$\mathbb{R}$EAL and $\mathbb{C}$OMPLEX DYNAMICS

Introduction

This volume of the "Collected Papers of John Milnor" is the seventh in the series and the second dedicated to his work in dynamical systems. The emphasis here is on papers in holomorphic dynamics, however there are two on real dynamics, and the first paper is focused on cellular automata.

Below are short introductions to each of these papers, with an attempt to describe the state of the art of the subjects studied by each one.

Notes on surjective cellular automaton-maps

(Manuscript of 1984, not previously published.)

In the 1980s, Stephen Wolfram introduced John Milnor to the study of ***cellular automata***. The present paper, written and privately circulated in 1984, and mildly modified for its publication here, is a survey of old and new results about one dimensional[1] cellular automata, emphasizing the surjective case. It relies strongly on the classical survey by HEDLUND [1969], which summarizes the unpublished work of many different mathematicians.

Let K be a finite "alphabet" with $k \geq 2$ elements, and let $K^{\mathbb{Z}}$ be the Cantor set consisting of all two-sided infinite sequences of elements of K. A continuous mapping $\mathbf{f} : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is called a ***cellular automaton-map*** if it commutes with the shift map. Equivalently, if $\mathbf{f}(\mathbf{a}) = \mathbf{b}$, then every entry b_j must be determined by a finite block of elements from the sequence $\mathbf{a}$ by a formula of the form

$$b_j \;=\; F(a_{j+c_0}\,,\; a_{j+c_0+1}\,,\; \ldots\,,\; a_{j+c_0+s})\;, \tag{1}$$

where c_0 and $s \geq 0$ are fixed integers. Here the associated ***block map*** F can be a completely arbitrary function from K^{s+1} to K.

Define the ***degree***[2] $d(\mathbf{f})$ to be the cardinality of the set $\mathbf{f}^{-1}(\mathbf{b})$ for a generic choice of $\mathbf{b}$. This is an integer in the range $0 \leq d(\mathbf{f}) \leq k^s$, satisfying the identity $d(\mathbf{f} \circ \mathbf{g}) = d(\mathbf{f})\, d(\mathbf{g})$. The map $\mathbf{f}$ is ***surjective*** if and only if it preserves the standard Bernoulli measure, or if and only if $d(\mathbf{f}) \geq 1$. Milnor notes that the overwhelming majority of block maps correspond to automaton maps which are

[1] Milnor's previously published papers on the subject: *"Directional entropies of cellular automaton-maps"* and *"On the entropy geometry of cellular automata"*, both in volume VI of this series, are concerned with higher dimensional automata.

[2] Hedlund, following Gleason and Welch, called this number M, and did not use the term "degree".

not surjective. On the other hand, if we consider only surjective maps, then he notes that the overwhelming majority have degree one, but are not bijective.

Following Boyle and Kreger [1987], Milnor defines the ***mean shift*** $v(\mathbf{f})$ of a surjective automaton-map. This is a real number satisfying the identity

$$v(\mathbf{f} \circ \mathbf{g}) \;=\; v(\mathbf{f}) \,+\, v(\mathbf{g})\,,$$

normalized so that the mean shift of the right shift map is equal to $+1$.

The topological and measure theoretic entropies of a cellular automaton-map are interesting invariants, but can be difficult to compute.[3] If we choose the constants c_0 and s of Equation (1) so that $c_0 \le 0 \le c_0 + s$, then it is not hard to check that

$$0 \;\le\; h_{\rm top}(\mathbf{f}) \;\le\; s\,\log(k)\,.$$

(In the paper, the factor of $\log(k)$ is eliminated by using logarithms to the base k.) In the surjective case, using the invariant Bernoulli measure, the measure theoretic entropy is also defined, and is less than or equal to the topological entropy.

Call an automaton-map $\mathbf{f}(\mathbf{a}) = \mathbf{b}$ ***left determined*** (respectively ***right determined***) if each entry b_i is uniquely determined by those a_j with $j \le i$ (respectively $j \ge i$). If an invertible map $\mathbf{f}$ is left determined and $\mathbf{f}^{-1}$ is right determined, then Milnor (again making use of the work of Boyle and Krieger) shows that both the topological entropy and the measure theoretic entropy are equal to the product $v(\mathbf{f})\log(k)$. The paper closes with a number of worked out examples.

Later Developments: For entropy computations in special cases, see for example: D'Amico, Manzini and Margara [2003], Akin [2003, 2008] and Margara and Manzini [1998]; and for entropies of automorphisms of a more general topological Markov shift, see Lind [1987].)

Tsujii's monotonicity proof for real quadratic maps

(Manuscript of 2000, not previously published.)

In this paper Milnor reformulates a proof of Masato Tsujii in the language of quadratic differentials, making it more accessible to the reader. The paper of Tsujii [2000], shows that:

> *The topological entropy of the real quadratic polynomial* $Q_c(x) = x^2 + c$ *varies monotonically with respect to the parameter* $c \in \mathbb{R}$.

Tsujii's proof uses a spectral property of a Ruelle operator. This result was originally stated by Douady, Hubbard and Sullivan (unpublished), and proved by means of kneading theory by Milnor and Thurston [1988], see Theorem 13.1 and Corollary 13.2 of their paper. There have been other proofs of this result; see for example Douady and Hubbard [1984-85], de Melo and van Strien [1993], and Douady [1995] (which also includes an unpublished proof by Sullivan). All of these rely on ideas from holomorphic dynamics. Tsujii's proof uses complex methods in a quite different way. As far as I know, there has been no completely real proof.

