CHAPTER 1

Introduction

1.1. Introduction and context

In this monograph we investigate well-posedness of boundary value problems associated with divergence-form elliptic equations

$$L_A u := \operatorname{div} A\nabla u = 0, \tag{1.1}$$

where the unknown u is a $\mathbb{C}^m$-valued function on the upper half-space

$$\mathbb{R}^{1+n}_+ := \{(t,x) \in \mathbb{R}^{1+n} : t > 0\}.$$

We work in ambient dimension $1+n \geq 2$, with $m \geq 1$. The equation (1.1) may be considered as system of m scalar equations. The value of m is quite irrelevant to everything that we do, so for simplicity the reader may assume $m = 1$ throughout.

We use the so-called 'first order approach,' following previous work by the second author with Hofmann, McIntosh, Mourgoglou, Rosén (Axelsson), and Stahlhut [5–8, 13, 15–17, 19]. In these articles, boundary value problems are considered with (boundary) data of regularity 0 (i.e., data in Lebesgue spaces L^p and Hardy spaces H^p) or of regularity -1 (i.e., in Sobolev spaces W^p_{-1}, Hölder spaces $\dot{\Lambda}^\alpha_{-1}$, or in $\dot{\mathrm{BMO}}_{-1}$). Here we address problems with *fractional regularity data*: that is, data in Hardy–Sobolev spaces $\dot{H}^p_\theta$ or Besov spaces $\dot{B}^{p,p}_\theta$ with $\theta \in (-1,0)$. These were considered in the previous articles only when $p = 2$.

Boundary value problems for the equation $L_A u = 0$ with data in Besov spaces have recently been studied by Barton and Mayboroda [22]. Under the additional assumptions that $m = 1$ and that solutions to (1.1) satisfy De Giorgi–Nash–Moser estimates (see (7.2)), they establish various well-posedness results via the method of layer potentials. One novelty of their approach is that they can also consider inhomogeneous problems $L_A u = f$, which we do not address. For homogeneous problems, however, our results are more general. The first order approach requires neither the existence of fundamental solutions (which is implied by De Giorgi–Nash–Moser estimates when $m = 1$, and which is crucial in setting up the method of layer potentials) nor the validity of trace theorems (which hold for Besov spaces, but not for Hardy–Sobolev spaces).

Our approach is based on an abstract framework of *adapted Besov–Hardy–Sobolev spaces.* For applications to boundary value problems with data of regularity 0 and -1, a theory of adapted Hardy spaces had been sufficiently developed by the second author and Stahlhut [19]. We extend this theory by exploiting properties of weighted tent spaces T^p_θ and their real interpolants, the *Z-spaces* Z^p_θ.

1.1.1. The elliptic equation. Consider again the elliptic equation (1.1). The gradient operator ∇ maps $\mathbb{C}^m$-valued functions in $1+n$ variables f to $\mathbb{C}^{m(1+n)}$-valued functions ∇f by writing $f = (f^j)_{j=1}^m$ as an m-tuple of $\mathbb{C}$-valued functions and acting as the usual gradient operator on each component f^j. The divergence operator div is similarly defined in terms of the usual divergence operator, sending $\mathbb{C}^{m(1+n)}$-valued functions to $\mathbb{C}^m$-valued functions. These differential operators are interpreted in the distributional sense. Vectors $v \in \mathbb{C}^{m(1+n)}$ are split into *transversal* and *tangential* parts $v = (v_\perp, v_\parallel)$ according to the decomposition

$$\mathbb{C}^{m(1+n)} = \mathbb{C}^m \oplus \mathbb{C}^{mn}, \tag{1.2}$$

and likewise $\mathbb{C}^{m(1+n)}$-valued functions f can be split into transversal and tangential parts $f = (f_\perp, f_\parallel)$, valued in $\mathbb{C}^m$ and $\mathbb{C}^{mn}$ respectively. We write $\nabla_\parallel$ and $\mathrm{div}_\parallel$ for the corresponding tangential restrictions of ∇ and div.

Throughout the monograph we assume that the coefficient matrix

$$A \in L^\infty\Big(\mathbb{R}^{n+1}_+ : \mathcal{L}\big(\mathbb{C}^{m(1+n)}\big)\Big)$$

is bounded, measurable, and *t-independent*, meaning that $A(t,x) = A(x)$ for almost every $(t,x) \in \mathbb{R}^{1+n}_+$. Thus we may consider A as an element of $L^\infty\big(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)})\big)$. We also assume that A is *strictly accretive on curl-free vector fields*, meaning that there exists $\kappa > 0$ such that

$$\mathrm{Re} \int_{\mathbb{R}^n} \big(A(x)f(x), f(x)\big)\, \mathrm{d}x \geq \kappa \|f\|_2^2 \tag{1.3}$$

for all $f \in L^2(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$ such that $\mathrm{curl}_\parallel(f_\parallel) = 0$. The round bracket in the integrand above is the usual Hermitean inner product on $\mathbb{C}^{m(1+n)}$. By $\mathrm{curl}_\parallel(f_\parallel) = 0$ we mean that

$$\partial_j f_k = \partial_k f_j \quad (1 \leq j,k \leq n,\ j \neq k),$$

where ∂_j is the distributional partial derivative in the jth coordinate direction of $\mathbb{R}^n$, acting componentwise on $\mathbb{C}^m$-valued functions. The strict accretivity condition (1.3) is weaker than the usual notion of pointwise strict accretivity

$$\mathrm{Re}(A(x)v, v) \geq \kappa |v|^2 \quad (v \in \mathbb{C}^{m(1+n)},\ x \in \mathbb{R}^n)$$

unless $m = 1$ and A is real, in which case these two notions are equivalent [8, §2].

As we have not assumed any regularity of A, we must consider weak solutions to (1.1): we say that a function $u \in W^2_{1,\mathrm{loc}}(\mathbb{R}^n : \mathbb{C}^m)$ solves (1.1) if for all $\varphi \in C_0^\infty(\mathbb{R}^{1+n}_+ : \mathbb{C}^m)$ we have

$$\iint_{\mathbb{R}^{1+n}_+} \big(A(x)\nabla u(t,x), \nabla\varphi(t,x)\big)\, \mathrm{d}x\, \mathrm{d}t = 0.$$

1.1.2. Formulation of boundary value problems. Various boundary value problems for the equation $L_A u = 0$ have been studied. Here we reformulate some of these problems, and introduce some new ones. In Definition 7.1 we concisely synthesise these problems into two categories, each parametrised by a set of exponents and two (related) families of spaces of boundary data.

First, for $1 < p < \infty$, following the groundbreaking work of Dahlberg on the Laplace equation on Lipschitz domains [29], we formulate the *L^p-Dirichlet problem* for L_A, denoted by $(\mathrm{D}_H)^p_{0,A}$:

$$(\mathrm{D}_H)^p_{0,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|N_* u\|_{L^p} \lesssim \|f\|_{L^p}, \\ \lim_{t\to 0} u(t,\cdot) = f \in L^p(\mathbb{R}^n : \mathbb{C}^m). \end{cases}$$

This should be read:

for all $f \in L^p(\mathbb{R}^n : \mathbb{C}^m)$,
there exists $u \in W^2_{1,\mathrm{loc}}(\mathbb{R}^{1+n}_+ : \mathbb{C}^m)$ solving $L_A u = 0$,
with $\|N_* u\|_p \lesssim \|f\|_p$ (the *interior estimate*),
and $u(t,\cdot) \to f$ in L^p as $t \to 0$ (the *boundary condition*).

Here N_* is the nontangential maximal function

$$N_* u(x) := \sup_{\substack{(t,y)\in\mathbb{R}^{1+n}_+ \\ y\in B(x,t)}} |u(t,y)|.$$

We say that the problem $(\mathrm{D}_H)^p_{0,A}$ is *well-posed* if for all $f \in L^p(\mathbb{R}^n : \mathbb{C}^m)$ there exists a unique u satisfying these conditions.[1]

Well-posedness is defined analogously for all boundary value problems that we consider: for all boundary data in the specified function space, there must exist a unique solution to (1.1)—up to an additive constant, for Regularity and Neumann problems—which satisfies the conditions of the boundary value problem. Of course, a boundary value problem may or may not be well-posed.

For $n/(n+1) < p < \infty$, we formulate the H^p*-Regularity problem* for L_A:

$$(\mathrm{R}_H)^p_{0,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\widetilde{N}_*(\nabla u)\|_{L^p} \lesssim \|\nabla_\| f\|_{H^p}, \\ \lim_{t\to 0} \nabla_\| u(t,\cdot) = \nabla_\| f \in H^p(\mathbb{R}^n : \mathbb{C}^{mn}), \end{cases}$$

where $\widetilde{N}_* u$ is the modified nontangential maximal function

$$\widetilde{N}_* u(x) := \sup_{\substack{(t,y)\in\mathbb{R}^{1+n}_+ \\ y\in B(x,t)}} \left(\fint\!\!\fint_{\substack{t/2<\tau<2t \\ \xi\in B(y,t)}} |u(\tau,\xi)|^2 \, \mathrm{d}\tau \, \mathrm{d}\xi \right)^{1/2} \quad (x \in \mathbb{R}^n), \tag{1.4}$$

and where $H^p(\mathbb{R}^n : \mathbb{C}^{mn})$ is the ($\mathbb{C}^{mn}$-valued) real Hardy space, which may be identified with $L^p(\mathbb{R}^n : \mathbb{C}^{mn})$ when $p > 1$. The restriction on p arises because a gradient $\nabla_\| f$ need not be in H^p for $p \leq n/(n+1)$, as it will only have one vanishing moment.

