Write

$$N^{\mathbf{p}}_{\mathbf{X},DB,\bullet} = N_\bullet \chi^+(DB_0)\chi^+(DB) + N_\bullet(\chi^+(DB) - \chi^+(DB_0)).$$

Note that

$$\begin{aligned}\chi^+(DB_0)\chi^+(DB) &= (\chi^+(DB_0) - \chi^+(DB) + \chi^+(DB))\chi^+(DB)\\ &= I - (\chi^+(DB) - \chi^+(DB_0))\end{aligned}$$

on $\mathbf{X}^{\mathbf{p},+}_{DB}$, so by the estimate (7.3), $\chi^+(DB_0)\chi^+(DB) \colon \mathbf{X}^{\mathbf{p},+}_{DB} \to \mathbf{X}^{\mathbf{p},+}_{DB_0}$ is invertible by a Neumann series if $\|B - B_0\|_\infty$ is sufficiently small. Since $N^{\mathbf{p}}_{\mathbf{X},DB_0,\bullet} \colon \mathbf{X}^{\mathbf{p},+}_{DB_0} \to \mathbf{X}^{\mathbf{p}}_\bullet$ is an isomorphism (by Theorem 7.5 and the assumptions), the composition

$$M \coloneqq N_\bullet \chi^+(DB_0)\chi^+(DB) = N^{\mathbf{p}}_{\mathbf{X},DB_0,\bullet}\chi^+(DB_0)\chi^+(DB) \colon \mathbf{X}^{\mathbf{p},+}_{DB} \to \mathbf{X}^{\mathbf{p}}_\bullet$$

is also an isomorphism. One can then write

$$N^{\mathbf{p}}_{\mathbf{X},DB,\bullet} = M\Big(I + M^{-1}N_\bullet\big(\chi^+(DB) - \chi^+(DB_0)\big)\Big)$$

and note that the second factor is also invertible by a Neumann series if $\|B - B_0\|_\infty$ is sufficiently small. Hence $N^{\mathbf{p}}_{\mathbf{X},DB,\bullet}$ is an isomorphism, and we are done. □

Remark 7.18. The proof above shows that the inverse of $N^{\mathbf{p}}_{\mathbf{X},DB,\bullet}$ is given by an expression coming from multiple Neumann series involving the operators $(N^{\mathbf{p}}_{\mathbf{X},DB_0,\bullet})^{-1}$ and $\chi^+(DB) - \chi^+(DB_0)$. Therefore, if the assumptions of Theorem 7.16 are satisfied for two exponents $\mathbf{p}, \mathbf{q}$, and if $(P_{\mathbf{X}})^{\mathbf{p}}_{A_0}$ and $(P_{\mathbf{X}})^{\mathbf{q}}_{A_0}$ are mutually well-posed, it follows that $(P_{\mathbf{X}})^{\mathbf{p}}_{A}$ and $(P_{\mathbf{X}})^{\mathbf{q}}_{A}$ remain mutually well-posed for $\|A - A_0\|_\infty$ sufficiently small.

Remark 7.19. There are three situations in which we can guarantee that the hypotheses of Theorem 7.16 are satisfied.

First, for any coefficients A, the hypotheses are satisfied provided that $\mathbf{p} \in I_{\min}$ (see Theorem 5.17; in the Besov space case we also require $\theta(\mathbf{p}) \in (-1, 0)$). The only thing which is not obvious here is uniformity of implicit constants: when $\theta(\mathbf{p}) = 0$ this is contained in the proof of [19, Proposition 7.1], and the general case follows by ♡-duality and interpolation (as in the derivation of the region $I_{\min}$).

Second: whenever $n = 1$ the hypotheses are satisfied for all coefficients and for all $\mathbf{p} \in I_{\max}$ (with $\theta(\mathbf{p}) \in (-1, 0)$ in the Besov space case): this follows from [19, Proposition 3.11 and Theorem 5.1], ♡-duality, and interpolation.