[3] Milnor tells me that he spent quite a bit of time trying to find a general method of computation, without success. Later Hurd, Kari and Culik [1992] showed that the computation is algorithmically impossible in the general case.

Local connectivity of Julia sets: Expository lectures

From: "The Mandelbrot set, theme and variations," Tan Lei (ed.).
London Math. Soc. Lecture Note Ser., **274**,
Cambridge Univ. Press, Cambridge, (2000) 67–116.

This is an expository paper describing the work of several mathematicians concerning local connectivity of polynomial Julia sets. The original version has been augmented (in Appendix B) by material from a lecture given in Cergy in 2006, in celebration of Douady's 70th birthday. The paper is divided into three nearly independent parts:

Section 2 describes part of the Yoccoz theory for quadratic polynomials. More explicitly, it shows that the Julia set $J(f_c)$ is locally connected if f_c belongs to the Mandelbrot set, is not renormalizable, and provided that both of the fixed points of f_c are repelling. Milnor's exposition follows the technique pioneered by Branner and Hubbard [1992], rather than the somewhat different argument of Yoccoz, which has never been published. (Compare Hubbard [1993] for a more complete presentation.)

Section 3 describes the original Branner-Hubbard theory, which is concerned with polynomials of degree three or more such that all but one of the critical orbits diverges to infinity. In this case, the Julia set can never be locally connected. In fact, it is either totally disconnected, or else contains countably many copies of an associated quadratic Julia set.

Section 4 describes examples of non-locally connected quadratic Julia sets which are infinitely renormalizable, based on unpublished work of Douady and Hubbard. (Compare Sørensen [2000], as well as Levin [2009].) Milnor uses the notation[4]

$$H\triangleright\ :\ M\ \to\ (H\triangleright M)\ \subset\ M$$

for the Douady-Hubbard ***tuning*** operation, a canonical homeomorphism from the Mandelbrot set onto a small copy of itself, associated with a hyperbolic component $H\subset M$. Thus a point $(f)\in M$ is ***infinitely renormalizable*** if and only if it belongs to the nested intersection

$$\bigcap_k (H_1\triangleright H_2\triangleright\cdots\triangleright H_k\triangleright M)$$

of successively smaller copies of the Mandelbrot set associated with some sequence of hyperbolic components H_j, each of period ≥ 2. Suppose in particular that each H_j is a satellite, attached to the main cardioid at internal angle $t_k=n_k/p_k$. The main theorem asserts that if the sequence $\{t_k\}$ tends to zero sufficiently fast, then this nested intersection consists of a single point (f), such that $J(f)$ is not locally connected.

An alternative and more detailed version of this material is presented in Appendix B. One notational difference is that Appendix B works rather with the "rounded Mandelbrot set" $\widehat{M}$, which can be defined for example as the set of multipliers μ in the half-plane $\Re(\mu)\leq 1$ such that the Julia set of the map $w\mapsto w^2+\mu w$ is

[4] In the original version of this paper, as published in 2000, the notation $H*M$ was used.

connected. One additional feature of this appendix is a “guess” that the t_k tend to zero fast enough for the proof to work if and only if

$$\sum |t_k|^{1/p_k} \;<\; \infty\ .$$

See LEVIN [2009, Theorem 2] for precise conditions, which include this inequality as an essential part. However, the problem of giving necessary and sufficient conditions for local connectivity of an infinitely renormalizable quadratic map seems to be still open.

The case of general polynomial maps with connected Julia sets and without irrationally neutral cycles has been studied in KIWI [2004]. He proves that in this case, the local connectivity of the Julia set is equivalent to the non-existence of wandering continua.

The situation in the case of rational maps is more complicated, and very different from the polynomial case. In ROESCH [2008], examples of maps with a locally connected Julia set but with a *Cremer point* or, with a *wandering continuum on the Julia set*, are given. (As shown by Kiwi, in the case of polynomial maps, these properties assure that the Julia set is non-locally connected.) For an update of the state of the art in the subject see DEZOTTI AND ROESCH [2014].

On rational maps with two critical points

Experiment. Math., **9** (2000) 481–522.

In this paper, Milnor studies rational maps from the Riemann sphere to itself of degree $n \geq 2$ which are ***bicritical***, in the sense of having exactly two different critical points. Every rational map of degree $n = 2$ is bicritical;[5] but for $n > 2$, the restriction of bicriticality is a strong one. He shows that the ***moduli space*** $\mathcal{M} = \mathcal{M}_n$, consisting of all conjugacy classes of degree n bicritical maps, is biholomorphic to $\mathbb{C}^2$. This *moduli space* can be described as the disjoint union of two sets: the ***connectedness locus*** $\mathcal{C}$, consisting of maps with connected Julia set; and the ***shift locus*** $\mathcal{S}$, consisting of maps for which the Julia set is totally disconnected. Milnor points out that this connectedness locus $\mathcal{C}$ contains an “essentially non-polynomial-like” region $\mathcal{C}_{NP}$, which he conjectures is a topological 4-cell. He gives a proof of this fact in the degree two case.