Remark 1.1. If f is a distribution with $\nabla_\| f \in H^p(\mathbb{R}^n : \mathbb{C}^{mn})$, then f may be identified with an element of $\dot{H}^p_1(\mathbb{R}^n : \mathbb{C}^m)$—the $\mathbb{C}^m$-valued homogeneous Hardy–Sobolev space of order 1, defined in Section 2.5—and the boundary condition $\lim_{t\to 0} \nabla_\| u(t,\cdot) = \nabla_\| f \in H^p(\mathbb{R}^n : \mathbb{C}^{mn})$ is equivalent to the condition

$$\lim_{t\to 0} u(t,\cdot) = f \in \dot{H}^p_1(\mathbb{R}^n : \mathbb{C}^m).$$

Therefore, by considering functions u rather than tangential gradients $\nabla_\| u$, the H^p-Regularity problem can be seen as a kind of $\dot{H}^p_1$-Dirichlet problem. Conversely,

[1] Use of the nontangential maximal function N_* is appropriate for $(\mathrm{D}_H)^p_{0,A}$ provided that solutions to $L_A u = 0$ have pointwise values. Otherwise, it is better to use the modified nontangential maximal function $\widetilde{N}_*$ defined in (1.4). Also, the boundary condition $\lim_{t\to 0} u(t,\cdot) = f$ is traditionally imposed as nontangential convergence almost everywhere rather than L^p convergence; we shall return to this point later. For now, observe that there is a priori no relation between the interior estimate and the boundary condition, as there are no trace theorems for the space $\{u : \|N_* u\|_p < \infty\}$.

by considering tangential gradients $\nabla_\parallel u$, the L^p-Dirichlet problem $(\mathrm{D}_H)^p_{0,A}$ can be seen as a kind of $\dot{H}^p_{-1}$-Regularity problem. For technical reasons we generally consider Regularity problems rather than Dirichlet problems.

For $n/(n+1) < p < \infty$, we also formulate the H^p-*Neumann problem* for L_A:

$$(\mathrm{N}_H)^p_{0,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\widetilde{N}_*(\nabla u)\|_{L^p} \lesssim \|\partial_{\nu_A} f\|_{H^p}, \\ \lim_{t\to 0} \partial_{\nu_A} u(t,\cdot) = \partial_{\nu_A} f \in H^p(\mathbb{R}^n : \mathbb{C}^m). \end{cases}$$

The A-*conormal derivative* ∂_{ν_A} of u is defined by

$$\partial_{\nu_A} u(t,\cdot) = e_0 \cdot A\nabla u(\cdot, t), \tag{1.5}$$

where $e_0 = (1, 0, \ldots, 0)$ is the normal vector to $\partial\mathbb{R}^{1+n}_+$ 'pointing in the t-direction'.[2]

The boundary value problems $(\mathrm{D}_H)^p_{0,A}$, $(\mathrm{R}_H)^p_{0,A}$, and $(\mathrm{N}_H)^p_{0,A}$ are all problems of *order zero*: in each of these problems boundary data are assumed to be in the Lebesgue space L^p or the Hardy space H^p. One can also formulate Regularity and Neumann problems of *order* -1. For $1 < p < \infty$, the $\dot{H}^p_{-1}$-*Regularity problem*, which is similar to the L^p-Dirichlet problem but with a different interior estimate[3] and a decay condition at infinity (see Remark 1.1), is

$$(\mathrm{R}_H)^p_{-1,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{T^p_{-1}} \lesssim \|\nabla_\parallel f\|_{\dot{H}^p_{-1}}, \\ \lim_{t\to\infty} \nabla_\parallel u(t,\cdot) = 0 \quad \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t\to 0} \nabla_\parallel u(t,\cdot) = \nabla_\parallel f \in \dot{H}^p_{-1}(\mathbb{R}^n : \mathbb{C}^{mn}). \end{cases}$$

Here $\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn})$ is the space of $\mathbb{C}^{mn}$-valued tempered distributions modulo polynomials, in which all homogeneous Hardy–Sobolev and Besov spaces are embedded. The space T^p_{-1} is a particular instance of a *weighted tent space*: for $0 < p < \infty$, $\theta \in \mathbb{R}$, and $f \colon \mathbb{R}^{1+n}_+ \to \mathbb{C}^N$ measurable with $N \in \mathbb{N}_+$ fixed, we have

$$\|f\|_{T^p_\theta} := \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} |t^{-\theta} f(t,y)|^2 \, \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \right)^{p/2} \mathrm{d}x \right)^{1/p}. \tag{1.6}$$

We can also formulate boundary value problems with data in BMO-type spaces and homogeneous Hólder spaces $\dot{\Lambda}_\alpha$, thus increase the range of exponents to '$p \geq \infty$'. For $0 < \alpha < 1$ we define

$$(\mathrm{R}_H)^{(\infty,\alpha)}_{-1,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{T^\infty_{-1;\alpha}} \lesssim \|\nabla_\parallel f\|_{\dot{\Lambda}_{\alpha-1}}, \\ \lim_{t\to\infty} \nabla_\parallel u(t,\cdot) = 0 \text{ in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t\to 0} \nabla_\parallel u(t,\cdot) = \nabla_\parallel f \in \dot{\Lambda}_{\alpha-1}(\mathbb{R}^n : \mathbb{C}^{mn}) \end{cases}$$

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[2] The Regularity and Neumann problems for Laplace's equation on domains do not require use of $\widetilde{N}_*$; in this case N_* works fine (see Dahlberg and Kenig [30] and Brown [24]). The modified nontangential maximal function $\widetilde{N}_*$ is needed for more general operators L_A. This modification was introduced by Kenig and Pipher [60], who obtained the first results in this direction.

[3] It is known that $\left\|\widetilde{N}_* u\right\|_{L^p} \lesssim \|\nabla u\|_{T^p_{-1}}$ in the range of p that we shall deal with (see [19]), so the two problems are related. The converse inequality is not known except for the case of real equations [42, Theorem 1.7].

and furthermore, with $\alpha = 0$,

$$(\mathrm{R}_H)^{(\infty,0)}_{-1,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{T^\infty_{-1;0}} \lesssim \|\nabla_\| f\|_{\dot{\mathrm{BMO}}_{-1}}, \\ \lim_{t\to\infty} \nabla_\| u(t,\cdot) = 0 \text{ in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t\to 0} \nabla_\| u(t,\cdot) = \nabla_\| f \in \dot{\mathrm{BMO}}_{-1}(\mathbb{R}^n : \mathbb{C}^{mn}). \end{cases}$$

The spaces $\dot{\mathrm{BMO}}_{-1}$ and $\dot{\Lambda}_{\alpha-1}$ are most conveniently represented as the homogeneous Triebel–Lizorkin space $\dot{F}^{\infty,2}_{-1}$ and Besov spaces $\dot{B}^{\infty,\infty}_{\alpha-1}$ respectively, as their negative orders prevent traditional characterisations in terms of smoothness. In these problems the limit in the boundary condition is imposed in the weak-star topology, which is possible since $\dot{\Lambda}_{\alpha-1}$ and $\dot{\mathrm{BMO}}_{-1}$ are the Banach duals of $\dot{H}^{n/(n+\alpha)}_1$ and $\dot{H}^1_1$ respectively. The spaces $T^\infty_{-1;\alpha}$ are again instances of weighted tent spaces: for $\theta \in \mathbb{R}$ and $\alpha \geq 0$, and for measurable $f\colon \mathbb{R}^{1+n}_+ \to \mathbb{C}^N$, we have the Carleson-type norm

$$\|f\|_{T^\infty_{\theta;\alpha}} := \sup_{B\subset\mathbb{R}^n} \frac{1}{r_B^\alpha} \left(\frac{1}{r_B^n} \int_0^{r_B} \int_{B(c_B, r_B - t)} |t^{-\theta} f(t,y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}t}{t} \right)^{1/2} \tag{1.7}$$

where the supremum is taken over all open balls $B = B(c_B, r_B) \subset \mathbb{R}^n$.

For p and α as above, we can define *order* -1 *Neumann problems* $(\mathrm{N}_H)^p_{-1,A}$, $(\mathrm{N}_H)^{(\infty,\alpha)}_{-1,A}$, and $(\mathrm{N}_H)^{(\infty,0)}_{-1,A}$ in the same way, with tangential gradients $\nabla_\|$ replaced by A-conormal derivatives ∂_{ν_A} in the boundary condition (keeping $\nabla_\|$ in the decay condition at infinity).

Note that in the 'order -1' problems above, we impose a tent space estimate on ∇u rather than a nontangential maximal function estimate. We also impose a decay condition on the tangential gradient $\nabla_\| u$ at infinity. For p sufficiently small the decay condition is implied by the other conditions, and if L_A satisfies a De Giorgi–Nash–Moser condition (see (7.2)) then it is implied for all $p < \infty$, and also for some range of $\alpha \geq 0$. (see Lemma 6.4).

