Preface

The present volume contains a selection of 32 articles by Sigurður Helgason, covering various aspects of geometric analysis on Riemannian symmetric spaces. The papers are arranged chronologically and cover a time span of more than 50 years, from 1956 to 2007.

Sigurður Helgason has had a profound impact on geometric analysis, more particularly in the geometry of homogeneous spaces, harmonic analysis on symmetric spaces, the theory of the Radon transform, and the study of invariant differential operators and their eigenfunctions. The collection is preceded by an introduction by Helgason. Here some background and perspective is given on the individual papers, and they are brought into a historical perspective as seen by the author himself today. The introduction organizes the articles according to the following list of topics, which describes the evolution of Helgason's work over the period:

1. Invariant differential operators,
2. Geometric properties of solutions,
3. Double fibrations in integral geometry, Radon transforms,
4. Spherical functions and spherical transforms,
5. Duality for symmetric spaces,
6. Representation theory,
7. Fourier transform on Riemannian symmetric spaces,
8. Multipliers.

Many people know the work of Helgason primarily from his outstanding books. The writing of Sigurður Helgason has been a model of exposition since the publication in 1962 of *Differential Geometry and Symmetric Spaces*, [B1], which quickly became a classic. It was not only used by students and researchers in Lie group theory, the first chapter was-and still is-an excellent introduction to Riemannian geometry. This book, and its successor from 1984, *Groups and Geometric Analysis*, [B6], was awarded the Steele price for expository writing in 1988.

In addition to these two classical and prize-winning books, Helgason has published several other books, notably *The Radon Transform* [B4] and *Geometric Analysis on Symmetric Spaces* [B7]. A complete list of books and articles up to 2008 is enclosed in this volume. The above citations refer to that list.

The exposition in the original papers is of the same outstanding clarity as that in the books. It is our hope that by collecting these papers, we can create renewed attention on all those aspects of his work which are not covered by the books.

We would like to thank Ed Dunne and Cristin Zannella at AMS for their invaluable help to get this project started and going. We also thank the publishers of the original articles for their cooperation and permission to include the articles in this volume.

Gestur Ólafsson and Henrik Schlichtkrull

Biographical note

Sigurður Helgason was born on September 30, 1927, in Akureyri, Iceland. He went to the Gymnasium in Akureyri 1939-1945. After a short period at the University of Iceland in Reykjavík, he began studies at the University of Copenhagen in 1946. Here he received the Gold Medal in 1951 for a paper summarized in [60] and the M.Sc. degree in 1952. He went to Princeton as a graduate student in 1952 and received the ph.D. from there in 1954. He was a Moore instructor at M.I.T. 1954-56, after which he returned to Princeton 1956-57. He was at University of Chicago 1957-59, and at Columbia University 1959-60. Since then he has been at M.I.T., where he was appointed full professor in 1965. The periods 1964-66, 1974-75, 1983 (fall) and 1998 (spring) he spent at the Institute for Advanced Study, Princeton, and the periods 1970-71 and 1995 (fall) at the Mittag-Leffler Institute, Stockholm. He has been awarded honorary doctor by University of Iceland, University of Copenhagen and University of Uppsala. In 1988 he received the Steele prize of the A.M.S. for expository writing. He carries the Major Knights Cross of the Icelandic Falcon since 1991.

He and his wife live in Belmont, Massachusetts. They have a son and a daughter, and 3 grandchildren.

Introduction

I am deeply grateful to the American Mathematical Society and to Gestur Ólafsson and Henrik Schlichtkrull for their interest and their proposal to produce this volume with selections of my mathematical papers. The editors and publisher kindly asked me to write some explanatory comments on these papers.

My interest in mathematics I trace to my mathematics teacher in the Gymnasium in Akureyri in North Iceland, Trausti Einarsson. Although an astronomer, his deep respect for mathematics was highly infectious. The geometry textbooks by the remarkable mathematician Ólafur Danielsson, the pioneering founder of mathematics education in Iceland, were written by a man with a real mission. The program was on a respectable level — for example the final exam, for students at age 17, included finding the radius of a sphere inscribed in a regular dodecahedron of edge length 1. Danielsson, who wrote several beautiful papers in geometry along with his demanding Gymnasium teaching, was profoundly influenced by the rich geometry tradition in Denmark, which started around 1870 with J. Petersen and H. Zeuthen.

When I entered the Gymnasium in Akureyri around 1940, the Icelandic population totaled about 125,000. Thus the Gymnasium program put considerable emphasis on the teaching of several foreign languages. Our French teacher, Thórarinn Björnsson, was particularly memorable because of his outspoken admiration of French authors like Maupassant, Daudet and Rolland, in addition to his fondness for the French language in which he left us with a decent reading ability. Thus my later difficulties in understanding Élie Cartan's papers were not of linguistic nature.