To be able to understand the limiting behavior as the rational maps become degenerate, Milnor introduces a compactification $\overline{\mathcal{M}}$ of the moduli space $\mathcal{M}$. This is a projective variety which consists of $\mathcal{M}$ together with a line of “points at infinity” L_∞, and together with a single additional point $\widehat{\infty}$ which is singular when $n > 2$. (CAUTION: the notation in the original version, as published on 2000, is slightly different.) The nonsingular subset $\overline{\mathcal{M}} \smallsetminus \{\widehat{\infty}\}$ can be described as a complex line bundle over the Riemann sphere, with Chern number equal to $n-1$. The projection map carries each fiber L_X to an invariant $X \in \widehat{\mathbb{C}}$ which can be described (up to sign) as a cross-ratio of critical points and critical values.

Milnor defines the curve $\mathrm{Per}_k(\lambda) \subset \overline{\mathcal{M}} \smallsetminus \{\widehat{\infty}\}$ as the set of conjugacy classes of maps with a periodic point of period k and multiplier λ. For $\lambda \neq 0$ this curve

[5]For more on the quadratic case, see for example REES [1992], MILNOR [1993], and EPSTEIN [2000].

forms a holomorphic section of the line bundle, but for $\lambda = 0$ it is rather a fiber L_0 of the line bundle, consisting of the conjugacy classes of unicritical polynomials $z \mapsto z^n + c$. (The fiber L_{-1} also has an immediate dynamic interpretation, as does L_∞, but it is not known whether any other fiber can be characterized dynamically.) Milnor also studies the *real subspace* of moduli space.

Later Developments: For a study of bicritical rational maps with a cluster cycle[6] see SHARLAND [2013], and for an extension of Milnor's results to more general fields, see SILVERMAN [1998, 2007].

Periodic orbits, externals rays and the Mandelbrot set: An expository account

Géométrie complexe et systémes dynamiques (Orsay, 1995).
(Dedicated to Adrien Douady)
Astérisque **261** (2000) 277–333.

DOUADY AND HUBBARD [1984-85] developed the combinatorial structure of the Mandelbrot set based on the study of *external rays* and their landing points. An important theorem in their theory is the following.

> *Every parabolic point $c \neq 1/4$ in the Mandelbrot set M, is the landing point for exactly two external rays $\mathcal{R}^M_{t_-}$ and $\mathcal{R}^M_{t_+}$ in parameter space, where the angles $t_\pm \in \mathbb{Q}/\mathbb{Z}$ are periodic under doubling.*

Inspired by the thesis of SCHLEICHER [1994] but with a different approach, Milnor's paper gives new proofs of the above and other theorems that play a fundamental role in the Douady and Hubbard theory. An essential ingredient of these proofs is the use of ***orbit portraits***. Let $\mathcal{O} = \{z_1, \ldots, z_p\}$ be a periodic orbit for a polynomial map f. Suppose that a dynamic external ray $\mathcal{R}^{K(f)}_t$ with rational angle t lands at some point of $\mathcal{O}$. The ***orbit portrait*** $\mathcal{P} = \mathcal{P}(\mathcal{O})$ is the set $\{A_1, \ldots, A_p\}$ where each A_i consists of all angles of dynamic rays which land at the orbit point z_i. The A_i are disjoint finite subsets of $\mathbb{Q}/\mathbb{Z}$, all with the same cardinality. This common cardinality is called the ***valence*** $v \geq 1$ of the portrait $\mathcal{P}$.

Consider a polynomial map f_{c_0} having an orbit with portrait $\mathcal{P} = \mathcal{P}(\mathcal{O})$ of valence $v \geq 2$. Then the v external rays landing at a given orbit point z_i divide the dynamic plane into v "sectors" with total angular width (as measured around the circle at infinity) equal to one. Among the pv sectors at the various orbit points, there is a unique sector of shortest angular width[7] called the ***critical value sector***. Let $0 < t_- < t_+ < 1$ be the angles of the rays bounding this critical value sector (so that the difference $t_+ - t_-$ is its angular width). A fundamental theorem asserts that:

[6]Let f be a bicritical rational map with the property that the two critical points belong to disjoint superattracting orbits with the same period. A ***cluster point*** of f is a repelling periodic point which is a common boundary point for the immediate basins of both critical cycles. A ***cluster*** is the union of the cluster point and the Fatou components meeting at it.

[7]Compare the discussion of the "minor leaf" of a quadratic invariant lamination in THURSTON [2009].

The corresponding parameter rays $\mathcal{R}^M_{t_-}$ *and* $\mathcal{R}^M_{t_+}$ *land at a single point* $\mathbf{r}_\mathcal{P}$ *of the parameter plane.*

These two rays together with their common landing point, divide the parameter plane into an open subset $W_\mathcal{P}$ of angular width $t_+ - t_-$ and a complementary set $\mathbb{C} \setminus W_\mathcal{P}$. By definition, this set $W_\mathcal{P}$ is called the ***$\mathcal{P}$-wake*** and $\mathbf{r}_\mathcal{P}$ is called its ***root point***. *The* ***$\mathcal{P}$-wake*** *consists of all parameters* c *such that the map* f_c *has a (necessarily unique) orbit with portrait* $\mathcal{P}$.