Remark 1.2. We have not imposed any nontangential convergence of solutions in the problems above. This is because the classification theorems of the second author and Mourgoglou, in particular [16, Corollaries 1.2 and 1.4], automatically yield almost everywhere (a.e.) nontangential convergence of Whitney averages (of either the solution or its conormal gradient, whichever is relevant). When L_A satisfies a De Giorgi–Nash–Moser condition this can be improved to a.e. nontangential convergence without Whitney averages.

Let us summarise the problems we have introduced so far. There are Dirichlet problems of order 0 and 1 (interpreting the H^p-Regularity problem as a $\dot{H}^p_1$-Dirichlet problem), Regularity problems of order 0 and -1, and Neumann problems of order 0 and -1.

In their recent memoir [22], Barton and Mayboroda consider problems of *intermediate order*. They formulate Dirichlet problems of order $\theta \in (0,1)$ and Neumann problems of order $\theta \in (-1,0)$ as follows. For $0 < \theta < 1$ and $n/(n+\theta) < p \leq \infty$,

$$(\mathrm{D}_B)^p_{\theta,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{L(p,\theta,2)} \lesssim \|f\|_{\dot{B}^{p,p}_\theta}, \\ \mathrm{Tr}\, u = f \in \dot{B}^{p,p}_\theta(\mathbb{R}^n : \mathbb{C}^m) \end{cases}$$

and

$$(\mathrm{N}_B)^p_{\theta-1,A}: \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{L(p,\theta,2)} \lesssim \|\partial_{\nu_A} f\|_{\dot{B}^{p,p}_{\theta-1}}, \\ \partial_{\nu_A} u|_{\partial\mathbb{R}^{1+n}_+} = \partial_{\nu_A} f \in \dot{B}^{p,p}_{\theta-1}(\mathbb{R}^n : \mathbb{C}^m). \end{cases}$$

The spaces $L(p,\theta,2)$ are defined by the norms

$$\|F\|_{L(p,\theta,2)} := \left(\iint_{\mathbb{R}^{1+n}_+} \left(\fint\!\!\fint_{\substack{t/2<\tau<2t \\ \xi\in B(x,t)}} |\tau^{1-\theta} F(\tau,\xi)|^2 \,\mathrm{d}\xi\,\mathrm{d}\tau \right)^{p/2} \mathrm{d}x \frac{\mathrm{d}t}{t} \right)^{1/p}$$

with the usual modification when $p = \infty$. We refer to these spaces as *Z-spaces* starting from Section 2.3 (the letter L already being overused), with an indexing convention such that $Z^p_\theta = L(p,\theta+1,2)$. The boundary condition for $(\mathrm{D}_B)^p_{\theta,A}$ is phrased in terms of the trace operator, which Barton and Mayboroda show to be bounded from $\{u : \nabla u \in L(p,\theta,2)\}$ to $\dot{B}^{p,p}_\theta$ when $p > n/(n+\theta)$ [22, Theorem 3.9]. A similar argument is used to define the boundary conormal derivative $\partial_{\nu_A} u|_{\partial\mathbb{R}^{1+n}_+}$.

Remark 1.3. We warn the reader that our indexing convention for boundary value problems is different to that in [22]: we index our problems by the order of the function space in which the boundary data is assumed to lie. Thus Barton and Mayboroda refer to $(\mathrm{N}_B)^p_{\theta-1,A}$ as $(\mathrm{N})^p_{\theta,A}$.

As we stated earlier, for technical reasons we consider Regularity problems rather than Dirichlet problems. Thus for $-1 < \theta < 0$ and $n/(n+\theta+1) < p \leq \infty$ we define

$$(\mathrm{R}_B)^p_{\theta,A}: \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{Z^p_\theta} \lesssim \|\nabla_\| f\|_{\dot{B}^{p,p}_\theta}, \\ \lim_{t\to\infty} \nabla_\| u(t,\cdot) = 0 \text{ in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t\to 0} \nabla_\| u(t,\cdot) = \nabla_\| f \in \dot{B}^{p,p}_\theta(\mathbb{R}^n : \mathbb{C}^{mn}) \end{cases}$$

and

$$(\mathrm{N}_B)^p_{\theta,A}: \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{Z^p_\theta} \lesssim \|\partial_{\nu_A} f\|_{\dot{B}^{p,p}_\theta}, \\ \lim_{t\to\infty} \nabla_\| u(t,\cdot) = 0 \text{ in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t\to 0} \partial_{\nu_A} u(t,\cdot) = \partial_{\nu_A} f \in \dot{B}^{p,p}_\theta(\mathbb{R}^n : \mathbb{C}^m), \end{cases}$$

replacing the trace conditions with limiting conditions for consistency with the 'endpoint order' problems that we have already defined, writing Z^p_θ instead of $L(p,\theta+1,2)$, and including a decay condition at infinity. As before, when $p = \infty$ we impose the boundary condition in the weak-star topology, using that $\dot{B}^{\infty,\infty}_\theta$ is the dual of $\dot{B}^{1,1}_{-\theta}$.

Remark 1.4. If we omit the decay condition, the Regularity problem $(\mathrm{R}_B)^p_{\theta,A}$ is equivalent to the Dirichlet problem $(\mathrm{D}_B)^p_{\theta+1,A}$ defined above by an argument similar to that of Remark 1.1, and the Neumann problem $(\mathrm{N}_B)^p_{\theta,A}$ is simply a rewriting of the previously-defined Neumann problem. As stated earlier, the decay condition is often redundant. Furthermore, the trace theorem of Barton and Mayboroda [22, Theorem 3.9] implies that the boundary limiting condition and the trace condition are equivalent given that $\nabla u \in Z^p_\theta$.

The Besov spaces $\dot{B}^{p,p}_\theta$ with $\theta \in (-1,0)$ are not the only function spaces situated between H^p_0 and $\dot{H}^p_{-1}$. One of our new contributions to this problem is that we also consider the Hardy–Sobolev spaces $\dot{H}^p_\theta$ with $\theta \in (-1,0)$. These are defined in Section 2.5; they may be identified with the homogeneous Triebel–Lizorkin spaces $\dot{F}^{p,2}_\theta$, whereas the Besov spaces $\dot{B}^{p,p}_\theta$ may be identified with $\dot{F}^{p,p}_\theta$ when $p < \infty$. We use Hardy–Sobolev spaces to formulate the following Regularity and Neumann problems, with $-1 < \theta < 0$ and $n/(n+\theta+1) < p < \infty$:

$$(\mathrm{R}_H)^p_{\theta,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{T^p_\theta} \lesssim \|\nabla_\parallel f\|_{\dot{H}^p_\theta}, \\ \lim_{t\to\infty} \nabla_\parallel u(t,\cdot) = 0 \text{ in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t\to 0} \nabla_\parallel u(t,\cdot) = \nabla_\parallel f \in \dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^{mn}) \end{cases}$$

and

$$(\mathrm{N}_H)^p_{\theta,A} : \begin{cases} L_A u = 0 \text{ in } \mathbb{R}^{1+n}_+, \\ \|\nabla u\|_{T^p_\theta} \lesssim \|\partial_{\nu_A} f\|_{\dot{H}^p_\theta}, \\ \lim_{t\to\infty} \nabla_\parallel u(t,\cdot) = 0 \text{ in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^{mn}), \\ \lim_{t\to 0} \partial_{\nu_A} u(t,\cdot) = \partial_{\nu_A} f \in \dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^{m}). \end{cases}$$

Recall that the T^p_θ (quasi-)norm is defined in (1.6). Furthermore, for $-1 < \theta < 0$ we formulate 'endpoint' problems $(\mathrm{R}_H)^\infty_{\theta,A}$ and $(\mathrm{N}_H)^\infty_{\theta,A}$ by replacing $\dot{H}^p_\theta$ with the homogeneous BMO-Sobolev space $\dot{\mathrm{BMO}}_\theta$, which may be identified with the homogeneous Triebel–Lizorkin space $\dot{F}^{\infty,2}_\theta$, and replacing T^p_θ with the weighted tent space $T^\infty_{\theta;0}$ (defined in (1.7)). In this case the boundary condition is imposed in the weak-star topology, using that $\dot{\mathrm{BMO}}_\theta$ is the dual of $\dot{H}^1_{-\theta}$. Unlike the function space $\{u : \nabla u \in Z^p_\theta\}$, there is no trace theorem for the function space $\{u : \nabla u \in T^p_\theta\}$.

Let us briefly summarise the Regularity and Neumann problems that we have introduced.