Later, at the University of Copenhagen, I benefited from splendid lectures by Harald Bohr, Børge Jessen and Werner Fenchel. The first two gave substantial courses in complex function theory and real analysis, respectively. With Fenchel I had also a kind of reading course in projective geometry. At that time I obtained a copy of Julius Petersen's remarkable book, "Methods and Theories for Solutions of Geometric Construction Problems" (1866). I worked on many of the 300 problems there. Example of such problems: Construct a triangle ABC from h_a, m_a and r. This was consistent with my expectation later to teach at a Gymnasium in Iceland.

One day I happened to notice a posting of the collection of prize questions which were posed by the university each year. The mathematics problem on the list called for a generalization to analytic almost periodic functions of the value-distribution theory of Ahlfors and Nevanlinna. Everyone under 30 was free to try for a year. A solution was to be submitted and judged anonymously so I could not consult anybody. Thus four months went by before I understood what the problem was about. My paper solving the problem became my masters thesis. A summary is given in paper [60] in this volume.

After completing my thesis in Princeton enjoying supportive and stimulating supervision by Bochner, my interest shifted to Lie group theory with the intention of studying Harish–Chandra's work. A related interest was geometric analysis, having received from my friend Leifur Ásgeirsson the page proofs of Fritz John's famous book *Plane Waves and Spherical Means*. Although the term 'group' does not appear in this book I sensed quickly that some of its themes were ripe for group-theoretic variations.

At an early stage it was a source of inspiration to me how at the hands of Sophus Lie, the study of differential equations led to the concept of Lie groups which in turn went on to penetrate numerous other fields of mathematics. Since Lie group theory had in the mid-fifties become so well developed I became interested in a kind of converse to Lie's program, namely to investigate differential operators and differential equations invariant under known groups. Examples were readily at hand: the Laplace–Beltrami operator on Riemannian or pseudo-Riemannian manifolds, the Dirac operator, the wave equation and its variations.

In the account that follows I describe some of my own work connected to analysis on homogeneous spaces. This is accompanied by brief mention of some prior and subsequent work which is *closely related*. These personal viewpoints do *not* represent any attempt at a historical account, the feature of relevance and of value being highly subjective. As Peter Lax aptly observed: "For a mathematician the central problem is the one he happens to be working on".

Inspired by Harish–Chandra's brilliant work on the representation theory of semisimple Lie groups G, my taste for geometry led me to a related topic, namely a study of the symmetric spaces of É. Cartan. Here enter several differential geometric features: geodesics, curvature, K-orbits, totally geodesic submanifolds and horocycles; for the compact dual spaces U/K we have also closed geodesics, especially those of minimal length, equators, antipodal manifolds, and totally geodesic spheres. Geometric analysis means, in this context, analysis formulated in terms of these geometric objects.

I have had the fortune of teaching for 50 years at M.I.T. but have also enjoyed the privilege of extended, stimulating visits at the Institute for Advanced Study in Princeton and at the Mittag–Leffler Institute in Djursholm, Sweden.

In conclusion I want to emphasize my wife's role in my life and work. Her support during these 50 years has been precious, and her tolerance, at times when nothing but mathematics mattered, has been admirable. She even typed my first two books in pre-TEX times, giving stylistic suggestions along the way.

1. Invariant Differential Operators.

Articles:[1] [6], *[9], [13], *[16], *[23], *[32], *[35], *[37], *[43], [44], [46], [49], [55], *[59], *[63], [64], *[71], *[81], *[82].

Books: [B2], [B6], [B7]

The general project outlined above suggested the task of describing the algebra $\mathbf{D}(G/H)$ of G-invariant differential operators on a homogeneous space G/H. For G/H reductive some preliminaries are done in [9]; for G/H of maximum mobility, namely two-point homogeneous, the algebra is easily shown to be generated by the Laplace–Beltrami operator. For G/K symmetric, Harish–Chandra's results for G implied that $\mathbf{D}(G/K)$ is commutative and could in fact be described rather explicitly. However, contrary to a statement appearing frequently in the literature, the operators in $\mathbf{D}(G/K)$ are not all induced from operators in the algebra $\mathbf{Z}(G)$ of bi-invariant differential operators on G. While this was verified in [16] to be true for all classical irreducible G/K, it fails for exactly four of the exceptional G/K [63]. However, even here, each D is induced by a fraction Z_1/Z_2 ($Z_i \in \mathbf{Z}(G)$), that is $\widetilde{Z}_1 D = \widetilde{Z}_2$ where $\widetilde{Z}_i \in \mathbf{D}(G/K)$ is induced by Z_i via the action of G on G/K. The algebra $\mathbf{D}(G/K)$ is the subject of Chapter II in [B6] and is related to the Iwasawa decomposition

$$G = NAK\,, \quad \mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}\,, \quad g = n \exp A(g) k\,, \quad A(g) \in \mathfrak{a}\,.$$