The set of all such wakes is partially ordered by inclusion. (In fact any two wakes are either nested or disjoint.) Note that $W_{\mathcal{P}_1} \subset W_{\mathcal{P}_2}$ if and only if each quadratic polynomial with an orbit of portrait $\mathcal{P}_1$ must also have an orbit with portrait $\mathcal{P}_2$. The intersection of $\overline{W}_\mathcal{P}$ with the Mandelbrot set M is called the ***$\mathcal{P}$-limb*** $M_\mathcal{P} \subset M$. Each such limb is connected, even if we remove its root point $\mathbf{r}_\mathcal{P}$. Furthermore, each such limb contains a unique hyperbolic component $H = H_\mathcal{P}$ of smallest period, and having $\mathbf{r}_\mathcal{P}$ as root point. Here $\mathcal{P}$ and H determine each other uniquely. The paper also contains a discussion of the combinatorics of the Douady-Hubbard ***tuning*** operation. In particular, each limb $M_\mathcal{P}$ contains a corresponding "tuned copy" $\mathcal{P} \triangleright M$ of the Mandelbrot set.

For more on the study of parameter rays of the Mandelbrot set see SCHLEICHER [2000]. For an elementary proof of the landing theorem of Douady-Hubbard for external parameter rays, see PETERSEN AND RYD [2000]. For a study of the set of angles of the parameter rays which land on the real slice $[-2, 1/4]$ of the Mandelbrot set and for a proof that this set has zero length but Hausdorff dimension one, see ZAKERI [2003]. A point $z \in J$ is ***accessible*** if there exists an external ray landing at z, and ***biaccessible*** if it is *accessible* through at least two distinct external rays. Douady and Hubbard conjectured that *for general polynomials with connected Julia set, the set of biaccessible points has Brolin measure zero unless the Julia set is an interval.* A special case of this conjecture is proved in ZAKERI [2000], while the general case is proved in ZDUNIK [2000].

Pasting together Julia sets: a worked out example of mating

Experiment. Math., **13** (2004) 55–92.

Given two monic polynomial maps $p, q : \mathbb{C} \to \mathbb{C}$ of the same degree $d \geq 2$, the ***mating*** operation, introduced by DOUADY [1983], attempts to construct a rational map of degree d by gluing together the filled Julia sets of p and q along their boundaries. In this paper, Milnor describes the classical definition of mating, discusses its subtleties, and makes a careful study of the self-mating of $f(z) = z^2 + c_0$, where c_0 is the landing point of the 1/4-ray in parameter space. This example seems quite paradoxical since it constructs a 2-dimensional Riemann sphere by pasting together two 1-dimensional dendrites. However it can be analyzed quite precisely, since the rational map obtained, in addition to its description as a mating, arises as a *Lattès example.* Besides the careful study of the self-mating of this map, Milnor presents in this paper quite a bit of foundational material regarding matings, Lattès examples, and what he calls *Lyubich measures*[8], with careful attention to computational issues. There are four appendices which discuss the Weierstrass $\wp$-function, the classification of quadratic Lattès examples which arise as matings, a recipe for computing external angles given Hubbard trees, and exotic topological conjugacies between quadratic polynomials.

This work inspired further work of D. Meyer, see MEYER [2009, 2013(a)].

REMARK. There is a serious problem in deciding when the mating operation is possible and when it is uniquely defined. In the case of postcritically finite quadratic polynomials, studied by REES [1992], TANLEI [1992], and SHISHIKURA [2000], it turns out that mating is always uniquely possible when the associated points of the Mandelbrot set do *not* belong to complex conjugate limbs, but that it is never possible otherwise. The "*Quadratic Mating Conjecture*" stated in MILNOR [1993], and revisited in the present paper, asks whether this statement remains true without assuming postcritical finiteness, at least in the locally connected case. One important test of this conjecture would be the problem of mating two polynomials with indifferent fixed points of multiplier μ and ν. The conjecture would imply that this mating is possible if and only $\mu\,\nu \neq 1$. (There is one anomalous exception, since there is a unique mating when $\mu = \nu = 1$.) For the parabolic case, see HAISSINSKY AND TAN LEI [2004], and for the case of Siegel disks with rotation number of bounded type, see YAMPOLSKY AND ZAKERI [2001]. Adam Epstein has shown that the mating of quadratic polynomials does not depend continuously on the two polynomials: Compare the discussion by Pilgrim and Tan Lei in [ROESCH, 2012, p. 1160].

Douady's definition applied only to polynomials with locally connected Julia sets. A more robust definition, which required only connected Julia sets, was outlined in MILNOR [1993]. This suggested definition, now called ***slow mating*** , was not developed for many years. But recently, through the work of Buff and Chéritat and others, it has proved quite useful. The basic idea is to replace each filled Julia set by a neighborhood bounded by an equipotential, then glue these neighborhoods together along their boundaries, and finally to take the limit as the equipotential neighborhood shrinks to the filled Julia set. For rather amazing movies illustrating slow mating, see

`http://www.math.univ-toulouse.fr/~cheritat/MatMovies/`

For the different notions of mating see MEYER AND PETERSEN [2012], and for the state of the art in this subject see volume XXI of the "Annales de la Faculté des Sciences de Toulouse, Série 6, Fascicule 5," edited by P. Roesch, which is a veritable "*Kama Sutra*" of mating. For unmatings of rational maps, see MEYER [2014].