- At order 0 we have problems $(\mathrm{R}_H)^p_{0,A}$ and $(\mathrm{N}_H)^p_{0,A}$ for $n/(n+1) < p < \infty$, with boundary data in H^p and a modified nontangential maximal estimate on the interior.
- At order -1 we have $(\mathrm{R}_H)^p_{-1,A}$ and $(\mathrm{N}_H)^p_{-1,A}$ for $1 < p < \infty$, with boundary data in $\dot{H}^p_{-1}$ and a T^p_{-1} interior estimate. Furthermore, for $0 \leq \alpha < 1$, we have $(\mathrm{R}_H)^{(\infty,\alpha)}_{-1,A}$ and $(\mathrm{N}_H)^{(\infty,\alpha)}_{-1,A}$ with boundary data in $\dot{\Lambda}_{\alpha-1}$ (or $\dot{\mathrm{BMO}}_{-1}$ when $\alpha = 0$) and a $T^\infty_{-1;\alpha}$ interior estimate.
- In between, i.e., for order $\theta \in (-1,0)$, and with $n/(n+\theta+1) < p \leq \infty$, we have $(\mathrm{R}_B)^p_{\theta,A}$ and $(\mathrm{N}_B)^p_{\theta,A}$ with boundary data in $\dot{B}^{p,p}_\theta$, and $(\mathrm{R}_H)^p_{\theta,A}$ and $(\mathrm{N}_H)^p_{\theta,A}$ with boundary data in $\dot{H}^p_\theta$ ($\dot{\mathrm{BMO}}_\theta$ when $p = \infty$). Here the interior estimates are in Z^p_θ and T^p_θ ($T^\infty_{\theta;0}$ when $p = \infty$) respectively.

For all problems of negative order we also impose a decay condition on $\nabla_\parallel u(t,\cdot)$ as $t \to \infty$ in the space $\mathcal{Z}'$ of tempered distributions modulo polynomials, which is redundant in many cases (see Lemma 6.4). Note that for $\theta \in (-1,0)$, the problems $(\mathrm{R}_H)^2_{\theta,A}$ and $(\mathrm{R}_B)^2_{\theta,A}$ (and likewise for Neumann problems) coincide, since $\dot{H}^2_\theta = \dot{B}^{2,2}_\theta$ and $Z^2_\theta = T^2_\theta$.

Since we assume no regularity of our coefficients, and since we consider weak solutions without further regularity properties, the only Regularity and Neumann problems which are meaningful are those of order between -1 and 0. Thus, within this context, this list of boundary value problems is essentially complete.

1.1.3. Perturbed Dirac operators and Cauchy–Riemann systems. Let D denote the differential operator on $\mathbb{C}^{m(1+n)}$-valued functions given by

$$D := \begin{bmatrix} 0 & \operatorname{div}_{\parallel} \\ -\nabla_{\parallel} & 0 \end{bmatrix}$$

with respect to the transversal/tangential splitting (1.2) of $\mathbb{C}^{m(1+n)}$. We refer to D as a *Dirac operator*, because D^2 acts as the Laplacian Δ on the range of D. When $B \in L^\infty\big(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)})\big)$ is a coefficient matrix satisfying the same assumptions as A, we refer to the composition DB as a *perturbed Dirac operator*.

The *Cauchy–Riemann system* associated with DB is the first-order partial differential system

$$(\mathrm{CR})_{DB} : \begin{cases} \partial_t F + DBF = 0 \\ \quad \operatorname{curl}_{\parallel} F_{\parallel} = 0 \end{cases} \quad \text{in } \mathbb{R}^{1+n}_+$$

interpreted in the weak (L^2_{loc}) sense: that is, we say that $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+ : \mathbb{C}^{m(1+n)})$ solves $(\mathrm{CR})_{DB}$ if for all test functions $\varphi \in C_c^\infty(\mathbb{R}^{1+n}_+ : \mathbb{C}^{m(1+n)})$

$$\iint_{\mathbb{R}^{1+n}_+} (F(t,x), \partial_t \varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathbb{R}^{1+n}_+} (F(t,x), B^*(x) D\varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t,$$

and for all $\psi \in C_c^\infty(\mathbb{R}^{1+n}_+ : \mathbb{C}^m)$ and $1 \le j,k \le n$, $j \ne k$,

$$\iint_{\mathbb{R}^{1+n}_+} (F_k(t,x), \partial_j \psi(t,x))\,\mathrm{d}x\,\mathrm{d}t = -\iint_{\mathbb{R}^{1+n}_+} (F_j(t,x), \partial_k \psi(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

The condition $\operatorname{curl}_{\parallel} F_{\parallel} = 0$ in $(\mathrm{CR})_{DB}$ is preserved for all limits, so the Cauchy–Riemann system may be considered as an evolution equation in a restricted space involving a differential structure.

The first-order approach to boundary value problems for elliptic equations $L_A u = 0$ exploits a correspondence between these elliptic equations and Cauchy–Riemann systems $(\mathrm{CR})_{DB}$. Recall that $A \in L^\infty\big(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)})\big)$. Write A in matrix form with respect to the transversal/tangential splitting (1.2) of $\mathbb{C}^{m(1+n)}$ as

$$A = \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix}, \tag{1.8}$$

and using this representation of A define auxiliary matrices

$$\overline{A} := \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ 0 & I \end{bmatrix} \quad \text{and} \quad \underline{A} := \begin{bmatrix} I & 0 \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix}$$

in $L^\infty\big(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)})\big)$. Strict accretivity of A implies that $A_{\perp\perp}$ is invertible in $L^\infty\big(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^m)\big)$, and so $\overline{A}$ is invertible in $L^\infty\big(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{m(1+n)})\big)$. Thus we may define

$$\hat{A} := \underline{A}\,\overline{A}^{-1}.$$

The transformed coefficient matrix $\hat{A}$ satisfies the same assumptions as A, and $\hat{\hat{A}} = A$ [8, Proposition 3.2].

The correspondence between elliptic equations $L_A u = 0$ and Cauchy–Riemann systems $(\mathrm{CR})_{DB}$ is given by the following theorem. See [5, Proposition 4.1; 8, §3; 16, Lemma 7.1; 68, §2] for proofs and discussions.

Theorem 1.5 (Auscher–Axelsson–McIntosh). *Let A be as above, and let $B = \hat{A}$. If u solves $L_A u = 0$, then the A-conormal gradient $\nabla_A u$ solves the Cauchy–Riemann system* $(\mathrm{CR})_{DB}$. *Conversely, if F solves* $(\mathrm{CR})_{DB}$, *then there exists a function u, unique up to an additive constant, such that $L_A u = 0$ and $F = \nabla_A u$.*

The *A-conormal gradient* $\nabla_A u$ of a function $u \colon \mathbb{R}^{1+n}_+ \to \mathbb{C}^m$ is defined by

$$\nabla_A u = \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_\| u \end{bmatrix}, \tag{1.9}$$

where the A-conormal derivative ∂_{ν_A} is defined in (1.5). The components of $\nabla_A u$ appear in the boundary conditions of the Regularity and Neumann problems; hence our preference for Regularity problems over Dirichlet problems.

Thus in our consideration of elliptic equations we may focus instead on Cauchy–Riemann systems. The principal advantage of Cauchy–Riemann systems over elliptic equations is that the Cauchy equation $\partial_t F + DBF = 0$ can be solved by semigroup methods. We sketch how this is done, following [8] and [5].

Consider D as an unbounded operator on $L^2 := L^2(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$ with natural domain, and consider B as a multiplication operator on L^2. Then we have the following theorem [8, Proposition 3.3 and Theorem 3.4]. This is a very deep result: it is part of the framework developed by Axelsson, Keith, and McIntosh [21], which encompasses the solution of the Kato square root problem [10].

Theorem 1.6 (Axelsson–Keith–McIntosh). *The perturbed Dirac operator DB is bisectorial and has bounded H^∞ functional calculus on the closure $\overline{\mathcal{R}(D)} = \overline{\mathcal{R}(DB)}$ of its range.*

These notions are properly discussed in Section 3.1. Using the direct sum decomposition

$$L^2 = \mathcal{N}(DB) \oplus \overline{\mathcal{R}(DB)}$$

(which follows from bisectoriality of DB) and the bounded H^∞ functional calculus of DB on $\overline{\mathcal{R}(DB)}$, we obtain a decomposition

$$L^2 = \mathcal{N}(DB) \oplus \overline{\mathcal{R}(DB)}^+ \oplus \overline{\mathcal{R}(DB)}^-. \tag{1.10}$$

The *positive and negative spectral subspaces* $\overline{\mathcal{R}(DB)}^\pm$ are the images of $\overline{\mathcal{R}(DB)}$ under the projections $\chi^\pm(DB)$, which are defined via the functions $\chi^\pm \colon \mathbb{C} \setminus \mathrm{i}\mathbb{R} \to \{0,1\}$ given by

$$\chi^\pm(z) := \mathbf{1}_{z : \pm \mathrm{Re}(z) > 0}.$$

These are the characteristic functions of the right and left half-plane, restricted to $\mathbb{C} \setminus \mathrm{i}\mathbb{R}$. They are bounded and holomorphic on every bisector, so they fall within the scope of the H^∞ functional calculus.

By way of the H^∞ functional calculus, we may define a generalised *Cauchy operator* $C^+_{DB} \colon \overline{\mathcal{R}(DB)} \to L^\infty(\mathbb{R}_+ : \overline{\mathcal{R}(DB)})$ by

$$(C^+_{DB} f)(t) = \mathrm{e}^{-tDB} \chi^+(DB) f,$$

corresponding to the family of functions $(z \mapsto \mathrm{e}^{-tz}\chi^+(z))$, which are bounded and holomorphic on every bisector. When restricted to the positive spectral subspace $\overline{\mathcal{R}(DB)}^+$, the Cauchy operator C^+_{DB} acts as a strongly continuous semigroup. The Cauchy operator is used in the following classification theorem, which is a combination of parts of [8, Theorem 2.3] and [5, Corollary 8.4].