Finally, write A in the transversal/tangential decomposition as in (1.8), and suppose that $A^*_{\parallel\parallel}$ satisfies the De Giorgi–Nash–Moser condition of exponent α in dimension n (see (7.2)). It is shown in [19, §13] that the hypotheses of Theorem 7.16 are satisfied for the exponent $(p, 0)$ whenever $p \in (n/(n + \alpha), p_+(DB))$, where $p_+(DB) > 2$. If in addition $A_{\parallel\parallel}$ satisfies the De Giorgi–Nash–Moser condition of exponent α, then it follows by ♡-duality and interpolation that the hypotheses of Theorem 7.16 hold for all $\mathbf{p}$ in the region pictured in Figure 7.2 (in the Besov space case, we again require $\theta(\mathbf{p}) \in (-1, 0)$).

7.4. The method of layer potentials

We conclude the monograph by explaining how the first-order approach relates to the method of layer potentials. In solving boundary value problems for L_A, it has been a standing question whether solutions constructed by different methods agree, and in particular whether solutions are always given by layer potentials when their defining integrals exist. A consequence of Rosén's identification of the layer potentials in terms of Cauchy operators for DB having assumed the De Giorgi–Nash–Moser condition on A and A^*, is that solutions constructed via Cauchy operators coincide with solutions constructed by the method of layer potentials, and satisfy a layer potential representation

$$u(t,x) = \mathcal{S}_t(\partial_{\nu_A} u|_{t=0})(x) - \mathcal{D}_t(u|_{t=0})(x) \tag{7.4}$$

in some appropriate sense (modulo constants) [68]. At the time, Rosén's identification applied when $X^{\mathbf{p}} = L^2_\theta$ for $-1 \le \theta \le 0$. As a consequence of our results, we obtain a third representation theorem for exponents in the classification region. We do not yet know what happens outside the classification region.

Theorem 7.20 (Layer potential representation). *For* $\mathbf{p} \in J(\mathbf{X}, DB)$, *any solution* u *of* $L_A u = 0$ *with the interior control* $\nabla u \in \widetilde{X^{\mathbf{p}}}$ *is given by the layer potential representation* (7.4) *modulo constants.*

We shall prove this theorem (and more) later in the section. In our general setting the operators in (7.4) are generalised layer potential operators which we define below. When A and A^* satisfy the De Giorgi–Nash–Moser condition, these are the more familiar integral operators that we review next. We will also explain what we mean by 'the method of layer potentials'. This method has a long history in the case of the Laplace equation, or of equations with smooth coefficients. In the case of elliptic equations $L_A u = 0$ with non-smooth coefficients, solvability by means of layer potentials was first successfully developed in [2]. See the references therein for historical background.

Suppose, for the moment, that A and A^* both satisfy the De Giorgi–Nash–Moser condition (7.2) of some exponent. Then for all $(t,x) \in \mathbb{R}^{1+n}$ there exists a fundamental solution $\Gamma_{(t,x)}$ for L_{A^*} in $\mathbb{R}^{1+n}$ with pole at (t,x).[9] The fundamental solution $\widetilde{\Gamma}_{(t,x)} = (\widetilde{\Gamma}^{\beta,\alpha}_{(t,x)})$ is a $\mathbb{C}^{m\times m}$-valued locally integrable function on $\mathbb{R}^{1+n}$ satisfying

$$\operatorname{div} A^* \nabla \widetilde{\Gamma}_{(t,x)} = \delta_{(t,x)} \mathbf{1} \quad \text{in } \mathbb{R}^{1+n}$$

in the usual weak sense, where $\delta_{(t,x)}$ is the Dirac mass at (t,x) and $\mathbf{1}$ is the identity matrix. The assumption on A^* guarantees existence and uniqueness of fundamental solutions, with various decay estimates. A formal application of Green's formula tells us that

$$u^\alpha(t,x) = \sum_{\beta=1}^m \int_{\mathbb{R}^n} \overline{\widetilde{\Gamma}^{\beta,\alpha}_{(t,x)}(0,y)}\, \partial_{\nu_A} u^\beta(0,y) dy - \sum_{\beta=1}^m \int_{\mathbb{R}^n} \overline{\partial_{\nu_{A^*}} \widetilde{\Gamma}^{\beta,\alpha}_{(t,x)}(0,y)}\, u^\beta(0,y)\, \mathrm{d}y$$

when $\alpha = 1, \dots, m$, with our convention of the normal vector pointing in the t-direction (that is, inward for the upper half-space).

