PART I: $\mathbb{R}$EAL DYNAMICS

This section will consist of the following:

The characteristics of a vector field on the two-sphere, Ann. of Math. **58** (1953) 253–257.

On the concept of attractor, Comm. Math. Phys. **99** (1985) 177–195.

On the concept of attractor: correction and remarks, Comm. Math. Phys. **102** (1985) 517–519.

Directional entropies of cellular automaton-maps, In: "Disordered Systems and Biological Organization (Les Houches, 1985)", NATO Adv. Sci. Inst. Ser. F. Comput. Systems Sci. **20**, Springer, Berlin, (1986) 113–115.

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Non-expansive Hénon maps, Adv. in Math. **69** (1988) 109–114.

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A monotonicity conjecture for real cubic map, (with S. P. Dawson, R. Galeeva and C. Tresser). In: "Real and Complex Dynamical Systems (Hillerød, 1993)", NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **464**, Kluwer Acad. Publ., Dordrecht, (1995) 165–183.

On entropy and monotonicity for real cubic maps, (with an appendix by A. Douady and P. Sentenac), Comm. Math. Phys. **209** (2000) 123–178.

Fubini foiled: Katok's paradoxical example in measure theory, Math. Intelligencer **19** (1997) 30–32.

Introduction

John Milnor made his first contribution to the field of dynamical systems and to the classical Poincaré-Bendixson theory in 1953, while still a graduate student. The paper "***The characteristics of a vector field on the two-sphere***," which he describes as "*an attempt to understand a lecture by Lefschetz,*" defines an ***asymptote*** to be a solution curve (i.e., a curve tangent to the vector field) which either begins or ends at a saddle point. He proves that:

> *The number of sources at which no asymptote begins equals the number of sinks at which no asymptote ends. Furthermore, the number of asymptotes which begin at sources equals the number which end at sinks.*

Towards the end of the paper, he gives an example which shows that the second statement fails in the case of a surface of positive genus: For a vector field on the torus, the number of asymptotes which begin at sources may be different from the number which end at sinks. Vector fields on the torus had been studied much earlier in CHERRY [1938], and examples of this type are known as *Cherry flows.* However, Milnor did not learn about Cherry's work until long after his paper was published.

Twenty four years had to pass before John Milnor's interest in the field of Dynamical Systems was reawakened by conversations with Bill Thurston. A joint paper with Thurston was written and circulated in the late seventies, but not published until ten years later, as described later in this introduction.

Meanwhile, Milnor's paper ***"On the concept of attractor"*** appeared. Clearly he was concerned about the lack of agreement about definitions for some of the most basic concepts of dynamics. After describing some of the many different ways in which the word "attractor" had been used in the literature, he proposed a definition which involves both topology and measure theory, and which is broad enough to encompass many varied examples:

By definition, a closed subset $\mathcal{A}$ of some smooth compact manifold M, is called a (***measure***)-***attractor*** for the dynamical system $f : M \to M$, if it satisfies the following two conditions:

(a). The ***realm of attraction*** $\varrho(\mathcal{A})$, consisting of all points $x \in M$ for which its omega limit set[1] $\omega(x)$ is contained in $\mathcal{A}$, must have strictly positive measure.
(b). There is no strictly smaller closed set $\mathcal{A}' \subset \mathcal{A}$ such that $\varrho(\mathcal{A}')$ coincides with $\varrho(\mathcal{A})$ up to a set of measure zero.

Both of these conditions are essential: the first guarantees that a randomly chosen point will have a positive probability of being attracted to $\mathcal{A}$, and the second guarantees that any strictly smaller closed set will have an essentially smaller realm of attraction.

There is always a maximal attractor $\mathcal{A}_{\max}$ which contains all others. He called this the "***likely limit set***".

Caution. Several terms introduced in this paper have never caught on. The "realm of attraction" is more often called the "***basin of attraction***". Similarly, the "likely limit set" is more often called the "***global attractor***" (or "maximal attractor"), and the term "reducible" in Appendix 2 is replaced by "***renormalizable***".

There is also a variant definition which is purely topological, with no reference to measure theory. In Appendix 1, the *"generic attractor"* is defined as the smallest closed set $\mathcal{A}'_{\max} \subset M$ such that the realm of attraction $\rho(\mathcal{A}'_{\max})$ contains a countable intersection of dense open sets.

[1]The ***omega-limit set*** $\omega(x)$ of a point $x \in M$ is the collection of all accumulation points for the sequence $x,\ f(x)\, f^{\circ 2}(x),\ \ldots$ of successive images of x under f.

Milnor provided some rather awkward examples for which $\mathcal{A}'_{\max} \neq \mathcal{A}_{\max}$. However, in Appendix 2 he asked what happens in more familiar cases, such as quadratic maps. In fact LYUBICH [1994] showed that these two definitions of attractor agree in the quadratic polynomial case. However some natural and dramatic examples of ***wild attractors***, with $\mathcal{A}'_{\max}$ much larger than $\mathcal{A}_{\max}$, were discovered even earlier. MISHA LYUBICH [1986, 1987], and MARY REES [1986] both showed that almost every orbit for the complex exponential map converges to the discrete set $\{0, 1, e, e^2, \ldots\}$ (that is, to the orbit of zero), although MICHAL MISIUREWICZ [1981] had shown that the exponential map is topologically transitive, so that $\mathcal{A}'_{\max}$ is the entire space. BRUIN, KELLER, NOWICKI AND VAN STRIEN [1996] described a polynomial map of the interval, which can be written as $f(x) = x^{2d} - c$ with large d, which is topologically transitive, and such that almost every orbit converges to a Cantor set (namely to the closure of the orbit of zero). This answers a question posed by Milnor in his 1985 paper (end of Appendix 2). Even more dramatic example of attractors with "***intermingled basins***" were described by ALEXANDER, KAN, YORKE AND YOU [1992]. (For other such examples see KAN [1994], BONIFANT AND MILNOR [2008], and BONIFANT, DABIJA AND MILNOR [2007]; and for a general discussion of attractors see MILNOR [2006].)

Note. In later work, Milnor has distinguished between an "***attractor***", which is required to contain a dense orbit, and the more general "***attracting set***", where no such condition is required. (Compare MILNOR [2006].) However, no such distinction is made in this original paper, although it does discuss the related concept of a *"minimal attractor"*, that is, an attractor such that no proper subset is an attractor.

One essential error in this paper was pointed out in the sequel: ***"On the concept of attractor: correction and remarks"***. SMALE [1967], and also RUELLE AND TAKENS [1971], had considered compact sets $\mathcal{A}$ having a neighborhood U such that the intersection of the forward images of U is equal to $\mathcal{A}$. Following an early paper of Besicovitch, Milnor had claimed that this condition did not imply that any orbit outside of $\mathcal{A}$ actually converged to $\mathcal{A}$. However Besicovitch was wrong, and much later published a retraction. In fact, this condition implies the apparently much stronger condition that $\mathcal{A}$ has a ***trapping neighborhood***, that is, a neighborhood N with $f(N)$ compactly contained in N, and with $\bigcap f^{\circ k}(N) = \mathcal{A}$. (See for example MILNOR [2006].)

