the mixed case by Pink [58]. The characterization of the integral model $\mathrm{M}({}^K\mathbf{X})$ itself is as follows. One requires the limit $\mathrm{M}^p(\mathbf{X})$ to satisfy an extension property very similar to the Neron property[6]. This idea appears first in [50, 51].

(3) The characterization of $\mathrm{P}({}^K\mathbf{X})$ can be reduced via functoriality, at least for a wide class of (mixed) Shimura data, to the case of the symplectic Shimura data, where a moduli problem in terms of 1-motives is available. It is then possible to show independence of an embedding into a symplectic Shimura datum directly. The author does not know of a better characterization which works in the integral case, too.

If a faithful representation $\rho\colon P_{\mathbf{X}} \hookrightarrow \mathrm{GL}(L_{\mathbb{Z}_{(p)}})$ is given, the objects (1.18)–(1.20) yield an l-adic sheaf (for every $l \neq p$), a vector bundle with connection, and a local system (in the analytic case), respectively, on the Shimura variety. Whenever it is possible to precise the moduli problem determined by this representation, these objects should be equal to the corresponding realizations of the universal mixed motive.

However, it should be possible to recover the filtration steps of the de Rham bundle and tensor constructions of them, too. This is seen as follows: If a moduli problem exists and $\mathrm{P}({}^K\mathbf{X})$ represents motives together with a trivialization of the

[6]in fact this *is* the Neron property for the first step in an unipotent extension

de Rham realization, the filtration on the realization yields a filtration on L_S, compatible with the $P_{\mathbf{X}}$-structure (determined by the tensors, polarization and weight filtration). Filtrations of this type on L_S with varying S are represented by a quasi-projective (projective, if $\mathbf{X}$ is pure) variety $\mathrm{M}^\vee(\mathbf{X})$, called the 'compact' dual. It is defined over $\mathcal{O}$ and independent of ρ, too. Hence we get a $P_{\mathbf{X}}$-equivariant morphism $\mathrm{P}({}^K\mathbf{X}) \to \mathrm{M}^\vee(\mathbf{X})$, or, in other words, a morphism of Artin stacks

$$\Xi\colon\ \mathrm{M}({}^K\mathbf{X}) \to [\mathrm{M}^\vee(\mathbf{X})/P_{\mathbf{X}}]. \tag{1.21}$$

This allows to associate with *every* $P_{\mathbf{X}}$-bundle $\mathcal{E}$ on $\mathrm{M}^\vee(\mathbf{X})$ a bundle $\Xi^*\mathcal{E}$ on $\mathrm{M}({}^K\mathbf{X})$ called an (integral) automorphic vector bundle. (In particular this holds for the bundles in the universal filtration associated with ρ.) The integral structure, however, is of course not pinned down by considering $\Xi^*\mathcal{E}$ as an abstract sheaf. The analytic comparison isomorphism, however, allows to compare this map with the Borel embedding

$$\mathbb{D}_{\mathbf{X}} \hookrightarrow \mathrm{M}^\vee(\mathbf{X})(\mathbb{C}).$$

Therefore, if $\mathcal{E}_{\mathbb{C}}|_{\mathbb{D}_{\mathbf{X}}}$ is equipped with a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant Hermitian metric h, we may define $\Xi^*(\mathcal{E}, h)$ (by slight abuse of notation). It is a well-defined Hermitian arithmetic vector bundle on $\mathrm{M}({}^K\mathbf{X})$.

For many purposes, in particular for the main purpose of this book, this is not sufficient because $\mathrm{M}({}^K\mathbf{X})$ is not proper. Desirable are toroidal compactifications $\mathrm{M}({}^K_\Delta\mathbf{X})$, depending on a rational polyhedral cone decomposition Δ of the conical complex $C_{\mathbf{X}}$ associated with $\mathbf{X}$. Furthermore, an extension $\mathrm{P}({}^K_\Delta\mathbf{X})$ of $\mathrm{P}({}^K\mathbf{X})$, or equivalently of the morphism (1.21), is needed to extend automorphic vector bundles. This would yield proper varieties and Hermitian automorphic vector bundles $\Xi^*(\mathcal{E}, h)$ on them. It turns out that there is only one meaningful way to extend $\mathrm{P}({}^K\mathbf{X})$, forced by the structure of Abelian unipotent extension as a torus torsor. In fact, this structure trivializes the standard principal bundle along this unipotent fibre and since the compactification along the unipotent fibre is defined by a torus embedding of the corresponding torus, the trivialization defines a 'trivial' extension of the bundle. This pins down the extensions in general, if one requires functoriality with respect to boundary maps (which are, in the algebraic setting, maps between formal completions). This functoriality also yields a 'q-expansion principle' for integral automorphic forms.

The extension of $\mathrm{P}({}^K\mathbf{X})$ determines canonical extensions of all automorphic vector bundles. In the rational case these are the same as described before by Mumford [55] (for fully decomposed bundles) and Deligne (for local systems).

The corresponding theory, developed in Part I of the thesis of the author [25], is presented in Sections 2.1–2.5.