For a (reasonable) function $f \colon \mathbb{R}^n \to \mathbb{C}^m$ and for $(t,x) \in \mathbb{R}^{1+n}$, with $t \neq 0$, one is led to define the *double layer potential* $\mathcal{D}_t f$ of f by

$$\mathcal{D}_t f(x)^\alpha := \int_{\mathbb{R}^n} \big(\partial_{\nu_{A^*}} \widetilde{\Gamma}^{\cdot,\alpha}_{(t,x)}(0,y), f(y)\big)\, \mathrm{d}y \quad (\alpha = 1, \dots, m),$$

[9] Fundamental solutions were constructed in dimension $n+1 \ge 3$ by Hofmann and Kim [43] at least for systems having pointwise accretivity bounds, and in dimension $n+1 \ge 2$ and accretivity in our sense by Rosén [68].

and the *single layer potential* $\mathcal{S}_t f$ of f by

$$\mathcal{S}_t f(x)^\alpha := \int_{\mathbb{R}^n} (\widetilde{\Gamma}^{\cdot,\alpha}_{(t,x)}(0,y), f(y))\,\mathrm{d}y \quad (\alpha = 1,\dots,m).$$

Here, $(z,\zeta) = \bar{z}\cdot\zeta$ for $z,\zeta \in \mathbb{C}^m$. The first question is whether these operators extend boundedly to spaces of boundary data: on $L^2(\mathbb{R}^n : \mathbb{C}^m)$ for $\mathcal{D}_t$, and from $L^2(\mathbb{R}^n : \mathbb{C}^m)$ into $L^2_1(\mathbb{R}^n : \mathbb{C}^m)$ for $\mathcal{S}_t$ (uniformly in $t > 0$). This was asked by Hofmann in [40]. Another formal observation is that $(\widetilde{\Gamma}_{(t,x)}(s,y))^* = \Gamma_{(s,y)}(t,x)$, where $\Gamma_{(s,y)}$ is the fundamental solution for L_A with pole at (s,y). Note that $\Gamma_{(0,y)}$ is a solution to the equation $L_A u = 0$ away from the pole $(0,y)$, so that the integrals 'defining' $\mathcal{D}_t$ and $\mathcal{S}_t$ may be used to construct solutions. Proving the invertibility of such operators is one way to solve boundary value problems (there are other ways; see below). More precisely, one can solve Dirichlet problems for L_A in $\mathbb{R}^{1+n}_+$ with boundary data φ by solving the double layer equation

$$\lim_{t\searrow 0} \mathcal{D}_t f = \varphi,$$

and likewise one can solve Neumann problems for L_A in $\mathbb{R}^{1+n}_+$ with boundary data φ by solving the single layer equation

$$\lim_{t\searrow 0} \partial_{\nu_A} \mathcal{S}_t f = \varphi.$$

The corresponding solutions u are then given by $u(t,x) = \mathcal{D}_t f(x)$ and $u(t,x) = \mathcal{S}_t f(x)$ respectively.

It was shown by Rosén [68] that these layer potential operators fall within the scope of the first-order framework, hence answering Hofmann's question on layer potential boundedness in the positive. Keeping the De Giorgi–Nash–Moser assumption on A and A^*, and writing $B = \hat{A}$ as usual, for all f in a dense class in $L^2(\mathbb{R}^n : \mathbb{C}^m)$ and $t \in \mathbb{R}\setminus\{0\}$, Rosén shows that

$$\mathcal{D}_t f = -\operatorname{sgn}(t)\left(\mathbb{P}_D \mathrm{e}^{-tBD}\chi^{\operatorname{sgn}(t)}(BD)\mathbb{P}_{BD}\begin{bmatrix} f \\ 0 \end{bmatrix}\right)_\perp \tag{7.5}$$

and

$$\nabla_A \mathcal{S}_t f = \operatorname{sgn}(t)\mathrm{e}^{-tDB}\chi^{\operatorname{sgn}(t)}(DB)\begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{7.6}$$

where the vectors $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]$ are written with respect to the transversal/tangential splitting. In terms of Cauchy operators, on $\mathbb{R}^{1+n}_\pm$ this amounts to

$$\mathcal{D}f = \mp\left(\mathbb{P}_D C^\pm_{BD}\mathbb{P}_{BD}\begin{bmatrix} f \\ 0 \end{bmatrix}\right)_\perp, \quad \nabla_A \mathcal{S} f = \pm C^\pm_{DB}\begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{7.7}$$