Several possible modifications to the basic definition of *attractor* have been suggested in ILYASHENKO [2005], who defines the concept of a ***statistical attractor*** by studying the behavior of *"most"* points of a typical orbit $x,\ f(x),\ f^{\circ 2}(x),\ \ldots$, rather than considering only the limiting behavior as the number of iterations of the map tends to infinity.

Under the influence of Stephen Wolfram, John Milnor turned his attention towards cellular automata, particularly in the one-dimensional case (as described in the unpublished survey MILNOR [1984]). However, his published papers concern higher dimensional cases. Given some fixed lattice L in n-dimensional Euclidean space, and a finite set K of symbols, consider the compact space K^L consisting of all functions

$$\mathbf{a} : L \to K\ .$$

In other words, a "***configuration***" $\mathbf{a} \in K^L$ is a function which labels each lattice point $x \in L$ by a symbol $\mathbf{a}(x) \in K$. An ***n-dimensional cellular automaton***

map (or briefly ***CA-map***) is a translation invariant continuous function f which assigns to each configuration $\mathbf{a} \in K^L$ some new configuration $\mathbf{a}' = f(\mathbf{a}) \in K^L$. More explicitly, $\mathbf{a}'$ must have the form

$$\mathbf{a}'(x) \;=\; F(x+y_1,\; x+y_2,\; \ldots,\; x+y_p)\;,$$

where the y_j are specified lattice points, and where the function $F : K^p \to K$ can be arbitrary.

A basic numerical invariant for any topological dynamical system is its ***topological entropy*** $h_{\mathrm{top}}(f)$, as defined by Adler, Konheim and McAndrew [1965]. Here f can be any continuous map. This is a nonnegative number which measures the rate of increase in dynamical complexity as the system evolves with time. Roughly speaking, it measures the exponential growth rate of the number of distinguishable orbits as time advances.

A related concept is the ***measure-theoretic entropy***[2] $h_\mu(f)$, where μ is an f-invariant probability measure. This concept was introduced earlier by Kolmogorov and Sinai (see Sinai [1959]), based on ideas from Shannon's information theory. (For a survey of this subject see Young [2003].) Both $h_\mu(f)$ and $h_{\mathrm{top}}(f)$ measure the exponential rates of growth of numbers of orbits of length n, but:

- h_μ weights orbits according to their probability, while
- h_{top} simply counts *all* distinguishable orbits.

In the case of an n-dimensional CA-map with $n > 1$, both *the topological entropy* and *the measure theoretic entropy* are usually infinite. In the brief announcement ***"Directional entropies of cellular automaton-maps"***, as well as the more detailed paper ***"On the entropy geometry of cellular automata"***, Milnor describes an n-dimensional ***directional entropy function*** which takes finite values, and hence measures possibly useful properties of the map f, even in the higher dimensional case. (Compare Smillie [1988].) This is closely related to the problem of studying entropy for a collection of n commuting maps from a compact space to itself. (In some cases, it is more appropriate to study a d-dimensional directional entropy function, where d is an appropriately chosen integer between 1 and n.)

The second paper also examines ***causality*** in the "***space-time lattice***" $\mathbb{Z} \times L$. Any CA-map f determines the space of ***complete histories*** for f. By definition, this is the closed subset consisting of all $\mathbf{h} \in (\mathbb{Z} \times L)^K$ such that $\mathbf{h}|_{(t+1)\times L}$ is obtained from $\mathbf{h}|_{t\times L}$ by applying the map f. Given the values of this complete history in some subset of $\mathbb{Z} \times L$, one can ask to what extent the values elsewhere in $\mathbb{Z} \times L$ are uniquely determined.

Milnor recalls:

> "I spent some time trying to find an effective algorithm for computing the topological entropy of a one-dimensional CA-map. My difficulties were explained when a group including my former student Lyman Hurd showed that this problem is algorithmically unsolvable".

[2]Often called *"metric entropy"*; but Milnor objects to this terminology.

(See HURD, KARI AND CULIK [1992], which shows that there is no algorithm[3] which will take a description of an arbitrary cellular automaton and determine whether it has zero topological entropy, or for any fixed $\epsilon > 0$ compute its topological entropy to a tolerance ϵ.)

Milnor next studied *Hénon maps.* A real ***Hénon map*** $f : \mathbb{R}^2 \to \mathbb{R}^2$ can be described as a quadratic diffeomorphism of the plane, with the property that no straight line maps to a parallel straight line. Such maps can always be put into the normal form

$$f(x,\, y) \;=\; (y\,,\;\, y^2 \,+\, \alpha \,-\, \delta x)$$

where $\delta \neq 0$ is the constant Jacobian determinant, and α is an arbitrary real number.

In the paper ***"Non-expansive Hénon maps"***, he first notes that the topological entropy $h_{\rm top}(f)$ of a Hénon map in this normal form depends continuously on the two parameters. In fact upper semicontinuity has been proved by YOMDIN [1987] and by NEWHOUSE [1988], while lower semicontinuity (in the two-dimensional case only) has been proved by Katok and Mendoza. (Compare KATOK AND HASSELBLATT [1995].) For each fixed determinant δ, Milnor notes that $h_{\rm top}(f)$ takes all values in the interval $[0,\, \log(2)]$.

A homeomorphism of a metric space is called ***expansive*** if there is a constant $\epsilon > 0$ so that, for any two points $\mathbf{x} \neq \mathbf{y}$ of the space, there exists a positive or negative integer n such that the distance between $f^{\circ n}(\mathbf{x})$ and $f^{\circ n}(\mathbf{y})$ is greater than ϵ. The main result in this paper states that: *For a fixed $\delta \neq 0$, excluding a countable number of values of α corresponding to cases where $h_{\rm top}(f)$ is the logarithm of an algebraic integer or where f has a degenerate periodic point, the map $f|_K$ is not expansive.* The proof depends on work of FRIED [1987].

Let $H(f)$ be the rate of exponential growth for the number of fixed points of $f^{\circ n}$. Katok has shown that $h_{\rm top}(f) \leq H(f)$ for any two-dimensional diffeomorphism. Milnor notes that $h_{\rm top}(f) = H(f)$ in the expansive case, and asks whether equality holds for all 2-dimensional diffeomorphisms. However, this is certainly false. (Compare KALOSHIN [2004].)

It is difficult to accurately compute the topological entropy of a Hénon map. In NEWHOUSE AND PIGNATARO [1993] the authors develop two methods for estimating the topological entropy of any smooth dynamical system. Their algorithms are based on estimates relating the entropy to the logarithmic growth rates of the volumes of disks in the phase space. Towards the end of this paper the authors estimate the entropy of the Hénon map for several values of δ and α. For a different approach, see CHEN, OTT, AND HURD [1991].

The next paper in our discussion is a classic. In a preliminary form, it was widely circulated in 1977, and cited by many authors. However, it was not actually published until eleven years later. Because it introduces ***kneading theory***[4] it is considered by some as the foundation of the field of *Unimodal Real Dynamics*. The paper we are referring to is: ***"On iterated maps of the interval"*** which Milnor wrote in collaboration with Bill Thurston, who this volume is dedicated to.