Theorem 1.7 (Auscher–Axelsson–McIntosh). *If* $f \in \overline{\mathcal{R}(DB)}^+$, *then the Cauchy extension* $C^+_{DB}f$ *solves* $(\mathrm{CR})_{DB}$, *with*

$$\|\widetilde{N}_*(C^+_{DB}f)\|_2 \simeq \|f\|_2 \qquad \textit{and} \qquad \lim_{t\to 0}(C^+_{DB}f)(t,\cdot) = f \quad \textit{in } L^2.$$

Conversely, if F *solves* $(\mathrm{CR})_{DB}$ *and* $\widetilde{N}_*(F) \in L^2$, *then* $F = C^+_{DB}f$ *for a unique* $f \in \overline{\mathcal{R}(DB)}^+$.

Combining this with Theorem 1.5 yields a characterisation of well-posedness of the boundary value problems $(\mathrm{R}_H)^2_{0,A}$ and $(\mathrm{N}_H)^2_{0,A}$. Consider the L^2-Regularity problem $(\mathrm{R}_H)^2_{0,A}$ and let $B = \hat{A}$. A function u solves $L_A u = 0$ with $\widetilde{N}_*(\nabla u) \in L^2$ (∇u and $\nabla_A u$ are interchangeable in this assumption) if and only if $\nabla_A u = C^+_{DB}g$ for some $g \in \overline{\mathcal{R}(DB)}^+$, hence $\nabla_\| u(t,\cdot) = (C^+_{DB}g)(t)_\|$ and $\lim_{t\to 0}\nabla_\| u(t,\cdot) = g_\|$. Therefore $(\mathrm{R}_H)^2_{0,A}$ is well-posed if and only if $g \mapsto g_\|$ is an isomorphism from $\overline{\mathcal{R}(DB)}^+$ to $L^2(\mathbb{R}^n : \mathbb{C}^{mn}) \cap \mathcal{N}(\mathrm{curl}_\|)$. By the same argument, $(\mathrm{N}_H)^2_{0,A}$ is well-posed if and only if $g \mapsto g_\perp$ is an isomorphism from $\overline{\mathcal{R}(DB)}^+$ to $L^2(\mathbb{R}^n : \mathbb{C}^m)$.

By characterising solutions to $(\mathrm{CR})_{DB}$ within various function spaces, we show that well-posedness of corresponding Regularity and Neumann problems is equivalent to the transversal and tangential projections being isomorphisms between certain 'boundary function spaces'. In this section we only described how to handle boundary value problems of order 0 with L^2 boundary data. We need to extend this technique to boundary value problems of more general order, and beyond L^2. In the argument we just described, abstract semigroup theory (accessed via holomorphic functional calculus) did a lot of the work for us. However, once we go beyond L^2, we cannot rely on abstract semigroup theory on general Banach spaces. This would only classify solutions F to $(\mathrm{CR})_{DB}$ such that $F(t)$ is in a fixed function space for all $t > 0$, ruling out consideration of many tent spaces and Z-spaces. Furthermore, abstract semigroup methods do not always allow us to move from initial value problems for $(\mathrm{CR})_{DB}$ to boundary value problems for $L_A u = 0$. On top of these defects we also need to consider quasi-Banach spaces, for which no semigroup theory seems to be available. This makes things difficult.

1.1.4. Adapted function spaces. 'Adapted' Hardy spaces H^p_L, with respect to which some operator L has good properties (such as bounded H^∞ functional calculus), have been developed in various contexts. For example, Hardy spaces of differential forms on Riemannian manifolds were constructed by the second author with McIntosh and Russ [14] (these are adapted to the Hodge-Dirac operator $d+d^*$ on the de Rham complex); Hardy spaces adapted to nonnegative self-adjoint operators satisfying Davies–Gaffney estimates on spaces of homogeneous type were studied by Hofmann, Lu, Mitrea, Mitrea, and Yan [44] (generalising the aforementioned example); Hardy spaces adapted to divergence-form elliptic operators on $\mathbb{R}^n$ were developed by Hofmann and Mayboroda [45] and also McIntosh [46]. Some further developments can be found, for example, in the work of Hytönen, van Neerven, and Portal [51], Jiang and Yang [52], Anh and Li [4], and Duong and Li [32]. This is a very small sample of the work that has been done.

Hardy spaces $\mathbf{H}^p_{DB}$ and Sobolev spaces $\mathbf{W}^p_{-1,DB}$ adapted to perturbed Dirac operators DB were introduced by the second author and Stahlhut [19] (See also Stahlhut's thesis [71], and for a different approach see Frey, McIntosh, and Portal

[35]). These spaces (defined along with more general spaces in Chapter 4) consist of $\mathbb{C}^{m(1+n)}$-valued functions (at least formally); the simplest case is

$$\mathbf{H}^2_D = \mathbf{H}^2_{DB} = \overline{\mathcal{R}(DB)} = \overline{\mathcal{R}(D)} \subset L^2(\mathbb{R}^n : \mathbb{C}^{m(1+n)}).$$

The bounded H^∞ functional calculus of DB on $\mathbf{H}^2_{DB}$ extends to $\mathbf{H}^p_{DB}$ and $\mathbf{W}^p_{-1,DB}$, yielding spectral decompositions

$$\mathbf{H}^p_{DB} = \mathbf{H}^{p,+}_{DB} \oplus \mathbf{H}^{p,-}_{DB}, \quad \mathbf{W}^p_{-1,DB} = \mathbf{W}^{p,+}_{-1,DB} \oplus \mathbf{W}^{p,-}_{-1,DB}$$

analogous to (1.10). Furthermore, the Cauchy operator C^+_{DB} on $\overline{\mathcal{R}(DB)}$ extends to operators on $\mathbf{H}^p_{DB}$ and $\mathbf{W}^p_{-1,DB}$, both of which we denote by $\mathbf{C}^+_{DB}$.

The main application of these spaces, which incorporates results from work of the second author with both Stahlhut [19] and Mourgoglou [16], is a classification of solutions to the Cauchy–Riemann system $(\mathrm{CR})_{DB}$ with L^p-type interior estimates, for p such that certain DB-adapted spaces may be identified with D-adapted spaces. For simplicity we only state results for $1 < p < \infty$ in this introduction; corresponding results for $p \leq 1$ and '$p \geq \infty$' (i.e., for boundary data in BMO-type and Hölder spaces) are also available, but these may not be stated so simply in terms of Hölder conjugate exponents.

Theorem 1.8 (Auscher–Mourgoglou–Stahlhut). *Let* $1 < p < \infty$.

(i) *Assume* $\mathbf{H}^p_{DB} \simeq \mathbf{H}^p_D$. *If* $f \in \mathbf{H}^{p,+}_{DB}$, *then* $\mathbf{C}^+_{DB} f$ *solves* $(\mathrm{CR})_{DB}$, *with*

$$\|\widetilde{N}_*(\mathbf{C}^+_{DB} f)\|_p \simeq \|f\|_{H^p} \qquad \text{and} \qquad \lim_{t\to 0} \mathbf{C}^+_{DB} f(t) = f \quad \text{in } H^p.$$

Conversely, if F *solves* $(\mathrm{CR})_{DB}$ *and* $\widetilde{N}_* F \in L^p$, *then* $F = \mathbf{C}^+_{DB} f$ *for a unique* $f \in \mathbf{H}^{p,+}_{DB}$.

(ii) *Assume* $\mathbf{H}^{p'}_{DB^*} \simeq \mathbf{H}^{p'}_D$. *If* $f \in \mathbf{W}^{p,+}_{-1,DB}$, *then* $\mathbf{C}^+_{DB} f$ *solves* $(\mathrm{CR})_{DB}$, *with*

$$\|\mathbf{C}^+_{DB} f\|_{T^p_{-1}} \simeq \|f\|_{\dot{W}^p_{-1}} \qquad \text{and} \qquad \lim_{t\to 0} \mathbf{C}^+_{DB} f(t) = f \quad \text{in } \dot{W}^p_{-1}.$$

Conversely, if $F \in T^p_{-1}$ *solves* $(\mathrm{CR})_{DB}$ *and* $\lim_{t\to\infty} F(t)_\parallel = 0$ *in* $\mathcal{Z}'(\mathbb{R}^n)$, *then* $F = \mathbf{C}^+_{DB} f$ *for a unique* $f \in \mathbf{W}^{p,+}_{-1,DB}$.

Furthermore, it is shown that for every B there exists an open interval containing 2, which we denote $I_0(\mathbf{H}, DB)$, such that $\mathbf{H}^p_{DB} \simeq \mathbf{H}^p_D$ for all $p \in I_0(\mathbf{H}, DB)$ [19, Theorem 5.1]. Thus there is a nontrivial range of exponents for which Theorem 1.8 applies.