Remark that vectors in the form $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]$ belong to $\overline{\mathcal{R}(D)}$ but not to $\overline{\mathcal{R}(BD)}$, hence the presence of the projection $\mathbb{P}_{BD}$.[10]

Now, we reverse the point of view. The right hand sides of the expressions in (7.7) are defined for all coefficients A, whether or not the De Giorgi–Nash–Moser assumptions are satisfied. Thus we may define abstract double and single layer potentials.

[10]This projection, and $\mathbb{P}_D$, were systematically forgotten in [19] by the abuse of notation explained in Section 5.6, as this does not affect the definition in L^2. The stated results for the extensions there were correct. There is also a typo there: the four formulae (81–84) are written with $\mathrm{e}^{+t\cdots}$ instead of $\mathrm{e}^{-t\cdots}$ when $t < 0$.

We first define the abstract double layer potential. We need only consider f in a dense class. For $f \in L^2(\mathbb{R}^n : \mathbb{C}^m)$, (7.5) is well-defined by the functional calculus of the bisectorial operator BD. When f belongs to the classical Sobolev space $W^2_1(\mathbb{R}^n : \mathbb{C}^m)$, $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] \in \mathbb{X}^{\mathbf{p}+1}_D$ with $\mathbf{p} = (2, 0)$ (Hardy-Sobolev and Besov spaces are the same in this case). It follows from Proposition 5.23 and Corollary 5.25 that (7.5) is well-defined as an element of $(\mathbb{X}^{\mathbf{p}+1}_{BD} \cap \mathcal{D}(D))_\perp = (\mathbb{X}^{\mathbf{p}+1}_D \cap \mathcal{D}(D))_\perp$. Furthermore, by the calculations in the proof of Theorem 6.13, we have that $\mathcal{D}f$ is a weak solution of the equation $L_A u = 0$ on each half-space.

Next, we turn to the abstract single layer potential. Assume $f \in L^2(\mathbb{R}^n : \mathbb{C}^m)$. Then $h = \left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] \in \overline{\mathcal{R}(D)}$, and we can define

$$\mathcal{S}f = \mp\left(D^{-1} C^{\pm}_{DB} \begin{bmatrix} f \\ 0 \end{bmatrix}\right)_\perp \tag{7.8}$$

as an element of $L^\infty(\mathbb{R}^\pm : \mathbf{X}^{(2,1)}_\perp)$, where D^{-1} is the inverse of the isomorphism $D\colon \mathbf{X}^{(2,1)}_D \to \overline{\mathcal{R}(D)}$ (a nice Fourier multiplier). By Corollary 5.24, we then have

$$C_{DB} h = D C_{BD} \mathbb{P}_{BD} D^{-1} h = D \mathbb{P}_D C_{BD} \mathbb{P}_{BD} D^{-1} h.$$

Applying D^{-1} and mapping into $\mathbf{X}^{(2,1)}_D$ yields

$$D^{-1} C_{DB} h = \mathbb{P}_D C_{BD} \mathbb{P}_{BD} D^{-1} h,$$

hence

$$(D^{-1} C_{DB} h)_\perp = (\mathbb{P}_D C_{BD} \mathbb{P}_{BD} D^{-1} h)_\perp = (C_{BD} \mathbb{P}_{BD} D^{-1} h)_\perp,$$

where the last equality follows from Corollary 5.25. Taking the sign function into account, this shows that

$$\mathcal{S}f = \mp(C^{\pm}_{BD} \mathbb{P}_{BD} D^{-1} h)_\perp$$

on $\mathbb{R}^\pm$. It follows by the calculation of Theorem 6.13 that $\mathcal{S}f$ is also a solution to $L_A u = 0$ in each half-space and that $\nabla_A \mathcal{S} f$ satisfies the equality in (7.7).