[8]A ***Lyubich measure*** is a generalization for rational maps from the Riemann sphere to itself of degree $d \geq 2$ of *Brolin's measure*. (See also FREIRE, LOPES AND MAÑÉ [1983] and LYUBICH [1983].) For a generalization to the case of Hénon maps see FORNÆSS AND SIBONY [1992] and for birational maps see FAVRE AND GUEDJ [2001]. For a proof of Brolin's theorem for preimages of points under holomorphic maps of $\mathbb{P}^k$ see BRIEND AND DUVAL [2001]. See also FAVRE AND JONSSON [2003].

On Lattès maps

In: "Dynamics on the Riemann sphere,"
(Dedicated to Bodil Branner)
P. Hjorth and C. L. Petersen (editors) Eur. Math. Soc., Zürich, (2006) 9–43.

In the paper ***On Lattès maps***, Milnor defines a ***Lattès map*** as a rational map f of degree $d \geq 2$ from the Riemann sphere to itself which is a quotient of an affine torus map. More precisely, there should exist an affine map $L(w) = aw + b$ from some torus $\mathcal{T} = \mathbb{C}/\Lambda$ onto itself, and a holomorphic map Θ from this torus onto the Riemann sphere which satisfies $f \circ \Theta = \Theta \circ L$. Here $\Lambda \subset \mathbb{C}$ can be any rank two lattice.

He provides a complete classification of Lattès maps up to conformal conjugacy. This classification is determined by four invariants. Milnor proves that since Lattès maps are postcritically finite, in the absence of exceptional points, they can be characterized by having parabolic orbifolds, or being linearized by a flat metric with singularities at the postcritical points. He gives a precise description of the possible crystallographic groups[9] $\widetilde{G}$ acting on $\mathbb{C}$, and of the corresponding orbifold geometries on $\mathbb{C}/\widetilde{G}$. Milnor calls a Lattès map ***flexible*** if it admits non-trivial deformations. He proves that a given Lattès map is *flexible if and only if* $n = 2$, *and the affine map* $L(\tau) = a\tau + b$ *has integer derivative* a. Thus flexible Lattès maps are precisely those rational maps which are double covered by an integral torus endomorphism. A Lattès map is flexible if and only if the multiplier of every periodic orbit is an integer. He notes that the family of all flexible Lattès maps of degree a^2 is connected when a^2 is even but that there are two connected components when a^2 is odd.

There is an important classical conjecture associated with such maps:

> **Fundamental conjecture:** *The flexible Lattès maps are the only rational maps* f *with a Julia set of positive measure which admits a measurable* f*-invariant line field.*

Mañé, Sad and Sullivan [1983] showed that if this conjecture holds it would imply the *density of hyperbolicity for rational maps,* that is, that every rational map can be approximated by a *hyperbolic map.*

Towards the end of the paper Milnor discusses recent developments in the subject, and ends the paper with a detailed analysis of some examples of Lattès and flexible Lattès maps.

For a characterization of Lattès maps using only measure theoretic properties, see Zdunik [1990], and for a generalization of these results to higher dimensions, see Berteloot and Dupont [2005]. For a characterization of Lattès maps in terms of the dynamics and geometry of the associated homogeneous polynomial map of $\mathbb{C}^2$, see Berteloot and Loeb [1998]. For a generalization of this results to higher dimensions see Berteloot and Loeb [2001] as well as Dupont [2003].

[9]Let $E(n)$ denote the group of rigid complex motions acting on $\mathbb{C}^n$ for any $n \geq 1$. A ***complex crystallographic group*** is, by definition, a discrete subgroup of $E(n)$ with compact quotient.

For a proof that *every flexible Lattès map can be approximated by strictly postcritically finite rational maps which are not Lattès maps*, see Buff and Gauthier [2011]. For a characterization of Lattès maps by their combinatorial expansion, see Yin [2011]. For a classification of Lattès maps in $\mathbb{P}^2$, see Rong [2010]. For various algebra-dynamical characterizations of Lattès maps in different contexts, see Cantat [2008].

Elliptic curves as attractors in $\mathbb{P}^2$, I: Dynamics

(with A. Bonifant and M. V. Dabija).
Experiment. Math., **16** (2007) 385–420.

In this paper, written in collaboration with Araceli Bonifant and Marius Dabija, Milnor and his coauthors explore the dynamical features of those rational maps f of the real or complex projective plane of degree $d \geq 2$, with a curve $\mathcal{C} = f(\mathcal{C})$ of genus one as invariant subset. Many such maps had been constructed by Bonifant and Dabija [2002]. Maps in one particularly interesting family were given the name of ***Desboves maps***. However the present paper uses the more accurate term ***modified Desboves maps*** since they arise from a perturbation of the *classical Desboves map* (Desboves [1886]), which is dynamically quite boring. The main objective of the Bonifant-Dabija-Milnor paper is the study of the extent to which, an invariant genus-one curve $\mathcal{C} \subset \mathbb{P}^2$ can be an "*attractor*." Note that such a curve always has a dense orbit. The authors distinguish three degrees of attraction. The elliptic curve $\mathcal{C}$ is called:

a ***trapped attractor*** if $\mathcal{C}$ has a compact trapping neighborhood;

a ***measure-theoretic attractor*** if $\mathcal{C}$ has an attracting basin of positive measure;

a ***global attractor*** if $\mathcal{C}$ is a measure-theoretic attractor whose basin of attraction has full measure.