[3]For similar results about entropy for piecewise affine maps of a 4-dimensional cube, see KOIRAN [2001].

[4]For a recent survey of this theory, see HALL [2009].

In its simplest form a ***kneading sequence*** for real quadratic polynomials, is a sequence of symbols *"left"* and *"right"* describing the location of the critical orbit with respect to the critical point.[5] In this paper Milnor and Thurston gave a precise criterion of which *"left/right"*-sequences occur as *kneading sequences* of *real quadratic polynomials*, or equivalently of any *unimodal real maps.*

More generally, they work with *piecewise monotone maps.* A continuous map f from the unit interval I to itself, is ***piecewise monotone*** if I can be subdivided into finitely many subintervals $I_j \subset I$ with $j \in [1, \ell]$ on which the map is *alternately strictly increasing or decreasing.* Each such maximal interval I_j is called a ***lap*** of f, the positive integer $\ell = \ell(f)$, is called the ***lap number***, and the $\ell - 1$ separating points between these maximal intervals are called ***turning points***. MISIUREWICZ AND SZLENK [1977] had observed that the topological entropy $h_{\rm top}(f)$ can be computed in terms of these lap numbers by the formula[6]

$$h_{\rm top}(f) \;=\; \log(s)\,, \qquad \text{where} \qquad s \;=\; \lim_{k\to\infty} \ell(f^{\circ k})^{1/k} \;=\; \inf_k \ell(f^{\circ k})^{1/k}\;.$$

Here s is called the ***growth number***, and can take any value in the interval $[1,\ \ell(f)]$.

Whenever $s > 1$, a basic theorem asserts that f is semi-conjugate to a piecewise linear function which has slope $\pm s$ everywhere, and has the same topological entropy as f. As an example, in the unimodal case with $\ell = 2$, the map f is semiconjugate to the modified tent map

$$F_s(x) \;=\; \min\Big(sx\,,\ s(1-x)\Big) \qquad \text{for} \qquad 0 \le x \le 1\;.$$

Define the ***address*** $A(x)$ of a point of the interval by setting $A(x) = I_j$ if $x \in I_j$ where x is not a turning point, and setting $A(c_j) = c_j$. The basic claim of kneading theory is that we can obtain a fairly complete description of the dynamics of f simply by following the orbit of each turning point c_k and looking at the associated sequence of addresses $A\big(f^{\circ p}(c_k)\big)$. (It is also important to keep track of the sign $\epsilon(I_j)$, which is defined to be $+1$ or -1 according as f is increasing or decreasing on I_j.) Out of this kneading data, the authors construct a formal power series $D \in \mathbb{Z}[[t]]$ called the ***kneading determinant***, which is closely related to a description of the periodic points of f. In fact this formal power series actually converges throughout the open disk $|t| < 1/s$, and extends to a meromorphic function in the unit disk which has a pole at $1/s$ whenever $s > 1$.

Another basic result from this paper is the entropy monotonicity theorem for quadratic polynomial maps, which can be stated as follows.

Monotonicity Theorem. *The topological entropy of a quadratic map of the form*

$$f_b(x) \;=\; b\,x(1-x)$$

is monotone increasing as a function of the parameter b *(but not strictly increasing). In fact, for each* $n > 0$ *the numbers of laps* $\ell(f_b^{\circ n})$ *is monotone increasing as a function of* b.

[5]Such sequences had been previously studied in METROPOLIS, STEIN AND STEIN [1973]. (See PARRY [1966] for an even earlier study of unimodal maps.)

[6]See also ROTHSCHILD [1971].

In the original prepublication form of this paper, this result was stated only as a conjecture. However, this monotonicity statement follows easily from results in the earlier version, together with the following.

> **Lemma.** *A quadratic polynomial having a critical orbit of period p is uniquely determined, up to affine conjugacy, by its kneading invariant, or equivalently by the linear order of the successive points in its critical orbit.*

Dennis Sullivan pointed this out to the authors, and also pointed that this lemma is an immediate corollary of Thurston's proof that a critically finite rational map is determined, up to Möbius conjugacy, by topological data. (Compare DOUADY AND HUBBARD [1993].) A different proof of this lemma is included in DOUADY AND HUBBARD [1985a, Corollary 6.3].

This is noteworthy as a theorem in real dynamics which has been proved using complex analytic methods. Quite different proofs have been given by DE MELO AND VAN STRIEN [1993], DOUADY [1995], and by TSUJII [2000], but all known proofs seem to make use of complex methods.

MILNOR AND THURSTON [1988, §14.8] use the rather awkward term ***Microimplantation*** for a construction which "implants" a small copy of an arbitrary unimodal function into a critically periodic unimodal function. This can be thought of as a precursor of the Douady-Hubbard theory of ***tuning***, which plays an important role in one-dimensional complex dynamics.[7] (Compare DOUADY AND HUBBARD [1985b].)

The topological entropy of a unimodal map is quite easy to compute, using kneading theory. (Compare BLOCK, KEESLING, LI AND PETERSON [1989].) For maps with three laps, it is a little harder. (See BLOCK AND KEESLING [1992]; and for compositions of two quadratic maps see RADULESCU [2008].) For general piecewise monotone maps see GÓRA AND BOYARSKY [1991], or BALMFORTH, SPIEGEL AND TRESSER [1994] (and possibly COLLET, CRUTCHFIELD AND ECKMANN [1983]). Hubbard trees provide very interesting examples of one-dimensional dynamical systems with a more complicated topology. For entropy studies in this case, see ALSEDÀ AND FAGELLA [2000], and TAO LI [2007].

In the case in which the filled Julia set a real quadratic map f is locally connected and has no interior the *kneading sequence* alone gives a complete topological description of the dynamics of f on its Julia set. (See BRUIN AND SCHLEICHER [2012].) In LAU AND SCHLEICHER [1994] the authors describe a version of kneading theory for complex quadratic polynomials, and raise the question of which sequences in $\{0,\ 1\}^{\mathbb{N}^*}$ occur as *kneading sequences of complex quadratic polynomials* or equivalently as *kneading sequences of angles*. This question was answered in BRUIN AND SCHLEICHER [2009, Theorem 4.2]. This result is a generalization of the Milnor-Thurston *kneading theory* from the real to the complex case.

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7 The Douady-Hubbard ***tuning*** construction assigns to each hyperbolic component H of the Mandelbrot set $\mathcal{M}$ a homeomorphism

$$\mathcal{M} \xrightarrow{\cong} (H \triangleright \mathcal{M}) \subset \mathcal{M}$$

from $\mathcal{M}$ onto a "small copy" of $\mathcal{M}$. Each $H\triangleright$ maps hyperbolic components to hyperbolic components, so that the period of $H \triangleright H'$ is equal to the product of the periods of H and H'.

Milnor [1992] proposed a corresponding conjecture for cubic polynomials:

Cubic Monotonicity Conjecture. *In the family of maps*

$$x \mapsto x^3 - 3Ax + b \qquad or \qquad x \mapsto -x^3 - 3Ax + b\ , \tag{1}$$

the set of parameter pairs for which the topological entropy takes some constant value[8] *is connected.*

A first step in this direction was given in the paper ***"A monotonicity conjecture for real cubic maps"***, written with Dawson, Galeeva and Tresser. The conjecture was then proved by Milnor and Tresser in the paper ***"On entropy and monotonicity for real cubic maps"***.