Remark 1.9. The assumption in part (ii) of the theorem is given in terms of the conjugate exponent p' and the adjoint coefficients B^*. As part of our theory we show—as a corollary of what will eventually be termed '♡-duality'—that this is equivalent to $\mathbf{W}^p_{-1,DB} \simeq \mathbf{W}^p_{-1,D}$, thus unifying the assumptions of parts (i) and (ii).

As we described in the case where $p = 2$, Theorem 1.8 implies a characterisation of well-posedness of various Regularity and Neumann problems, both of order 0 and order -1, in terms of certain transversal and tangential projections being isomorphisms. We will not explicitly state this characterisation now; instead, we state our extension of this result in Theorem 1.11.

The technical heart of this monograph is an extension of Theorem 1.8 to fractional order $\theta \in (-1, 0)$, incorporating both Hardy–Sobolev spaces and Besov spaces. To this end, we introduce Hardy–Sobolev spaces $\mathbf{H}^p_{\theta,L}$ and Besov spaces $\mathbf{B}^p_{\theta,L}$ adapted to operators L satisfying 'Standard Assumptions,' which are satisfied in particular by the perturbed Dirac operators DB and BD. We define extension operators

$$(\mathbb{Q}_{\varphi,L}f)(t) = \varphi(tL)f \quad (t > 0,\ f \in \overline{\mathcal{R}(L)})$$

for appropriate holomorphic functions φ, and the adapted Hardy–Sobolev and Besov norms are then, roughly speaking, defined by

$$\|f\|_{\mathbf{H}^p_{\theta,L}} := \|\mathbb{Q}_{\varphi,L}f\|_{T^p_\theta}, \quad \|f\|_{\mathbf{B}^p_{\theta,L}} := \|\mathbb{Q}_{\varphi,L}f\|_{Z^p_\theta}.$$

These definitions are reminiscent of the φ-transform characterisations of Triebel–Lizorkin and Besov spaces due to Frazier and Jawerth [34] (the letter φ has a different meaning there), with functional calculus and tent/Z-spaces replacing discretised Littlewood–Paley decompositions and sequence spaces respectively.

Chapters 3 to 5 are occupied with the construction of a sufficiently rich general theory of adapted Besov–Hardy–Sobolev spaces. The theory is relatively straightforward once enough preliminaries have been collected, but getting to this stage takes some time. We emphasise in particular the amount of work needed to quantify independence on φ of the spaces $\mathbf{H}^p_{\theta,L}$ and $\mathbf{B}^p_{\theta,L}$ (essentially all of Sections 3.3 and 3.4) and the care which must be taken in discussing completions (Section 4.3), which is necessary to discuss and fully exploit interpolation.

1.1.5. Classification of solutions to CR systems, and applications to well-posedness. Our main theorem is the following classification of solutions to the Cauchy–Riemann system $(\mathrm{CR})_{DB}$. In this statement we restrict ourselves to $1 < p < \infty$. As with our statement of Theorem 1.8, this is a simplification of the full result (Theorems 6.8 and 6.9): our theorem also allows for $p \leq 1$ and '$p \geq \infty$,' but the corresponding results are better stated in terms of the 'exponent notation' that we introduce in Section 2.1.

Theorem 1.10. *Let $-1 < \theta < 0$ and $1 < p < \infty$.*

(i) *Suppose that $\mathbf{H}^p_{\theta,DB} \simeq \mathbf{H}^p_{\theta,D}$. If $f \in \mathbf{H}^{p,+}_{\theta,DB}$, then $\mathbf{C}^+_{DB}f$ solves $(\mathrm{CR})_{DB}$, with*

$$\|\mathbf{C}^+_{DB}f\|_{T^p_\theta} \simeq \|f\|_{\dot{H}^p_\theta} \qquad \text{and} \qquad \lim_{t\to 0} \mathbf{C}^+_{DB}f(t) = f \quad \text{in } \dot{H}^p_\theta,$$

and furthermore $\lim_{t\to\infty} \mathbf{C}^+_{DB}f(t)_\| = 0$ in $\mathcal{Z}'(\mathbb{R}^n)$. Conversely, if $F \in T^p_\theta$ solves $(\mathrm{CR})_{DB}$ and $\lim_{t\to\infty} F(t)_\| = 0$ in $\mathcal{Z}'(\mathbb{R}^n)$, then $F = \mathbf{C}^+_{DB}f$ for a unique $f \in \mathbf{H}^{p,+}_{\theta,DB}$.

(ii) *Suppose that $\mathbf{B}^p_{\theta,DB} \simeq \mathbf{B}^p_{\theta,D}$. If $f \in \mathbf{B}^{p,+}_{\theta,DB}$, then $\mathbf{C}^+_{DB}f$ solves $(\mathrm{CR})_{DB}$, with*

$$\|\mathbf{C}^+_{DB}f\|_{Z^p_\theta} \simeq \|f\|_{\dot{B}^{p,p}_\theta} \qquad \text{and} \qquad \lim_{t\to 0} \mathbf{C}^+_{DB}f(t) = f \quad \text{in } \dot{B}^{p,p}_\theta,$$

and furthermore $\lim_{t\to\infty} \mathbf{C}^+_{DB}f(t)_\| = 0$ in $\mathcal{Z}'(\mathbb{R}^n)$. Conversely, if $F \in Z^p_\theta$ solves $(\mathrm{CR})_{DB}$ and $\lim_{t\to\infty} F(t)_\| = 0$ in $\mathcal{Z}'(\mathbb{R}^n)$, then $F = \mathbf{C}^+_{DB}f$ for a unique $f \in \mathbf{B}^{p,+}_{\theta,DB}$.

Parts (i) and (ii) of this theorem are essentially identical, the only modifications being the replacement of (adapted) Hardy–Sobolev spaces with (adapted) Besov spaces, and of tent spaces with Z-spaces. In fact, our arguments apply equally well to both parts, so we prove them simultaneously. Although the theorem can be thought of as 'intermediate to' Theorem 1.8, it does not follow by any interpolation argument. It is proven similarly, but the underlying techniques must be generalised, and this takes a considerable amount of work. Neither direction is easy, but the 'converse' direction—finding a 'trace' f given a solution F—is certainly more difficult.

A consequence of this theorem (and of the abstract theory that we construct) is that certain solution spaces for the equation $L_A u = 0$ form interpolation scales (Section 6.5). This provides another viewpoint on the structure of the solution spaces, although it is weaker than the explicit description that we obtain. Another consequence is a representation theorem for solutions u of $L_A u = 0$, rather than for their conormal gradients $\nabla_A u$ (which solve the Cauchy–Riemann system $(\mathrm{CR})_{DB}$). This result is much more complicated to state; we refer the reader to Theorem 6.13.

Identifying regions of exponents where Theorem 1.10 applies is important. These are regions in which we can identify DB-adapted spaces with D-adapted spaces, so we call them *identification regions.* Starting from information on the intervals $I_0(\mathbf{H}, DB), I_0(\mathbf{H}, DB^*) \ni 2$ (as given by the second author and Stahlhut), a procedure of '♡-duality' and interpolation yields nontrivial regions of exponents (p, θ) for which Theorem 1.10 applies (Section 5.3).

With Theorem 1.10 as a springboard, we extend the characterisation of well-posedness of Regularity and Neumann problems—described for $p = 2$ after the statement of Theorem 1.5 and then extended to $p \neq 2$ and $\theta \in \{-1, 0\}$ by the second author with Mourgoglou and Stahlhut—as follows. For all exponents (p, θ), we show that $\mathbf{H}^p_{\theta,D}$ is equal to the set of those $f \in \dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^{m(1+n)})$ with $\mathrm{curl}_\parallel f_\parallel = 0$. Let $N_\perp$ and $N_\parallel$ denote the projections from $\mathbf{H}^p_{\theta,D}$ onto $\mathbf{H}^p_{\theta,\perp} := \dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^m)$ and $\mathbf{H}^p_{\theta,\parallel} := \dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^{mn}) \cap \mathcal{N}(\mathrm{curl}_\parallel)$ respectively. For (p, θ) as in Theorem 1.10 we have an identification of $\mathbf{H}^{p,+}_{\theta,DB}$ as a subset of $\mathbf{H}^p_{\theta,D}$, and so we can use $N_\perp$ and $N_\parallel$ to define

$$N^{(p,\theta)}_{H,DB,\parallel} \colon \mathbf{H}^{p,+}_{\theta,DB} \to \mathbf{H}^p_{\theta,\parallel} \quad \text{and} \quad N^{(p,\theta)}_{H,DB,\perp} \colon \mathbf{H}^{p,+}_{\theta,DB} \to \mathbf{H}^p_{\theta,\perp}.$$

Corresponding definitions of $N^{(p,\theta)}_{B,DB,\parallel}$ and $N^{(p,\theta)}_{B,DB,\perp}$ are also made for Besov spaces. It is important to understand that we can define these restrictions on DB-adapted spaces only after we have identified them with D-adapted spaces, on which the projections are initially defined (the only exception to this is when B has a block structure). These operators carry the well-posedness of Regularity and Neumann problems, as shown by the following theorem. Again, this is a simplification of the full result (Theorem 7.5). The $\theta \in \{-1, 0\}$ endpoints follow from Theorem 1.8.