Having defined the layer potentials abstractly on dense subspaces, the results of Section 5.2 yield bounded extensions of $\mathcal{S}$ and $\mathcal{D}$ on the classical smoothness spaces $\mathbf{X}^{\mathbf{p}}$ when $\mathbf{X}^{\mathbf{p}}_{DB} = \mathbf{X}^{\mathbf{p}}_D$. More precisely, we can compute them as

$$\mathcal{S}f = \mp\left(D^{-1} C^{\pm}_{DB} \begin{bmatrix} f \\ 0 \end{bmatrix}\right)_\perp \quad (f \in \mathbf{X}^{\mathbf{p}})$$

and

$$\mathcal{D}f = \pm\left(\mathbb{P}_D C^{+}_{BD} \mathbb{P}_{BD} \begin{bmatrix} f \\ 0 \end{bmatrix}\right)_\perp \quad (f \in \mathbf{X}^{\mathbf{p}+1}) \tag{7.9}$$

using the appropriate extensions from Sections 5.2 and 5.4. Note that it is necessary to use $\mathbb{P}_D$ in (7.9) for arbitrary $f \in \mathbf{X}^{\mathbf{p}+1}$, although this does not matter for $f \in L^2$. Thus for $\mathbf{p} \in J(\mathbf{X}, DB)$, we get the uniform bounds

$$\sup_{t \neq 0} \|\nabla_A \mathcal{S}_t f\|_{\mathbf{X}^{\mathbf{p}}} + \|\mathcal{S}_t f\|_{\mathbf{X}^{\mathbf{p}+1}} \lesssim \|f\|_{\mathbf{X}^{\mathbf{p}}} \tag{7.10}$$

and

$$\sup_{t \neq 0} \|\nabla_A \mathcal{D}_t f\|_{\mathbf{X}^{\mathbf{p}}} + \|\mathcal{D}_t f\|_{\mathbf{X}^{\mathbf{p}+1}} \lesssim \|f\|_{\mathbf{X}^{\mathbf{p}+1}}. \tag{7.11}$$

Compare these bounds with those of Barton and Mayboroda [22, Chapter 3], in particular (3.11–16). The inequalities (7.10) and (7.11) encapsulate a number of concrete inequalities; see the table in Section 2.7.

We also obtain limits for these operators as $t \to 0^{\pm}$ (in $\mathbf{X}^{\mathbf{p}}$ or $\mathbf{X}^{\mathbf{p}+1}$ accordingly, and in the strong or the weak-star topology depending on whether $\mathbf{p}$ is finite). In particular we can also recover the jump relations with this formalism. The equation

$$\nabla_A \mathcal{S}_{0+} f - \nabla_A \mathcal{S}_{0-} f = \big(\chi^+(DB) + \chi^-(DB)\big) \begin{bmatrix} f \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{7.12}$$

encodes both the jump relation of the conormal derivative of $\mathcal{S}_t$ and the continuity of $\mathcal{S}_t$ across the boundary. Here we used that $\chi^+(DB) + \chi^-(DB) = I$ on $\mathbf{X}^{\mathbf{p}}_D$, and that $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] \in \mathbf{X}^{\mathbf{p}}_D$ when $f \in \mathbf{X}^{\mathbf{p}}$. For the double layer operator, we have

$$\begin{aligned} \mathcal{D}_{0+} f - \mathcal{D}_{0-} f &= -\left(\mathbb{P}_D(\chi^+(BD) + \chi^-(BD))\mathbb{P}_{BD} \begin{bmatrix} f \\ 0 \end{bmatrix}\right)_{\perp} \\ &= -\left(\mathbb{P}_D \mathbb{P}_{BD} \begin{bmatrix} f \\ 0 \end{bmatrix}\right)_{\perp} = -f. \end{aligned} \tag{7.13}$$

This time we used that $\chi^+(BD)+\chi^-(BD) = \mathbb{P}_{BD}$ on $\mathbf{X}^{\mathbf{p}+1}_{BD}$ and that (5.10) applies to $\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right] \in \mathbf{X}^{\mathbf{p}+1}_D$ when $f \in \mathbf{X}^{\mathbf{p}+1}$.