Before describing the idea, it is convenient to replace each of the families of Equation (1) by a more conveniently parametrized family. Consider cubic polynomial maps f from the unit interval to itself such that both critical points $0 \le c_1 \le c_2 \le 1$ are real, and belong to this interval. Then the corresponding critical values $v_j = f(c_j)$ form a convenient pair of parameters. To fix our ideas, consider the case where the third derivative f''' is positive. Then these parameters vary over the triangle $0 \le v_2 \le v_1 \le 1$. For each period $p \ge 2$ and for each of the two critical points c_j, there is a smooth curve in the parameter triangle consisting of all (v_1, v_2) such that the critical point c_j is periodic of period p. By definition, each connected component of such a critically periodic curve is called a ***bone***. A fundamental lemma asserts that: *each bone is a simple arc joining two points on the boundary of the triangle.* The difficult part of the proof is the assertion that there are no *"bone loops"*:

There is no simple closed curve consisting of parameter values for which one critical point is periodic.

For the proof of this essential statement, the authors relied on the thesis of Christopher Heckman, which has never been published. (See Heckman [1996].) However, the situation has since become much clearer. It is not difficult to prove that the region enclosed by a bone loop can not contain any hyperbolic map. Hence this *"no bone loop"* statement follows easily from the generic hyperbolicity theorem for real polynomial maps, which was proved seven years later:

Theorem of Kozlovski, Shen and van Strien [2007]. *Every real polynomial map can be approximated by a hyperbolic polynomial map of the same degree.*

In fact an entropy monotonicity theorem for real polynomial maps of arbitrary degree[9] has been announced in Bruin and van Strien [2009]. (For the special case of compositions of two quadratic maps, see Radulescu [2007].)

One important ingredient of these proofs is the use of ***stunted sawtooth map***. These are piecewise linear maps from the unit interval to itself which have slope alternately $\pm\ell$ along ℓ subintervals, which are separated by $\ell - 1$ plateaus where the slope is zero. Furthermore, the boundary $\{0, 1\}$ is required to map into itself.

[8] The sets defined in this way are known as ***isentropes***.

[9] In the cubic case, Milnor and Tresser mention that each *isentrope is actually a cellular set.* However, as far as I know it is not known whether this is true for higher degrees.

For a survey of one-dimensional dynamics see VAN STRIEN [2010] and for a survey of results and questions about Milnor's conjecture on monotonicity of topological entropy see VAN STRIEN [2012].

In the paper ***"Fubini Foiled: Katok's Paradoxical Example in Measure Theory".*** Milnor describes a topological foliation of the open unit square by a family of smooth real analytic curves Γ_β whose, with the following strange property. There is a set E of full Lebesgue measure in the square such that each Γ_β intersects E in at most one point. (It follows from Fubini's Theorem that this could never happen for a *smooth* foliation.) He also gives a dynamical interpretation of this construction.

In fact examples of this type arise naturally when studying the ergodic theory of smooth dynamical systems. The first example constructed by A. Katok was based on a family of degree-two Blaschke products[10] mapping the unit circle to itself. Milnor remarks in his note that a different version of the construction, based on tent maps, was given by J. Yorke. In SHUB AND WILKINSON [2000], the authors came across the same phenomenon when looking for volume preserving non-uniformly hyperbolic systems in the neighborhood of a diffeomorphism that is not homotopic to an Anosov diffeomorphism and which has a zero exponent at every point.

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PART II: $\mathbb{C}$OMPLEX DYNAMICS

This section will consist of the following:

Dynamical properties of plane polynomial automorphisms (WITH S. FRIEDLAND), Ergodic Theory Dynam. Systems **9** (1989) 67–99.

Self-similarity and hairiness in the Mandelbrot set, "Computers in geometry and topology (Chicago, IL, 1986)", Lecture Notes in Pure and Appl. Math., **114**, Dekker, New York, (1989) 211–257.

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Geometry and dynamics of quadratic rational maps (WITH AN APPENDIX BY THE AUTHOR AND TAN LEI), Experiment. Math. **2** (1993) 37–83.

The Fibonacci unimodal map (WITH M. LYUBICH), J. Amer. Math. Soc. **6** (1993) 425–457.

Fixed points of polynomial maps II, Fixed point portraits (WITH L. GOLDBERG), Ann. Sci. École Norm. Sup. **26** (1993) 51–98.

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Introduction

I have special feelings for the Friedland and Milnor paper I am about to introduce since it was the first paper I read when I started my studies in complex dynamics in higher dimensions. I believe this paper has had a big influence in the study of this subject, since it showed that the methods used to study Hénon maps would apply to a much bigger family of maps, the family of ***polynomial automorphisms***[1] of $\mathbb{R}^2$ (or $\mathbb{C}^2$) with arbitrary degree.

[1] A ***polynomial automorphism*** is a polynomial mapping which is one-to-one and onto, and whose inverse is also a polynomial mapping.

Hénon [1976] studied a two-parameter family of polynomial automorphisms $H:\mathbb{R}^2\to\mathbb{R}^2$ which can be written as

$$H(x,\,y) \;=\; (y,\;\; y^2\,+\,\alpha\,-\,\delta x)\;,$$

showing that they display very complicated dynamics. Here $\delta\neq 0$ is the constant Jacobian determinant of H and α is any real number.

In the paper ***"Dynamical properties of plane polynomial automorphisms"*** Friedland and Milnor extend Hénon's construction by discussing the group G of all polynomial automorphisms of the real or complex plane. To fix ideas, I will concentrate on the complex case; however the real case is completely analogous. Note that the Jacobian determinant for any polynomial automorphism must be a non-zero constant,[2] since this determinant is a polynomial function with no zeros in the $(x,\,y)$-plane.

This group of polynomial automorphisms has long been object of study. Jung [1942] proved the basic result that G is generated by two subgroups A and E, as follows:

- A is the 6-dimensional subgroup consisting of all affine automorphisms, and
- E is the infinite dimensional subgroup consisting of all automorphisms which map horizontal lines to horizontal lines, and hence have the form

$$e(x,\;y)\;=\;(a\,x\,+\,b(y),\;\;c\,y\,+\,d)\;, \tag{1}$$

where $b(y)$ is any polynomial function, and $a\,c\neq 0$. (For another proof of Jung's Theorem see McKay and Wang [1988], and for an extension to fields of arbitrary characteristic, see van der Kulk [1953]. Note that the group of ***holomorphic*** automorphisms of $\mathbb{C}^2$ is much larger. As one example. in the expression (1) we can replace a and $b(y)$ by $e^{a(y)}$ and $b(y)$, where $a,b:\mathbb{C}\to\mathbb{C}$ can be arbitrary holomorphic functions.)

Later authors have sharpened this result by showing that G is actually the amalgamated free product of A and E. Friedland and Milnor refine this statement to obtain a more useful normal form. Define a ***generalized Hénon transformation*** to be a map of the form

$$g(x,\,y)\;=\;(y\,,\;\;p(y)\,-\,\delta x)\;, \tag{2}$$

where $\delta\neq 0$ is again the Jacobian determinant, and where $p(y)$ is a polynomial of degree $d\geq 2$. They prove the following.