A very important tool in the Bonifant-Dabija-Milnor paper, is the ***transverse Lyapunov exponent*** $\mathrm{Lyap}_{\mathcal{C}}$ along the invariant genus-one curve $\mathcal{C}$. *The sign of this exponent is the main indicator of whether or not this elliptic curve is a measure-theoretic attractor.* The authors provide examples both in the real and complex case in which the curve $\mathcal{C}$ is a measure theoretic attractor. (Methods to actually compute this exponent are given in Bonifant, Dabija and Milnor [2014].)

A rational self-map f of the real or complex projective space $\mathbb{P}^2$ with $\deg(f) \geq 1$ is called ***elementary*** with *center* ρ_0 if it leaves invariant the pencil of lines passing through ρ_0. In the special case of elementary maps, Bonifant-Dabija-Milnor were able to prove that: *if the transverse Lyapunov exponent along the elliptic curve $\mathcal{C}$ is negative, then its attracting basin has positive measure;* conjecturing in the general case that:

> *an f-invariant elliptic curve $\mathcal{C}$ is a measure-theoretic attractor if and only if the transverse Lyapunov exponent along $\mathcal{C}$ is negative*

The "if" part of this conjecture was later verified by Taflin [2010].

Bonifant-Dabija-Milnor give examples of *elementary maps* in the modified Desboves family displaying the phenomena of ***intermingled basins***, that is two disjoint basins of positive measure with the same topological closure. (Compare Alexander, Kan, Yorke and You [1992].)

The authors also study elliptic curves of higher degree. They give examples of *singular real* quartic curves of genus-one that are trapped attractors under suitable rational maps. However, they prove that:

A complex genus-one curve can never be a trapped attractor.

In the case of *real modified Desboves maps*, they provide examples where there appear to be attracting invariant circles which deform smoothly under perturbation on the map. (Since the rotation number varies under perturbation, there must actually be gaps corresponding to each rational rotation number.) Each such attracting circle appears to be the real slice of an attracting Herman ring. One of the last sections of this paper provides a more general discussion of one complex dimensional Herman rings or Siegel disks in $\mathbb{P}^2$. For such "rotation curves," the transverse Lyapunov exponent is no longer constant, as it was in the case of an elliptic curve, but is rather a convex piecewise linear function on the ring or disk, constant on each invariant circle. In the region where this exponent is negative (respectively positive), the rotation curve is uniformly attracting (or repelling).

The paper concludes with an outline of open problems.

Schwarzian derivatives and cylinder maps

(with A. Bonifant)

In: "Holomorphic dynamics and renormalization,"
M. Lyubich and M. Yampolsky (editors). Fields Inst. Commun., **53**,
Amer. Math. Soc., Providence, RI, (2008) 1–21.

In this paper Bonifant and Milnor study the dynamics of a class of skew product maps F from the cylinder $\mathscr{C} = (\mathbb{R}/\mathbb{Z}) \times I$, to itself, where $I = [0,1]$. These maps have the form $F(x,y) = \big(kx\,,\ f_x(y)\big)$, where $k \geq 2$ is an integer and each $f_x : I \to I$ is a diffeomorphism which fixes the two endpoints of the interval. As in the previous paper, the authors are interested in the attracting or repelling properties of the object under study. In this case, there are two invariant circles (the two boundary circles), and each one has a well defined ***transverse Lyapunov exponent***. In particular, they study the case where the ***Schwarzian derivatives*** of the maps f_x have constant sign, and prove that:

The sign of the Schwarzian derivative of f_x is equal to the sign of the sum of the Lyapunov exponents of the two boundary circles.

Moreover, when *the Schwarzian derivative is negative and the transverse Lyapunov exponents of both boundary circles are also negative,* the authors prove that *the basins of attraction of the two boundary circles are intermingled*. This result generalizes the work of Kan [1994].

In the case when the *Schwarzian derivative is positive and both Lyapunov exponents are strictly positive* Bonifant and Milnor prove that the map F *admits* *an asymptotic measure ν. In other words, there is a uniquely defined probability measure ν on $\mathscr{C}$ such that Lebesgue almost every orbit is **uniformly distributed** with respect to ν.*

Finally, in the case of *zero Schwarzian derivative*, the authors prove in some cases and conjecture in others that almost all orbits spend most of the time close to one of the two cylinder boundaries; but that each such orbit passes from the ϵ-neighborhood of one boundary circle to the ϵ-neighborhood of the other infinitely many times on such an irregular schedule that there is no asymptotic measure.

For a related study of this phenomenon see Ilyashenko [2008, 2010].

Cubic polynomial maps with periodic critical orbit, I

In: "Complex Dynamics: Families and Friends,"
(Dedicated to John Hamal Hubbard)
D. Schleicher (ed.) A. K. Peters, Wellesley, MA (2009) 333–411.

This paper has influenced the study of the complicated two-dimensional parameter space for cubic polynomials for many years. A preliminary draft was circulated in 1991, but the paper was not published until 2009, when Dierk Schleicher persuaded Milnor to include it in the volume dedicated to John Hubbard, making it available to a wider audience in the field.

In this paper Milnor continues the study of the complex two-dimensional parameter space of cubic polynomials maps, initiated by the papers of Branner [1986, 1993] and Branner and Hubbard [1988, 1992]. Milnor undertakes this investigation by looking at complex one-dimensional slices of this parameter space, studying cubic maps with a superattracting orbit of period $p \geq 1$. He begins the paper with the case $p = 1$, studying the dynamics of a cubic polynomial map F which has a superattracting fixed point.