For all exponents $\mathbf{p} \in J(\mathbf{X}, DB)$, we have proven the tent/Z-space estimates

$$\|C^+_{DB} h\|_{X^{\mathbf{p}}} \lesssim \|h\|_{\mathbb{X}^{\mathbf{p}}_D}$$

(see Theorems 4.34 and 5.26 and those in [19, §12.3].) Applied to $h = \left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]$ and taking extensions, these immediately yield

$$\|\nabla \mathcal{S} f\|_{X^{\mathbf{p}}} \lesssim \|f\|_{\mathbf{X}^{\mathbf{p}}} \tag{7.14}$$

and

$$\|\nabla \mathcal{D} f\|_{X^{\mathbf{p}}} \lesssim \|f\|_{\mathbf{X}^{\mathbf{p}+1}}. \tag{7.15}$$

Similar bounds for layer potentials on the lower half-space corresponding to (7.14) and (7.15) can also be derived. Compare these results with those of Barton and Mayboroda [22, Theorem 3.1] concerning $\mathcal{S}$ and $\mathcal{D}$: again, we recover their Besov space results (without the De Giorgi–Nash–Moser assumption that they use) and we obtain new Hardy–Sobolev space results.

With these results at hand, the proof of Theorem 7.20 is easy. We mostly reproduce calculations of [15, Lemma 8.1]. By Theorem 6.11 and the assumption on $\mathbf{p}$, we have the semigroup representation for the conormal gradient $\nabla_A u(t,x) = \mathrm{e}^{-tDB}\chi^+(DB)h(x)$, with $h \in \mathbf{X}^{\mathbf{p},+}_{DB}$ being the trace of $\nabla_A u$ at the boundary. Thus $h_\perp = \partial_{\nu_A} u|_{t=0}$ and $h_\parallel = \nabla_\parallel u|_{t=0}$. Write

$$\mathrm{e}^{-tDB}\chi^+(DB)h = \mathrm{e}^{-tDB}\chi^+(DB)\begin{bmatrix} h_\perp \\ 0 \end{bmatrix} + \mathrm{e}^{-tDB}\chi^+(DB)\begin{bmatrix} 0 \\ h_\parallel \end{bmatrix}$$

and, writing $u|_{t=0} = f$, we know that

$$\mathrm{e}^{-tDB}\chi^+(DB)\begin{bmatrix} 0 \\ h_\parallel \end{bmatrix} = \nabla_A\left(\mathbb{P}_D \mathrm{e}^{-tBD}\chi^+(BD)\mathbb{P}_{BD}\begin{bmatrix} f \\ 0 \end{bmatrix}\right)_{\perp}$$

as in the proof of Theorem 6.13. The second term is precisely $-\nabla_A \mathcal{D}_t f$.

Putting everything together, this gives us (7.4).

Let us make some comments on the use of invertibility of layer potentials to solve boundary value problems. As mentioned, one can use invertibility of the operator $\mathcal{D}_{0+}$ for Dirichlet problems and that of the operator $\partial_{\nu_A}\mathcal{S}_{0+}$ for Neumann problems. This is obtained (for example) for self-adjoint real equations ($m = 1$) for $\mathbf{p} = (2, 0)$, i.e., spaces of L^2 Dirichlet/Neumann data, in [2, Theorem 1.13], and for the lower half-space as well (looking at the 0^- limits). Prior to this was the work of Verchota for Laplace's equation on Lipschitz domains [79].

Such invertibility is not necessary for well-posedness as characterised in Theorem 7.5. But this is not the only way to prove well-posedness via layer potentials. For regularity problems, one can use invertibility of the layer potential $\mathcal{S}_0$, as is done in [2, Theorem 1.13] for real symmetric equations with L^2 regularity data, in [41] for real equations with L^p regularity data (for some $p > 1$), and in [15] for H^p regularity data for some $p < 1$. For Neumann problems, one can use invertibility of $\partial_{\nu_A}\mathcal{D}_0$. We give a quick sketch and refer to the proof of [15, Theorem 1.11] for details. The operator $\operatorname{sgn}(DB)$, written in matrix form according to the transversal/tangential decomposition, is

$$\operatorname{sgn}(DB) = \begin{bmatrix} \cdot & -2T \\ 2\nabla_x \mathcal{S}_0 & \cdot \end{bmatrix}.$$