Theorem 1.11. *Let $B = \hat{A}$, $-1 \leq \theta \leq 0$, and $1 < p < \infty$. Suppose that $\mathbf{H}^p_{\theta,DB} \simeq \mathbf{H}^p_{\theta,D}$. Then $(\mathrm{R}_H)^p_{\theta,A}$ (resp. $(\mathrm{N}_H)^p_{\theta,A}$) is well-posed if and only if $N^{(p,\theta)}_{H,DB,\parallel}$ (resp. $N^{(p,\theta)}_{H,DB,\perp}$) is an isomorphism. The same results hold* mutatis mutandis *for problems with Besov boundary data.*

The notion of well-posedness can be refined when considering boundary value problems with different exponents. Consider $-1 \leq \theta_0, \theta_1 \leq 0$ and $1 < p_0, p_1 < \infty$. We say that the problems $(\mathrm{R}_H)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_H)^{p_1}_{\theta_1,A}$ are *mutually well-posed* if they are both well-posed, and if for all $\nabla_\parallel f \in \dot{H}^{p_0}_{\theta_0} \cap \dot{H}^{p_1}_{\theta_1}$ the unique solutions to $(\mathrm{R}_H)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_H)^{p_1}_{\theta_1,A}$ with boundary data $\nabla_\parallel f$ are equal. This definition simply extends to all the boundary value problems that we consider. Two well-posed boundary value problems need not be mutually well-posed: this phenomenon was first observed by Axelsson [20]. The concept of mutual well-posedness extends the notion of compatible well-posedness introduced by Barton and Mayboroda [22, §2.4]. More precisely, mutual well-posedness defines an equivalence relation on the set of exponents (p, θ) for which a problem (e.g. $(\mathrm{R}_H)^p_{\theta,A}$) is well-posed, and compatible well-posedness corresponds to the equivalence class of the 'energy exponent' $(2, -\frac{1}{2})$. The problems $(\mathrm{R}_H)^2_{-1/2,A}$ and $(\mathrm{N}_H)^2_{-1/2,A}$ are always well-posed [13, Theorems 3.2 and 3.3].

By Theorem 1.11, $(\mathrm{R}_H)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_H)^{p_1}_{\theta_1,A}$ are mutually well-posed if and only if $N^{(p_0,\theta_0)}_{H,DB,\|}$ and $N^{(p_1,\theta_1)}_{H,DB,\|}$ are isomorphisms whose inverses are equal on the intersection $\mathbf{H}^{p_0}_{\theta_0,\|} \cap \mathbf{H}^{p_1}_{\theta_1,\|}$ (and likewise for Neumann problems, and with Besov boundary data). This allows us to interpolate mutual well-posedness as a straightforward corollary of Theorem 1.11. Furthermore, by using real interpolation instead of complex interpolation, we can deduce mutual well-posedness of boundary value problems with Besov boundary data from that of those with Hardy–Sobolev boundary data. The following theorem makes this precise. The full result is Theorem 7.7.

Theorem 1.12. *Suppose $-1 \leq \theta_0, \theta_1 \leq 0$, $1 < p_0, p_1 < \infty$, and $\alpha \in (0,1)$, and let*

$$\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1} \quad \text{and} \quad \theta = (1-\alpha)\theta_0 + \alpha\theta_1.$$

(i) *If $\mathbf{H}^{p_j}_{\theta_j,DB} \simeq \mathbf{H}^{p_j}_{\theta_j,D}$ for $j \in \{0,1\}$, and if $(\mathrm{R}_H)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_H)^{p_1}_{\theta_1,A}$ are mutually well-posed, then $(\mathrm{R}_H)^{p}_{\theta,A}$ is mutually well-posed with both $(\mathrm{R}_H)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_H)^{p_1}_{\theta_1,A}$, and furthermore if $\theta_0 \neq \theta_1$ then $(\mathrm{R}_B)^{p}_{\theta,A}$ is mutually well-posed with both $(\mathrm{R}_H)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_H)^{p_1}_{\theta_1,A}$.*

(ii) *If $\mathbf{B}^{p_j}_{\theta_j,DB} \simeq \mathbf{B}^{p_j}_{\theta_j,D}$ for $j \in \{0,1\}$, and if $(\mathrm{R}_B)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_B)^{p_1}_{\theta_1,A}$ are mutually well-posed, then $(\mathrm{R}_B)^{p}_{\theta,A}$ is mutually well-posed with both $(\mathrm{R}_B)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_B)^{p_1}_{\theta_1,A}$.*

Corresponding results are also true for Neumann problems.

Since invertibility is stable in complex interpolation scales, well-posedness of our boundary value problems is also stable in a way that preserves mutuality. This is made precise in the following simplification of Theorem 7.8.

Theorem 1.13. *Let $-1 < \theta < 0$ and $1 < p < \infty$, and suppose that $\mathbf{H}^p_{\theta,DB} = \mathbf{H}^p_{\theta,D}$. Suppose also that $(\mathrm{R}_H)^p_{\theta,A}$ is well-posed. Then $(\mathrm{R}_H)^{p_1}_{\theta_1,A}$ and $(\mathrm{R}_H)^p_{\theta,A}$ are mutually well-posed for all (p_1, θ_1) in some neighbourhood of (p, θ). Similar results hold for Neumann problems, and for problems with Besov boundary data.*

This theorem does not apply when $\theta \in \{-1, 0\}$. However, the result is true if $\theta \in \{-1, 0\}$ and $\theta_1 = \theta$, with p_1 in a neighbourhood of p. This is implicit in the work of the second author with Mourgoglou and Stahlhut, and the proof of Theorem 1.13 still applies in this case.

Finally, we have a duality result for well-posedness (this is a simplification of Theorem 7.11).

Theorem 1.14. *Let $-1 \leq \theta_0, \theta_1 \leq 0$ and $1 < p_0, p_1 < \infty$, and suppose that $\mathbf{H}^{p_0}_{\theta_0,DB} \simeq \mathbf{H}^{p_0}_{\theta_0,D}$ and $\mathbf{H}^{p_1}_{\theta_1,DB} \simeq \mathbf{H}^{p_1}_{\theta_1,D}$. Then $(\mathrm{R}_H)^{p_0}_{\theta_0,A}$ and $(\mathrm{R}_H)^{p_1}_{\theta_1,A}$ are mutually well-posed if and only if $(\mathrm{R}_H)^{p_0'}_{-\theta_0-1,A^*}$ and $(\mathrm{R}_H)^{p_1'}_{-\theta_1-1,A^*}$ are mutually well-posed. Similar results hold for Neumann problems and with Besov spaces.*

We may take $(p_0, \theta_0) = (p_1, \theta_1)$ in this result. The mapping $(p, \theta) \mapsto (p', -\theta - 1)$ is the reflection about the point $(\frac{1}{2}, -\frac{1}{2})$ in the $(1/p, \theta)$-plane, which corresponds to the aforementioned 'energy exponent'. This reflection describes what we call '$\heartsuit$-duality of exponents'.

These theorems can be used to derive new well-posedness results for Regularity problems $(\mathrm{R}_H)^p_{\theta,A}$ with fractional order $\theta \in (-1, 0)$, and also to derive known results for $(\mathrm{R}_B)^p_{\theta,A}$ which were recently obtained by different methods by Barton and Mayboroda [22] under the De Giorgi–Nash–Moser assumption. For details see Section 7.2.

Remark 1.15. All of our results can be reformulated with the lower half-space replacing the upper half-space. The main difference is that in this case positive spectral subspaces must be replaced with negative spectral subspaces.

1.2. Summary of the monograph

In Chapter 2 we discuss two types of function spaces. First, the 'ambient spaces': tent spaces, Z-spaces, and slice spaces. Many of the results here are new, or have not been used in this context. We then consider the 'smoothness spaces': Hardy–Sobolev spaces, Besov spaces, and so on. After a quick review of these spaces, we characterise them in terms of tent spaces and Z-spaces (Theorem 2.57). We also introduce a new system of notation for exponents: these are written as boldface letters, typically $\mathbf{p}$ and $\mathbf{q}$, and encode both integrability and regularity information. This is not strictly necessary, but it truly cleans up the exposition of later parts of the monograph and makes the flow of ideas more transparent.

In Chapter 3 we discuss basic operator theoretic notions. The operators that we use in applications (i.e., the perturbed Dirac operators DB and BD) are bisectorial, with bounded H^∞ functional calculi on their ranges, and satisfy certain off-diagonal estimates. Most of our abstract theory applies to any operator A satisfying these 'Standard Assumptions,' so we work with such operators until we are forced to use more specific properties of perturbed Dirac operators. We establish the boundedness of certain integral operators between tent spaces and Z-spaces. Particular examples of these operators are given in terms of 'extension' and 'contraction' operators $\mathbb{Q}_{\varphi,A}$ and $\mathbb{S}_{\psi,A}$, which we discuss. This chapter culminates in Theorem 3.19, which quantifies when operators of the form $\mathbb{Q}_{\psi,A}\eta(A)\mathbb{S}_{\varphi,A}$ are bounded between different tent/Z-spaces, where η is a holomorphic function (not necessarily bounded) on an appropriate bisector.