Given $D \in \mathbf{D}(G/K)$ and its radial part under the action of N on $X = G/K$, it is proved in [16] that the Abel transform (the Radon transform on K-invariant functions on X) is a simultaneous transmutation operator for all $D \in \mathbf{D}(G/K)$, that is, "converts" them into constant coefficient operators on A. Combining this with Harish–Chandra's work on the spherical transform and Hörmander's divisibility theorem for tempered distribution on $\mathbf{R}^n$, I showed in [16] that each $D \in \mathbf{D}(G/K)$ has a fundamental solution. The later Paley–Wiener theorem furnishes a shorter proof but requires the surjectivity of a constant coefficient differential operator on $\mathcal{D}'(\mathbf{R}^n)$.

The next step was to prove the surjectivity of each $D \in \mathbf{D}(G/K)$ on the space $\mathcal{C}^\infty(G/K)$, i.e., the existence of a global solution to the equation $Du = f$ for arbitrary smooth f. Already in some notes in 1964 I had reduced this to a support theorem for the horocycle Radon transform on X. I used Harish–Chandra's expansion for Eisenstein integrals but was stymied by singularities on $\mathfrak{a}_c^*$ whose location seemed unpredictable. But by looking

[1] Papers in this volume are indicated by an asterisk.

at the simplest possible examples I discovered, but not until 1972, that these singularities could be canceled out by a certain intertwining procedure. This yielded the support theorem for the horocycle Radon transform and thereby the surjectivity

$$D\,\mathcal{C}^\infty(X) = \mathcal{C}^\infty(X)$$

follows in [37]. The local solvability of each $Z \in \mathbf{Z}(G)$ is also proved there invoking methods of Raïs and Harish-Chandra. General local solvability was later proved by Duflo. The global solvability however fails (Cerèzo–Rouvière, *Sem. École Polytech.* (1973)) although it holds for the Casimir operator (Cerèzo–Rouvière loc. cit., Rauch and Wigner, *Ann. of Math.* (1976)). Other surjectivity results are proved in [35], [37], [43]. The non-Riemannian symmetric case has been investigated by Duflo, van den Ban and Schlichtkrull and others. The support theorem above was given a new proof by Gonzalez and Quinto. This approach has led Quinto and collaborators to various interesting support theorems in much more general context.

2. Geometric Properties of Solutions.

Articles: *[9], *[15], *[16], *[23], [25], *[32], *[40], [55], *[71], [78], *[81], *[82].

In his paper (*C.R. Acad. Sci.* Paris (1952)) Godement defined a harmonic function on G/K as a solution to $Du = 0$ for all $D \in \mathbf{D}(G/K)$ annihilating the constants. He proved that they had a characterization in terms of mean values over orbits of K and their translates. In [9] I proved a similar extension of Ásgeirsson's mean-value theorem for solutions of $L_x(u(x,y)) = L_y(u(x,y))$ on $\mathbf{R}^n \times \mathbf{R}^n$ (L-Laplacian) to the system $D_x u = D_y u$ on $G/K \times G/K$ for $D \in \mathbf{D}(G/K)$. Inspired by work of Ásgeirsson and Günther on Huygens' principle I became interested in studying the wave equation on symmetric spaces G/K and on Lorentzian manifolds of constant curvature. In both cases the natural equation is $L_x + c = \frac{\partial^2}{\partial t^2}$, where c is a constant related to the scalar curvature. Papers [16], [45], [55], [58], [71], [78] and a chapter in [B7] deal with explicit solution formulas for the natural wave equation on G/K and the occurrence of Huygens' principle. My work on these wave equations is related to extensive contributions by Branson, Ólafsson, Lax, Phillips, Schlichtkrull, Shahshahani, Solomatina and Ørsted. Papers [81] and [82] (the latter with Schlichtkrull) deal with a hyperbolic system on G/K introduced by Semenov–Tian–Shansky, *Izvestija* (1976). The time variable has dimension $\operatorname{rank}(X)$ and for $\operatorname{rank}(X) = 1$ it reduces to the above wave equation. Developing further the work of Semenov–Tian–Shansky, Shahshahani and Phillips, paper [81], gives an explicit solution of the Cauchy problem in terms of the Radon transform and in terms of the Fourier transform. Paper [82] with Schlichtkrull deals with an associated range problem.

For a Lorentzian manifold X of constant curvature, paper [9] sets up an analog of Riesz potentials and used properties of these to show that a function u on X is determined by a formula

$$u = c \lim_{r \to 0} r^{n-2} Q(\Box) M^r u \,,$$

where $(M^r u)(x)$ is the "mean value" of u over a "Lorentzian sphere" of radius r and center x, $Q(\Box)$ being a polynomial in the Laplacian $\Box$ on X. This can be used to deduce a result of Schimming and Schlichtkrull (*Acta Math.* (1994)) that each factor $\Box + c$ in $Q(\Box)$ satisfies Huygens' principle.