The operator T will be discussed below. The operator $\mathcal{S}_0$ is the single layer potential at $t = 0$, given by

$$\mathcal{S}_0 f := \lim_{t \to 0} \mathcal{S}_t f,$$

recalling that (7.12) shows that the limits $t \to 0^{\pm}$ are the same for this operator. For $\mathbf{p} \in J(\mathbf{X}, DB)$, well-posedness of $(\mathrm{R}_{\mathbf{X}})^{\mathbf{p}}_A$ on both half-spaces is equivalent to invertibility of $\nabla_x \mathcal{S}_0 \colon \mathbf{X}^{\mathbf{p}}_{\perp} \to \mathbf{X}^{\mathbf{p}}_{\parallel}$ (a multiple of the bottom left component of $\operatorname{sgn}(DB)$), or equivalently, invertibility of the single layer potential $\mathcal{S}_0$ from $\mathbf{X}^{\mathbf{p}}$ (which is the same here as $\mathbf{X}^{\mathbf{p}}_{\perp}$) onto $\mathbf{X}^{\mathbf{p}+1}$ as in [15, Theorem 1.11]. Then, for $f \in \mathbf{X}^{\mathbf{p}+1}$, the solutions of the regularity problems with data $\nabla_{\parallel} f$ (that is, f is the Dirichlet data) on the upper and lower half-spaces are respectively given by

$$u^{\pm} = \mathcal{S}_t \mathcal{S}_0^{-1} f \quad (t \in \mathbb{R}^{\pm}).$$

There is also a notion of mutual well-posedness on *both* half-spaces, given by the conjunction of mutual well-posedness on each half-space separately. As we did before, this is akin to demanding the agreement of inverses of the single layer potential for different $\mathbf{p}$. We leave this to the reader.

There are corresponding statements for Neumann problems, using the invertibility of the top right component $-2T$ of $\operatorname{sgn}(DB)$. A calculation shows that $T(\nabla f) = \partial_{\nu_A}\mathcal{D}_0 f$ (there is no jump here) and so invertibility of T corresponds to invertibility of the conormal derivative of the double layer potential $\partial_{\nu_A}\mathcal{D}_0 \colon \mathbf{X}^{\mathbf{p}+1} \to \mathbf{X}^{\mathbf{p}}$ (again the latter space should be thought of as the transversal component). The solutions to the Neumann problems in the upper and lower half-spaces with Neumann data g are given respectively by

$$u^{\pm} = \mathcal{D}_t (\partial_{\nu_A}\mathcal{D}_0)^{-1} g \quad (t \in \mathbb{R}^{\pm}).$$

We remark that for the block diagonal case (see Section 7.2.4) both $\mathcal{S}_0$ and $\partial_{\nu_A}\mathcal{D}_0$ are invertible in the range where well-posedness was established: this is because $\operatorname{sgn}(DB)$ is an involution with zero diagonal entries (and this is because A is diagonal if and only if $\hat{A}$ is diagonal, and then in this case $\operatorname{sgn}(DB) = DB[DB]^{-1}$ has zero diagonal entries).

For the triangular block form, it is either the invertibility of $\partial_{\nu_A}\mathcal{D}_0$ or that of $\mathcal{S}_0$ that holds, depending on the boundary value problem and on the value of $\mathbf{p}$ in the range of well-posedness. The results of [13] show this explicitly for the Regularity and Neumann problems with L^2 data.

We mention without proof that our results on stability in the coefficients (see Section 7.3) allow one to perturb invertibility of boundary layer operators. This was already observed in [2] in the following form: invertibility of *a collection of* boundary layer operators is stable under perturbation. Here, we observe that invertibility is preserved for each individual boundary layer operator, which is a little stronger.

In conclusion, we see that there are many different ways to obtain well-posedness (compatible or not) for boundary value problems associated with $L_A u = 0$ under our current assumptions on A. The first order method essentially covers all results in the literature except for that in [42], concerning a Dirichlet problem with data in L^p for large p, which uses harmonic measure methods for real equations. We still lack the understanding to establish well-posedness for more general coefficients, with any currently available approach.