> **Theorem.** *Every polynomial automorphism $g\in G$ is conjugate, within G, to either*
> - *an element of E, or*
> - *a composition of one or more generalized Hénon transformations* (2).

[2] The converse proposition, that every polynomial map of $\mathbb{C}^n$ with non-zero constant Jacobian determinant must be an automorphism, is known as the ***Jacobian Conjecture***, and is still open. See Keller [1939]; and for surveys see Bass, Connell and Wright [1982], van den Essen [2000], and de Bondt and van den Essen [2005].

Automorphisms conjugate to an element of E are called ***elementary automorphisms***, and have relatively uninteresting dynamics (see §6 of the paper under discussion). However, the compositions of generalized Hénon maps are well worth studying. One simple estimate applies to all cases.

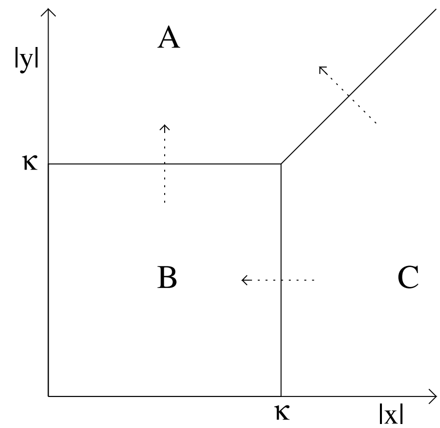

Figure 1. The arrows indicate possible transitions under a composition of generalized Hénon maps.

Given a constant $\boldsymbol{\kappa} > 0$, divide $\mathbb{C}^2$ into three regions **A**, **B**, **C** as illustrated in Figure 1:

A: $|\mathrm{y}| \ \geq \ \max(\boldsymbol{\kappa},\ |\mathrm{x}|)$,

B: $\boldsymbol{\kappa} \ \geq \ \max(|\mathrm{x}|\,,\ |\mathrm{y}|)$,

C: $|\mathrm{x}| \ \geq \ \max(\kappa,\ |\mathrm{y}|)$.

Then for any composition g of generalized Hénon maps, one can choose $\boldsymbol{\kappa}$ large enough so that:

- g maps the upper left region **A** into itself, with both $|\mathrm{x}|$ and $|\mathrm{y}|$ tending to infinity along all orbits.

- The inverse map g^{-1} sends the lower right region **C** into itself, with both $|\mathrm{x}|$ and $|\mathrm{y}|$ tending to infinity along all orbits.

It follows that every periodic orbit (and more generally any orbit whose orbit under both g and g^{-1} is bounded) must be contained in the the lower left region **B**, which is a compact bidisk.

For any composition of generalized Hénon maps, Friedland and Milnor obtain the following bounds for the ***topological entropy*** of g,

$$0 \ \leq \ h_{\mathrm{top}}(g) \ \leq \ \limsup_{n\to\infty} \frac{\log^{+}\big|\mathrm{Per}_n(g)\big|}{n} \ \leq \ \log(d)\,,$$

where $|\mathrm{Per}_n(g)|$ is the number of period n points, and where d is the algebraic degree.[3] This is a best possible estimate for the entropy in the real case. In the

[3] The ***algebraic degree*** of g is defined to be the maximum of the degrees of the polynomials g_1 and g_2, where $g(x,y) = \big(g_1(x,y),\ g_2(x,y)\big)$.

complex case, Smillie [1990] has proved that entropy is precisely equal to the logarithm of the degree.

For more on the dynamics of Hénon maps, or of polynomial automorphisms of $\mathbb{C}^2$, see Fornæss and Sibony [1992], Hubbard and Oberste-Vorth [1994, 1995], Bedford and Smillie [1991a, 1991b, 1992, 1998a, 1998b, 1999, 2002, 2006] and Bedford, Lyubich and Smillie [1993a, 1993b]. For a classification of degree two polynomial automorphisms of $\mathbb{C}^3$, see Fornæss and Wu [1998].

The next paper deals with the concept of *self-similarity* in the Mandelbrot set. In this paper Milnor posed several important conjectures in the field of complex dynamical systems in one variable. A closed set $X \subset \mathbb{C}$ is said to be ***self-similar*** about a point x_0 if there is a complex constant λ with $|\lambda| > 1$ such that the translated and magnified images $\lambda^n(X - x_0)$ converge to a limit in the Hausdorff topology for subsets of the Riemann sphere. Tan Lei [1990] showed that the Mandelbrot set $\mathcal{M}$ is self-similar about any Misiurewicz point.[4] (See also Douady and Hubbard [1985a].) In her case, the expansion constant λ is equal to the multiplier of the postcritical periodic orbit, that is, the first derivative of $f^{\circ p}$ at an orbit point, where p is the period. (For an earlier related result, see Eckmann and Epstein [1985].)

In the paper ***"Self-similarity and hairiness in the Mandelbrot set"***, Milnor studies a weaker form of self-similarity, not at Misiurewicz points, but rather at points in $\mathcal{M}$ which are infinitely renormalizable of stationary type.

First some notations and definitions. Let s range over the collection of all ***critically periodic points*** in the Mandelbrot set. Each such s is the "center" point of an associated hyperbolic component, which may be denoted by H_s. According to the ***tuning construction*** of Douady and Hubbard [1985b], to each s one can associate a homeomorphism from $\mathcal{M}$ onto a small copy of $\mathcal{M}$. Let us denote this tuning embedding of $\mathcal{M}$ into itself by

$$c \mapsto s * c \in s * \mathcal{M} \subset \mathcal{M} .$$

In particular, if s and s' are critically periodic points of periods p and p', then the "star product" $s * s'$ is critically periodic of period $p p'$, with $s * H_{s'} = H_{s*s'}$. *In fact, the collection of all critically periodic s forms a free non-commutative monoid, which acts as a monoid of embeddings of $\mathcal{M}$ into itself.*

Given any infinite sequence $s_1, s_2, \cdots$ of critically periodic points $s_j \neq 0$, we can form the infinite nested sequence

$$\mathcal{M} \supset s_1 * \mathcal{M} \supset s_1 * s_2 * \mathcal{M} \supset s_1 * s_2 * s_3 * \mathcal{M} \supset \cdots$$

of small copies of $\mathcal{M}$. The fundamental problem in this field is the conjecture that ***the intersection of this nested sequence of compact sets always consists of just one point.*** This problem was given special importance when Yoccoz proved that this is true if and only if the Mandelbrot set is locally connected. (Compare Hubbard [1993].) There has been a great deal of progress in special cases of this fundamental conjecture. (See for example Sullivan [1992], McMullen [1996], Lyubich [2002], Kahn and Lyubich [2009].) However, the general case remains open.

[4] Following Douady and Hubbard, a point μ in the Mandelbrot set $\mathcal{M}$, is called ***Misiurewicz*** if the orbit of zero under the quadratic map $q_\mu : z \mapsto z^2 + \mu$ is *eventually periodic* but *not periodic.*

In the paper under discussion, Milnor formulates a number of related conjectures, all concerning the ***stationary*** case where $s_1 = s_2 = s_3 = \cdots$. I will discuss the first four of these which can be stated as follows, using the notation s^{*n} for the n-fold star product $s * \cdots * s$.