In Chapter 4 we consider Besov–Hardy–Sobolev spaces adapted to an operator A satisfying the aforementioned Standard Assumptions. In Section 4.1 we introduce 'pre'-Besov–Hardy–Sobolev spaces $\mathbb{H}^{\mathbf{p}}_A$ and $\mathbb{B}^{\mathbf{p}}_A$ and establish their basic properties. Mapping properties of the holomorphic functional calculus between these spaces, including boundedness for H^∞ functions of A and 'regularity shifting' estimates for operators such as powers of A, are collected in Section 4.2. These all follow from Theorem 3.19. In Section 4.3 we discuss completions. This issue is more subtle than it initially seems. We define 'canonical completions' $\psi\mathbf{H}^{\mathbf{p}}_A$ and $\psi\mathbf{B}^{\mathbf{p}}_A$ in terms of auxiliary functions ψ, and show how these can be used to formulate satisfactory duality and interpolation results (Proposition 4.23 and Theorem 4.28). We also introduce 'inclusion regions' of exponents $\mathbf{p}$ such that $\mathbb{H}^{\mathbf{p}}_{A_0} \hookrightarrow \mathbb{H}^{\mathbf{p}}_{A_1}$ (likewise for Besov spaces) for two operators A_0, A_1 satisfying the Standard Assumptions with

$\overline{\mathcal{R}(A_0)} = \overline{\mathcal{R}(A_1)}$. An interpolation result for these regions (Theorem 4.32) is proven. Finally, in Section 4.4 we show that the extended Cauchy operator $\mathbf{C}_A$ produces strong solutions of the Cauchy problem for A with initial data in any completion of any adapted pre-Besov–Hardy–Sobolev space, and we also show the quasinorm equivalence

$$\|f\|_{\mathbb{H}^{\mathbf{p}}_A} \simeq \|C_A f\|_{T^{\mathbf{p}}} \quad (f \in \mathbb{H}^{\mathbf{p},\pm}_A) \tag{1.11}$$

when $\mathbf{p} = (p, s)$ with $p \leq 2$ and $s < 0$, and likewise for Besov spaces and Z-spaces (Corollary 4.36).

Up until this point, we work with $\mathbb{C}^N$-valued functions for an arbitrary $N \in \mathbb{N}$, as in this abstract setting we gain nothing from the transversal/tangential structure of $\mathbb{C}^{m(1+n)}$.

In Chapter 5 we consider the case when A is a perturbed Dirac operator of the form DB or BD (and so we finally specialise to $\mathbb{C}^{m(1+n)}$-valued functions). We show that for all exponents $\mathbf{p}$ the spaces $\mathbb{H}^{\mathbf{p}}_D$ and $\mathbb{B}^{\mathbf{p}}_D$ are equal to projections of classical smoothness spaces intersected with L^2 (Theorem 5.3), and so we may take projections of these classical smoothness spaces (without intersecting with L^2) as completions. We denote the resulting spaces by $\mathbf{H}^{\mathbf{p}}_D$ and $\mathbf{B}^{\mathbf{p}}_D$. Then we define 'identification regions' $I(\mathbf{H}, DB)$ and $I(\mathbf{B}, DB)$, consisting of exponents $\mathbf{p}$ for which we can identify $\mathbf{H}^{\mathbf{p}}_D$ and $\mathbf{B}^{\mathbf{p}}_D$ as completions of $\mathbb{H}^{\mathbf{p}}_{DB}$ and $\mathbb{B}^{\mathbf{p}}_{DB}$ respectively. These regions turn out to be open (with some minor restrictions; see Theorem 5.18) and stable under interpolation and $\heartsuit$-duality (in a sense which interchanges B and B^*; Corollary 5.14). Finally, in Theorem 5.26 we show that for $\mathbf{p} = (p, s) \in I(\mathbf{H}, DB)$ with $p > 2$ and $s < 0$ we have boundedness of the Cauchy operator C^+_{DB} from $\mathbb{H}^{\mathbf{p}}_{DB}$ to $T^{\mathbf{p}}$, extending the 'abstract' estimate (1.11) (and likewise for Besov spaces and Z-spaces). This is a long argument which requires various ad-hoc estimates. The result is known to fail for $s = 0$, so it does not follow by interpolation.

In Chapter 6 we turn our attention to differential equations. After presenting some basic properties of gradients of solutions to $L_A u = 0$ (or equivalently solutions of $(\mathrm{CR})_{DB}$) we prove Theorems 6.8 and 6.9, which classify solutions to $(\mathrm{CR})_{DB}$ in tent/Z-spaces with a decay condition at infinity (the decay condition is removed for certain exponents in Section 6.2). This leads us to a range of exponents, related to the identification region, called the *classification region*. The argument is quite long, particularly for exponents $\mathbf{p} = (p, s)$ with $p > 2$, and uses all of the preceding material. We have been (perhaps excessively) pedantic in citing dependence on previous results, so it should be possible to treat certain technical lemmas as 'black boxes' in initial readings. Although these results are 'intermediate to' Theorem 1.8 and proven by similar arguments, they do not follow by any interpolation procedure. The results must be reproven manually. As a corollary of Theorems 6.8 and 6.9 we prove that certain solution spaces for the equation $L_A u = 0$ are interpolation scales (Theorem 6.41), and that the Whitney averages of such solutions have nontangential boundary limits (Theorem 6.42).

In Chapter 7 we present applications to boundary value problems. Most of these have already been summarised in the introduction (Subsection 1.1.5). In particular, we derive ranges of well-posedness for Regularity and Neumann problems for various classes of coefficients in Section 7.2. Our results for real coefficient scalar equations with boundary data in Hardy–Sobolev spaces are new. In Section 7.3 we show that well-posedness of a boundary value problem is stable under perturbation of the coefficients, given certain a priori assumptions which are known to hold in some cases. Finally, we show the relationship between our approach and the method of layer potentials in Section 7.4. For exponents in the classification region, all solutions to boundary value problems with gradients in the corresponding tent/Z-space and with the appropriate decay condition are given by (generalised) layer potentials.

1.3. Notation

The following notation will be used throughout the monograph.

We let $\mathbb{N} := \{0, 1, 2, \dots\}$ denote the natural numbers (including 0), and $\mathbb{N}_+ := \{1, 2, \dots\}$ denote the positive natural numbers.

For $a, b \in \mathbb{R}$ and $t > 0$ we write

$$m_a^b(t) := \begin{cases} t^a & (t \leq 1) \\ t^{-b} & (t \geq 1). \end{cases}$$

For $0 < p, q \leq \infty$, we define the number

$$\delta_{p,q} := \frac{1}{q} - \frac{1}{p},$$

with the interpretation $1/\infty = 0$.

We write the Euclidean distance on $\mathbb{R}^n$ as $d(x, y) = d(y, x) := |x - y|$, the open ball with centre $x \in \mathbb{R}^n$ and radius $r > 0$ by $B(x, r) := \{y \in \mathbb{R}^n : d(x, y) < r\}$, and the (half closed, half open) annulus with centre $x \in \mathbb{R}^n$, inner radius $r_0 > 0$, and outer radius $r_1 > r_0$ by

$$A(x, r_0, r_1) := B(x, r_1) \setminus B(x, r_0) = \{y \in \mathbb{R}^n : r_0 \leq d(x, y) < r_1\}.$$

For subsets $E, F \subset \mathbb{R}^n$ we write

$$d(E, F) := \operatorname{dist}(E, F) = \inf\{d(x, y) : x \in E, y \in F\}.$$

We let $L^0(\Omega : E)$ denote the set of strongly measurable functions from a measure space Ω to a Banach space E. As usual, we identify two functions if they agree almost everywhere.

We write quasinorms as either $\|f\|_X$ or $\|f \mid X\|$ according to typographical need. For two quasinormed spaces X and Y, we write $X \hookrightarrow Y$ to mean that $X \subset Y$ (possibly after some identification has been made) and that the identity map is bounded. We write $X \simeq Y$ to mean that $X \hookrightarrow Y$ and $Y \hookrightarrow X$, i.e., to mean that X and Y are isomorphic, and $X = Y$ to mean that the sets X and Y are equal and that the associated quasinorms are equivalent. Often we refer to norms as 'quasinorms' even though they are actually norms; for example, we refer to the L^p quasinorm when $p \in (0, \infty]$, even though this is a norm when $p \geq 1$. For a quick introduction to quasi-Banach spaces the reader can consult the early sections of [54]. When necessary, we label dual pairings by the space on the left: for example, by $\langle f, g \rangle_{L^p}$, we mean the usual duality pairing between L^p and $L^{p'}$, with $f \in L^p$ and $g \in L^{p'}$.

When a and b are real numbers, we write $a \lesssim b$ to mean that $a \leq Cb$ for some $C > 0$ which is independent of a and b, and which may vary from line to line. If C depends on some other quantities $c_1, c_2, \dots$, we write $a \lesssim_{c_1, c_2, \dots} b$. We write $a \simeq b$ to mean that $a \lesssim b$ and $b \lesssim a$. Context prevents us from confusing this meaning of the symbol $\simeq$ and that introduced in the previous paragraph (representing isomorphism of quasinormed spaces).

We use a new and extensive notation for exponents, which appear in boldface $\mathbf{p}$. This is described in Section 2.1.

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