He works with the following normalization of the map $F : \mathbb{C} \to \mathbb{C}$,

$$F(z) \;=\; F_{a,v}(z) \;=\; z^3 \,-\, 3a^2 z \,+\, (2a^3 + v)\,,$$

where a is the *marked critical point*, and $v = F(a)$, the corresponding *critical value.* He goes on to define the ***superattracting period p-curve*** $\mathcal{S}_p$ as the locus of all polynomial maps $F = F_{a,v}$ for which the *marked critical point* a, is periodic of period p. In particular, $\mathcal{S}_1$ will stand for the set of maps for which the *marked critical point* a, is fixed. The set of parameters $(a, v) \in \mathcal{S}_p$ for which the Julia set of $F_{a,v}(z)$ is connected is called the ***connectedness locus*** of $\mathcal{S}_p$, and it is denoted by $\mathcal{C}(\mathcal{S}_p)$. In the first sections of this paper Milnor gives a careful study of the geometry and topology of $\mathcal{S}_1$. (Other studies in this direction have also been performed by Faught [1992], who proved that the boundary of the immediate attracting basin of the superattracting point a is a simple closed curve; and by Roesch [2007] who simplified Faught's argument. With this result in hand and by means of a renormalization argument P. Roesch proves that the boundary of every hyperbolic component of the connectedness locus of $\mathcal{S}_1$ is a Jordan curve.)

The penultimate section of the paper studies the geometric and topological properties of the curve $\mathcal{S}_p$ for $p \geq 1$. Milnor proves that

Each $\mathcal{S}_p$ is a smooth affine algebraic curve.

(It is conjectured that all of these curves are connected, or equivalently irreducible; but this has been proved only for $p \leq 4$.)

Milnor defines the ***escape locus***, as the complement $\mathcal{S}_p \smallsetminus \mathcal{C}(\mathcal{S}_p)$, consisting of maps for which one critical orbit escapes to infinity. He calls each connected component of this complement an ***escape region*** $\mathcal{E} \subset \mathcal{S}_p$. (These escape regions are studied in the sequel to this paper, as described below.) Milnor gives a recursive formula to compute the degree of the curve $\mathcal{S}_p$, and proves that this degree is asymptotic to 3^{p-1}. He defines the ***multiplicity*** of each escape region, and proves that the number of escape regions in $\mathcal{S}_p$, counted with multiplicity, is equal to the degree of $\mathcal{S}_p$. He describes two preliminary invariants which help to distinguish the different escape regions in $\mathcal{S}_p$. The first is the ***kneading sequence***, a sequence of zeros and ones with period q dividing p. The second is the ***associated quadratic map*** (as described in Branner and Hubbard [1992]) which is a critically finite quadratic map of period n dividing p.

The last section of Milnor's paper studies quadratic Julia sets in the cubic parameter space. (See also Buff and Henriksen [2001].) The author also includes an appendix on Hubbard trees.

For studies of other families of polynomial maps see for example Roesch [2010], Avila, Kahn, Lyubich and Shen [2009], Arteaga and Alves [2013], and Silverman [2012].

Cubic polynomial maps with periodic critical orbit, II: Escape regions

(with A. Bonifant and J. Kiwi)

Conform. Geom. Dyn. **14** (2010) 68–112; erratum pp. 190–193).

In this paper Bonifant-Kiwi-Milnor continue the study of the parameter space $\mathcal{S}_p$ of monic centered cubic polynomials maps with a marked critical point of period p.

One of the main difficulties in the study of $\mathcal{S}_p$ had been the lack of good parametrizations that would allow the drawing of computer pictures that will help to gain some geometrical intuition. In this paper Bonifant-Kiwi-Milnor provide what they call a ***canonical parametrization*** of $\mathcal{S}_p$, given as solutions of a Hamiltonian differential equation. This parametrization is ***local*** and ***uniquely defined up to translation***.

In ***Part I*** of this paper, as discussed above, Milnor started the classification of the escape regions of $\mathcal{S}_p$ in terms of their ***kneading sequences*** and their ***associated quadratic maps***. For periods $p \leq 3$, these invariants suffice to give a complete classification of the escape regions of $\mathcal{S}_p$. However for larger periods p, they only provide a partial classification. In their quest for such invariants Bonifant-Kiwi-Milnor review the *Branner-Hubbard puzzle* and its associated *marked grid*, introducing a *pseudometric* that captures some of the basic properties of the puzzle. Each escape region is associated with an essentially unique ***Puiseux series***, a power series in fractional powers which parametrizes a neighborhood of its point at infinity by expressing the critical value v as a function of the critical point a. More generally, each point in the orbit of a can be expressed as a Puiseux series in a.