Conjecture 1.1 (Geometric Convergence). For any $s \neq 0$, the sequence of points $s^{*n} \in \mathcal{M}$ always converges to a limit $s^\infty \in \mathcal{M}$. Furthermore the convergence is geometric, in the sense that the sequence of difference ratios

$$\frac{s^{*n} - s^{*n-1}}{s^{*n+1} - s^{*n}}$$

always converges to a complex number δ with $|\delta| > 1$. (Conjecture 3.1 makes the sharper statement that the map $s* : \mathcal{M} \to \mathcal{M}$ has a unique fixed point, equal to this limit s^∞ , and has a well defined derivative, equal to $1/\delta$, at this point.)

Conjecture 1.2 (Convergence in Measure). Within any bounded region U of the plane, the sequence of magnified images $(\mathcal{M} - s^\infty)\delta^n$ converges in measure to some measurable limit set X. In other words, the symmetric difference between X and this magnified image, intersected with U, has area tending to zero.

Conjecture 1.3. (Hairiness). This limit set X is everywhere dense. In other words, for any $z_0 \in \mathbb{C}$, the distance between z_0 and the magnified image $(\mathcal{M} - s^\infty)\delta^n$ tends to zero as $n \to \infty$.

Conjecture 1.4 (Sparsity). The limit set X is extremely sparse in some regions of the plane, so that for any $\epsilon > 0$ one can find a disk of radius one which intersects X is a set of area less than ϵ.

The first and third conjectures have been proved by Lyubich [1999] for real values of s. I am not aware of any work which explicitly considers the remaining two, although Lyubich's paper certainly seems relevant to Conjecture 1.2.

We now leave the discussion of the Mandelbrot set, and of complex quadratic polynomials, to focus on *real or complex cubic polynomial maps*, as studied in Milnor's paper ***"Remarks on Iterated Cubic Maps"***.

The study of cubic polynomial maps in the complex setting was initiated in Branner and Hubbard [1988, 1992]. In these papers the authors defined the ***connectedness locus*** for the family of cubic polynomial maps, to be the set of those polynomials $p(z)$ of degree three for which none of the finite critical points are attracted to infinity. This set can also be defined as the set of those polynomial maps for which the filled Julia set $K_{p(z)}$ is connected. Their work is based on the normal form

$$p(z) \ = \ z^3 - 3a^2 z + b \ , \tag{3}$$

with critical points $\pm a$.

In the first part of his paper, Milnor describes the parameter space for *real cubic polynomials*, noting that a complex polynomial of the form (3) is conjugate to a polynomial with real coefficients if and only if the two invariants $A = a^2$ and $B = b^2$ are real. Given any real cubic map $f : \mathbb{R} \mapsto \mathbb{R}$, let I be the "real filled Julia set", that is, the smallest closed interval which contains all points with bounded orbit. Then the real $(A,\ B)$-plane can be divided into four regions, according as the graph of f intersected with $I \times I$ has zero, one, two or three non-degenerate connected components.

In both the real and complex cases, Milnor divides the possible hyperbolic components into four types: $\mathcal{A}$: *Adjacent Critical Points*; $\mathcal{B}$: *Bitransitive*; $\mathcal{C}$: *Capture*; and $\mathcal{D}$: *Disjoint Periodic Sinks.* (This is very similar to the classification used by Rees [1990] for quadratic rational maps.) Corresponding to three of these four types, there is a typical shape, which is repeated infinitely many times in the $(A,\ B)$-plane. For the bitransitive type, there is a "***swallow***" shape, in the capture case an "***arch***" shape, and in the disjoint case a "***product***" shape. On the other hand, there are only two hyperbolic components of type $\mathcal{A}$ in the real $(A,\ B)$-plane. (One of these is centered at the map $z \mapsto z^3$, and the other is centered at the map with complex normal form $z \mapsto z^3 \pm i$, represented by the real map $z \mapsto 1 - z^3$.)

In the case of a real map with complex critical points, the map from $\mathbb{R}$ to itself is strictly monotone, with rather boring dynamics. However, if we think of it as a complex map with real coefficients, then the situation becomes much more interesting. We can search for hyperbolic components by following the orbit of either complex critical point, and find two new shapes: the "***tricorn***" in the bitransitive case, and the "***Mandelbrot shape***" in the disjoint case.

A dramatic new phenomenon appears for such complex maps with real coefficients. Douady and Hubbard had proved the equivalence of the ***MLC conjecture***, which asserts that the Mandelbrot set (= quadratic connectedness locus) is locally connected, and the ***generic hyperbolicity conjecture*** for quadratic polynomials, which asserts that every quadratic polynomial map can be approximated by a hyperbolic one. (These conjectures seem very likely, but remain unproved.) For cubic polynomial maps, pictures show that the cubic connectedness locus, intersected with the real $(A,\ B)$-plane, is manifestly not locally connected.[5] (Compare Epstein and Yampolsky [1999].) In fact, a proof that the full cubic connectedness locus is not locally connected was given in Lavaurs [1989]. Yet it seems quite possible that generic hyperbolicity does hold, not only in the quadratic case, but much more generally.

See Milnor [2009, 2012], as well as Bonifant, Kiwi and Milnor [2010], for further studies of cubic polynomial maps and of hyperbolic components.

We now go from cubic polynomial maps to the study of quadratic rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. It might seem that these two theories are totally different, but in fact they are closely related.

In the paper ***"Geometry and Dynamics of Quadratic Rational Maps"***, Milnor uses the notation Rat_2 for the space of all quadratic rational maps from the Riemann sphere to itself. This space can be identified with a Zariski open subset of the complex projective space $\mathbb{CP}^5$. He notes that Rat_2 has the rational homology of a three-sphere, and that its universal covering has $\mathbb{S}^3 \times \mathbb{S}^2$ as deformation retract.

However, the main focus of the paper is on the ***moduli space*** $\mathcal{M}_2$, which he defines as the quotient space, consisting of Möbius conjugacy classes $\langle f \rangle$ of elements $f \in \mathrm{Rat}_2$. An elementary argument shows that $\mathcal{M}_2$ is canonically biholomorphic to

[5] Milnor had conjectured the non-local-connectivity of the cubic connectedness locus in 1986, in a preprint which seems to be an ancestor of the paper under discussion. (See Branner and Hubbard [1988, p. 145].)

the complex coordinate space $\mathbb{C}^2$. The proof uses elementary symmetric functions of the multipliers at the three fixed points of f as coordinates.

Although there are many other ways of choosing a biholomorphic map between $\mathcal{M}_2$ and $\mathbb{C}^2$ (compare the Friedland-Milnor paper), this particular choice has very useful properties, as follows. For each complex number η, let $\mathrm{Per}_1(\eta) \subset \mathcal{M}_2$ be the set of all conjugacy classes $\langle f \rangle$ of maps which have a fixed point z_0 of multiplier $f'(z_0) = \eta$.

> *Then the correspondence* $\mathcal{M}_2 \xrightarrow{\cong} \mathbb{C}^2$ *sends each* $\mathrm{Per}_1(\eta)$ *to a complex line.*

(As one example, the family of all quadratic ***polynomial*** maps can be identified with the line $\mathrm{Per}_1(0) \subset \mathcal{M}_2$.) Using this construction, it is quite easy to make computer pictures illustrating the dynamics in any one of the lines $\mathrm{Per}_1(\eta)$.

More generally, for any $p > 1$, define $\mathrm{Per}_p(\eta)$ to be the ***closure*** of the set of $\langle f \rangle$ with a period p orbit of multiplier η. He shows that each $\mathrm{Per}_p(\eta)$ is an algebraic curve which intersects the polynomial locus $\mathrm{Per}_1(0)$ transversally, and hence has degree equal to the number of hyperbolic components of period p in the Mandelbrot set. Of particular interest are those curves $\mathrm{Per}_n(\eta)$ for which η is a root of unity, since they contain faces where two or more hyperbolic components of $\mathcal{M}_2$ come together along a common boundary.

Corresponding to the isomorphism $\mathcal{M}_2 \cong \mathbb{C}^2$, Milnor also proposes a compactification[6] $\widehat{\mathcal{M}_2} \cong \mathbb{CP}^2$, consisting of $\mathcal{M}_2$ together with a two-sphere of *ideal points* at infinity. Elements of this two-sphere can be thought of as limits of quadratic rational maps as they degenerate towards a fractional linear or constant map, with conjugacy class $\langle z \mapsto \mu\, z \rangle$. Of course such a limit cannot be uniform over the entire Riemann sphere. What happens as we tend to the sphere of ideal points is that the two critical points and one increasingly repelling fixed point all crash together. The product of the remaining two fixed point multipliers converges to $+1$, and the limit map can be identified with the conjugacy class $\langle z \mapsto \mu\, z \rangle$ of a linear map.

For any period $p \geq 2$ and any multiplier $\eta \in \mathbb{C}$, he proves the following.

> *The only possible limit points of the curve* $\mathrm{Per}_p(\eta) \subset \mathcal{M}_2$ *on the two-sphere at infinity are ideal points which correspond to linear maps of the form* $z \mapsto \mu\, z$ *where* μ *is a* q*-th root of unity, with* $q \leq p$.

Milnor conjectured that the case $q = 1$ can not occur, but that there are no other restrictions. In other words, for any p and η, the ideal point $\langle z \mapsto \mu z \rangle$ should occur as a limit point of $\mathrm{Per}_p(\eta)$ if and only if μ is a primitive q-th root of unity for some $2 \leq q \leq p$. In the special case $\eta = 0$, this has been proved by Stimson [1993]; but I am not aware of a proof (or counter-example) in the general case.

Perhaps an explicit example will help to explain the meaning of this result. For any point in the curve $\mathrm{Per}_3(0)$, there is a unique representative map f such that

[6] Compare Epstein [2000]. For other compactifications see Silverman [1998], and DeMarco [2005, 2007].

the origin is a period three critical point with orbit $0 \mapsto \infty \mapsto 1 \mapsto 0$. A brief computation then shows that f must have the form

$$f(z) \;=\; 1 \;-\; \frac{1+c}{z} \;+\; \frac{c}{z^2}\;.$$

For most values of the parameter c, this is a well defined quadratic rational map. However, as c tends to zero, it converges (in the finite plane) to the fractional linear map $z \mapsto 1 - 1/z$; and this limit is Möbius conjugate to the linear map $z \mapsto e^{2\pi i/3} z$. The behavior as $|c|$ tends to infinity is harder to understand. However, if we rescale by choosing a square root of c and setting

$$g(w) \;=\; \frac{f(w\sqrt{c})}{\sqrt{c}} \;=\; \frac{1}{\sqrt{c}} - \frac{1+c}{w\,c} + \frac{1}{w^2\sqrt{c}}\;,$$

then as $|c| \to \infty$ the map g converges (again in the finite plane) to the fractional linear map $w \mapsto -1/w$, which is Möbius conjugate to the linear map $z \mapsto -z$. Thus the cube roots of unity and the square root of unity appear as the only multipliers in the limit.

It is often important to keep track of the critical points or the fixed points of a rational map. Milnor also considers quadratic rational maps which have been "marked" by providing an ordered list, either of the two critical points, the three fixed points, or of both. In particular, he studies the ***critically marked moduli space*** $\mathcal{M}_2^{\rm cm}$, and also the ***fixed-point marked moduli space*** $\mathcal{M}_2^{\rm fm}$, as well as the ***totally marked moduli space*** $\mathcal{M}_2^{\rm tm}$.

The second part of this paper studies the dynamics of quadratic rational maps. Following Rees [1990], Milnor describes a preliminary classification of hyperbolic components into four different types. Three of these types are completely analogous to the corresponding types for cubic polynomial maps, namely the "*bitransitive type*", the "*capture type*", and the "*disjoint attractor type*". Each of these hyperbolic components is a topological 4-cell,[7] with a unique post-critically finite "center" point. The corresponding rational maps have connected Julia set. There is one important further distinction: Some of these hyperbolic components are bounded, and some are unbounded. (Compare Epstein [2000].) For an analogous discussion of hyperbolic components for higher degree rational maps, see Milnor [2012].

There is also just one anomalous hyperbolic component called the ***hyperbolic shift***[8] ***locus***, consisting of hyperbolic maps with totally disconnected Julia set. It has a slightly more complicated topology, and has no post-critically finite point. (However, if we work in the compactified moduli space $\widehat{\mathcal{M}}_2$, then this anomalous hyperbolic component can be extended to a topological 4-cell with a prefered center point.)

Douady and Hubbard were the first to note that many quadratic rational maps with two disjoint attracting cycles can be constructed by pasting together the boundaries of the filled Julia sets for two quadratic polynomial maps. This ***mating*** construction has been studied by many authors. (See for example Douady [1983],

[7] However, if we work in the critically marked moduli space $\mathcal{M}_2^{\rm cm}$, then there is a singular point $\langle z \mapsto 1/z^2 \rangle$ which lies at the center of one hyperbolic component.

[8] The term "***escape locus***" is used in the paper. However, Milnor feels that this is poor terminology, since there is no place to escape to in the Riemann sphere.

WITTNER [1988], TAN LEI [1992], SHISHIKURA [2000], SHISHIKURA AND TAN LEI [2000], CHÉRITAT [2012], EPSTEIN [2012], and for open problems BUFF ET AL. [2012].)

Since it is very hard to get a feeling for a complex two-dimensional parameter space, much of the more detailed dynamic discussion concerns suitably chosen one-dimensional slices. However, if we consider only quadratic rational maps with real coefficients, then we obtain a real two-dimensional parameter space, which is much easier to deal with.

First consider consider real quadratic rational maps from the topological circle $\mathbb{RP}^1$ to itself. There is a rough classification into seven different types. Two of these types represent covering maps of degree +2 or −2 from the circle onto itself, with no real critical points. The remaining types represent maps which carry $\mathbb{PR}^1$ onto an interval $I \subset \mathbb{PR}^1$ which is bounded by the two critical values. There are five different ways that this interval I can map into itself.