The paper of KIWI [2006] makes a study of the dynamics of polynomials over the completion of the field of formal Puiseux series, and shows that it is closely related to classical dynamics. Using non-Archimedean methods, KIWI [2014] proves that: *The Puiseux series associated to an escape region is uniquely determined by the leading monomials of the Puiseux series for the differences* $f^{\circ j}(a) - a$ *associated with the periodic critical orbit.* In more dynamic terms, this means that: *each escape region is uniquely determined by the asymptotic behavior of these differences as* a *tends to infinity.*

In the present paper, Bonifant-Kiwi-Milnor show that the Euler characteristic of the affine curve $\mathcal{S}_p$ is given by the simple formula

$$\chi(\mathcal{S}_p) \;=\; (2 \,-\, p)\,\text{degree}\,(\mathcal{S}_p)\;,$$

and hence is asymptotic to $-p\,3^{\,p-1}$ as $p \to \infty$. The proof is based on a careful study of the relationship between the Puiseux series, the Branner-Hubbard marked grid, and the differential of the canonical parametrization which is a well defined holomorphic 1-form. The Euler characteristic of the non-singular compactification $\overline{\mathcal{S}}_p$ can be described as the sum of $\chi(\mathcal{S}_p)$ and the number of escape regions. No formula is known for the latter number. However, it has been computed numerically for $p \le 26$ by DEMARCO-SCHIFF [2013].

In the special case $p = 4$, Bonifant-Kiwi-Milnor show that $\mathcal{S}_4$ is a (connected) Riemann surface of genus 15 with 20 punctures (corresponding to the 20 escape regions). The proof involves an explicit classification of the escape regions in this case, and an explicit cell structure with the puncture points as vertices. The paper concludes with a discussion of the corresponding problem for real cubic maps.

Hyperbolic components

(WITH AN APPENDIX BY A. POIRIER)

Contemp. Math., **573**, Conformal dynamics and hyperbolic geometry,
(Dedicated to Linda Keen)
Amer. Math. Soc., (2012) 183–232.

This is another paper which was circulated long ago (in 1992), and only recently published. (The published version includes substantial modifications, including a final section on rational maps.)

Milnor uses the notation $\mathcal{P}^d$ for the parameter space of ***monic centered*** complex polynomial maps of degree $d \ge 2$, i.e., polynomials of the form

$$a_d\, z^d \;+\; a_{d-1}\, z^{d-1} \;+\; \cdots \;+\; a_1\, z \;+\; a_0$$

with $a_d = 1$ and $a_{d-1} = 0$. The ***connectedness locus*** $\mathcal{C}^d \subset \mathcal{P}^d$ is the subset consisting of maps for which the filled Julia set is connected. A polynomial or rational map is called ***hyperbolic*** *if the orbit of every critical point converges to an attracting cycle.* The connected components H of the *hyperbolic locus* which are contained in $\mathcal{C}^d$ are called ***bounded hyperbolic components***.

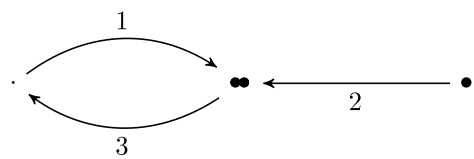
Milnor associates to each $f \in H$ a *mapping scheme* S_f which describes the induced dynamics on the finite set consisting of all critical and postcritical Fatou components. Similarly, the *reduced mapping scheme* $\overline{S}_f$ describes the induced dynamics on the set of critical Fatou components. He shows that all $f \in H$ have *canonically isomorphic* mapping schemes and reduced mapping schemes. Thus one can just as well write S_H or $\overline{S}_H$ in place of S_f or $\overline{S}_f$.

By definition a ***hyperbolic mapping scheme*** S consists of a finite set of "vertices," together with a map F from the set $|S|$ of vertices to itself, and an integer valued ***critical weight function*** $\mathbf{w}(s) \geq 0$ (representing the number of critical points associated with the vertex s), satisfying two conditions:

- *Any vertex of weight zero is the iterated forward image of some vertex of positive weight.*
- (Hyperbolicity condition) *Every periodic orbit under* F *contains at least one vertex of positive weight.*

The scheme is called ***reduced*** if the value of the *critical weight function* at each vertex is always greater or equal to one.

The following diagram shows an example of a hyperbolic mapping scheme with vertices of critical weight zero, two, and one. The label on the arrow from s to $F(s)$ represents the degree of the associated map between Fatou components, and is equal to $w(s)+1$.



Basic results proved in this paper are the following:

Each hyperbolic component is a topological cell with a preferred center point.

Any two hyperbolic components with the same reduced mapping scheme are canonically biholomorphic.

To obtain a canonical model for hyperbolic components with a specified reduced mapping scheme, Milnor first constructs a (non-holomorphic) topological model, with each Fatou component represented by a copy of the unit disk, and each map between Fatou components by a Blaschke product. In order to promote this to a biholomorphic model, he modifies this construction by representing each Fatou component by a copy of $\mathbb{C}$, and each map of degree d between Fatou components by a monic centered polynomial map of degree d.

There follows a section concerning analogous results for polynomial mappings with real coefficients, and more generally for *real forms* of complex polynomial mappings, and a section concerning critically marked polynomial maps.

The last section of this paper deals with the case of rational maps. Since there does not seem to be any good analog for the space of monic centered polynomial maps, he rather works with the moduli space for ***fixed point marked rational maps***, proving analogs of the main results for polynomial maps.

The paper ends with an appendix written by Alfredo Porier, in which he proves that *every reduced mapping scheme can be realized by a postcritically finite polynomial.* To prove this result Porier constructs appropriate Hubbard trees which mimic the dynamics of the scheme.

For an application of this topological model in the study of the combinatorics and topology of straightening maps of degree $d \geq 2$ see Inou and Kiwi [2012].

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