On the other hand, if we consider quadratic rational maps from the entire Riemann sphere onto itself which have real coefficients, then the situation is more complicated. In particular, it is again of interest to consider hyperbolic components.

The next paper, ***"The Fibonacci Unimodal Map"*** by Lyubich and Milnor, is unusual in that it studies just one specific example

$$f(x) \;=\; x^2 \;-\; 1.8705286\cdots$$

of a real quadratic polynomial map.[9] Here the parameter value is carefully chosen so that each closest return of the critical orbit $0 \mapsto x_1 \mapsto x_2 \mapsto \cdots$ to zero occurs after a Fibonacci number of iterations. (In fact these closest returns are arranged along the real axis according to the following pattern,

$$x_1 \;<\; x_5 < x_8 \;<\; x_{34} < x_{55} \;<\; \cdots \;\to\; 0 \;\leftarrow\; \cdots \;<\; x_{21} < x_{13} \;<\; x_3 < x_2 \;,$$

where the convergence to zero is extremely fast.)

The reason for the careful study of this particular example is that it seemed to be an excellent candidate for a real quadratic map with exotic dynamics. (Compare HOFBAUER AND KELLER [1990].) However, the main result of the paper is that this example is not at all exotic. In fact it has an absolutely continuous invariant probability measure, with support equal to the interval $[x_1, x_2]$, and with positive entropy. The same result applies more generally to any C^2-smooth unimodal map with negative Schwarzian derivative, with $f''(0) \neq 0$, and with the Fibonacci recursion pattern.

Here the condition that $f''(0) \neq 0$ ***is crucial.*** In fact, a few years later BRUIN, KELLER, NOWICKI AND VAN STRIEN [1996] proved that a higher degree unimodal map of the form $g(x) = x^{2n} - c$ with exactly this Fibonacci recursion pattern does have a ***wild attractor***.[10] More explicitly, they showed that almost every orbit converges to the closure of the critical orbit, which is a Cantor set, although the map is topologically transitive on the interval $[x_1, x_2]$.

[9] This paper is included here among the papers on complex dynamics since complex estimates play a crucial role in the argument.

[10] Compare the discussion of "*On the concept of attractor*", earlier in this volume.

The proof in this Lyubich-Milnor paper is rather complicated. One essential step is the proof that the geometry "degenerates", in the following sense: If $u(n)$ denotes the n-th Fibonacci number, then the ratio $x_{u(n+1)}/x_{u(n)}$ converges exponentially fast to zero as $n \to \infty$. The proof also depends on a comparison of quadratic Fibonacci maps with cubic Fibonacci maps with one escaping critical orbit, as studied earlier by Branner and Hubbard [1992].

Lyubich [1994] later proved the much more general result that no quadratic polynomial map can have a wild attractor, using closely related methods. In fact in Lyubich [1997, page 189], where he studies a form of "Fibonacci renormalization" to study more general quadratic maps, he comments that "*The proof of the moduli growth for the Fibonacci combinatorics is the heart of the whole paper.*" For a later study of Fibonacci renormalization, see Buff [2000].

The next paper, ***"Fixed points of polynomial maps. Part II: Fixed point portraits"***, written with Lisa Goldberg, classifies polynomial maps of degree $d \geq 2$ in terms of the ***external rays*** which land at their fixed points. Part I of this paper (Goldberg [1992]), was a detailed analysis of degree d ***rotation sets*** . By definition these are finite subsets of the circle $\mathbb{R}/\mathbb{Z}$ which are invariant, and have a well defined rotation number, under multiplication by d,

In the present paper, Goldberg and Milnor define the ***rational type*** $T \subset \mathbb{R}/\mathbb{Z}$ *of a fixed point* z as the set of angles of ***rational*** external rays which land at z. According to basic results of Douady, Hubbard and Sullivan, any repelling or parabolic periodic point, for a monic polynomial with connected Julia set, is the landing point of one or more rational external rays. (See for example Milnor [2006]. There can be no such rays in the case of an attracting, Siegel, or Cremer periodic point.)

Define the ***fixed point portrait***[11] of a monic polynomial to be the collection $\{T_1, \ldots, T_k\}$ consisting of the rational types $T_j \neq \emptyset$ of its k distinct repelling or parabolic fixed points. Every fixed point portrait for a monic degree d polynomial with connected Julia set must satisfy the following four conditions:

- $\mathbf{P_1}$. Each T_j is a rotation set, with a well defined rotation number p_j/q_j.
- $\mathbf{P_2}$. The T_j are disjoint and pairwise unlinked (i.e., contained in disjoint intervals).
- $\mathbf{P_3}$. The union of those T_j which have rotation number zero is precisely equal to the set of angles which are fixed by the d-tupling map.
- $\mathbf{P_4}$. Each pair $T_i \neq T_j$ with non-zero rotation number must belong to different connected components of the complement $(\mathbb{R}/\mathbb{Z}) \smallsetminus T_\ell$ for some T_ℓ with rotation number zero.

In the case $k = d$, where there are d distinct repelling fixed points, Goldberg and Milnor prove that these conditions are necessary and sufficient. (In fact, even when $k < d$ there always exists a post-critically finite polynomial with the given fixed point portrait. Compare Poirier [1991], Hu and Jiang [1994].)

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[11] Compare the somewhat similar concept of "***critical portrait***", as described for example in Bielefeld, Fisher and Hubbard [1992]. However, the fixed point portrait is a much cruder invariant. For example, in the quadratic case, any two points in the same limb of the Mandelbrot set will have the same fixed point portrait.

The discussion makes use of a basic construction which separates the various fixed points. Let f be a polynomial of degree $d \geq 2$ which is required to satisfy the following very weak hypothesis: ***Each of the*** $d-1$ ***external rays which are*** f***-invariant must land at some*** (***necessarily fixed***) ***point in the Julia set.*** (Note that this condition is always satisfied if the Julia set is connected.)

> **Theorem.** *If this condition is satisfied, then these* $d-1$ *invariant rays, together with their landing points, divide the plane into finitely many "**basic regions**", each of which has either exactly one fixed point, or exactly one* f*-invariant parabolic basin* (*but not both*)*, in its interior.*

(For a more conceptual proof which extends this result to some transcendental cases, see Benini and Fagella [2011].) To illustrate the force of this statement, consider the following.

> **Corollary (Poirier).** *If the Julia set of* f *is connected, then there cannot be any Cremer point which lies on the boundary of a bounded Fatou component of* f.

It would probably be quite difficult to prove this statement directly; but it follows easily from the theorem above. In the special case where the Cremer point z_0 is a fixed point, and the basin $U = f(U)$ is f-invariant, it follows immediately. (Note that U must be either an attracting or parabolic basin, or a Siegel disk.) The general case then follows, since one can choose some iterate $g = f^{\circ n}$ so that $z_0 = g(z_0)$ lies on the boundary of $g(U) = g(g(U))$. □

The last paper in this collection, ***"The mathematical work of Curt McMullen"***, provides a brief description of the work in complex dynamics, hyperbolic geometry and Teichmüller theory, for which McMullen was awarded the Fields Medal in 1998.

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