It was proved by Furstenberg and Karpelevich that a bounded solution of the Laplace equation $Lu = 0$ on the symmetric space G/K is necessarily harmonic (in Godement's sense above). This led them to a Poisson type integral formula for such functions. While the analog of the Schwarz theorem for continuous boundary values is easy, in [23] Korányi and I proved the corresponding Fatou theorem for such functions $u(x)$; namely

$$\lim_{t \to \infty} u(\gamma(t)) \quad \text{exists}$$

for almost all geodesics, starting at the origin $o = eK$. The result has many refinements, variations and generalizations due to Knapp, Korányi, Lindahl, Michelson, Schlichtkrull, Sjögren, Stein, Williamson and others.

3. Double Fibrations in Integral Geometry. Radon Transforms.

Articles: *[9], *[15], [17], *[18], *[19], *[20], *[28], [52], [53], *[62], *[69], *[74], [78], *[91], *[92].

Books: [B4], [B6], [B7].

In Radon's pioneering paper (1917) a function f on $\mathbf{R}^2$ is explicitly determined by its line integrals. In projective geometry points and lines are put on equal footing. Observing that $\mathbf{R}^2$ and the set $\mathbf{P}^2$ of lines are both permuted transitively by the same group $\mathbf{M}(2)$, the isometry group of $\mathbf{R}^2$, the following generalization was proposed in [20]. Given a locally compact group and K and H two closed subgroups satisfying some natural conditions we consider the coset spaces $X = G/K$, $\Xi = G/H$. Elements $x = gK$, $\xi = \gamma H$ are (by Chern (1942)) said to be *incident* if the sets gK and γH intersect as subsets of G. Given $x \in X$, $\xi \in \Xi$ let

$$\check{x} = \{\xi \in \Xi : x, \xi \ \text{ incident}\} \,,$$

$$\widehat{\xi} = \{x \in X : x, \xi \ \text{ incident}\} \,.$$

Then we have two integral transforms

$$f \in C_c(X) \to \widehat{f} \in C(\Xi) \,, \, \varphi \in C_c(\Xi) \to \check{\varphi} \in C(X)$$

given by

$$\widehat{f}(\xi) = \int\limits_{\widehat{\xi}} f(x)\,dm(x)\,,\ \check{\varphi}(x) = \int\limits_{\check{x}} \varphi(\xi)\,d\mu(\xi)$$

with canonically defined measures dm and $d\mu$. In group-theoretic terms with $L = K \cap H$, dk_L , dh_L invariant measures on K/L and H/L we have

$$\widehat{f}(\gamma H) = \int\limits_{H/L} f(\gamma hK)\,dh_L\,, \quad \check{\varphi}(gK) = \int\limits_{K/L} \varphi(gkH)\,dk_L\,.$$

These geometrically dual transforms are also adjoint:

$$\int\limits_X f(x)\check{\varphi}(x)\,dx = \int\limits_\Xi \widehat{f}(\xi)\varphi(\xi)\,d\xi\,,$$

where dx and $d\xi$ are invariant measures, $dx = dg_K$, $d\xi = dg_H$.

The principal problems for these transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ would be

(i) Injectivity
(ii) Inversion Formulas
(iii) Range Questions
(iv) Support Theorems. These are results of the following type:

$C_c(X) = \{f \in C(X) : \widehat{f} \in C_c(\Xi)\}$, but there are many variations.

In addition to considering the classical case $X = \mathbf{R}^n$, $\Xi = \mathbf{P}^n$, the space of hyperplanes, where I determined the image of $\mathcal{D}(\mathbf{R}^n)$ under the Radon transform, I obtained in [9] and [62] inversion formulas for the case when X is a Riemannian manifold of constant curvature and Ξ the family of totally geodesic submanifolds of a given dimension. This was later extensively studied by Semyanisty, Rubin, Palamodov, Rouvière, Kurusa, Berenstein and Casadio-Tarabusi. Substantial generalization to totally geodesic submanifolds in a noncompact symmetric space $X = G/K$ was done by Ishikawa (2003). Paper [18] contains an inversion formula for the case when X is a compact symmetric space of rank 1 and Ξ the set of antipodal submanifolds. The extensive calculation was later simplified (see *Integral Geometry and Radon Transforms*, book to appear) using Rouvière's method for the noncompact analog (*Enseign. Math.* (2001)). This implies the injectivity of the X-ray transform on any compact symmetric space (the case rank$(X) > 1$ being settled by Strichartz and Gindikin using Fourier series on a torus). For $X = G/K$ noncompact there is, in addition to the injectivity, a support theorem for the X-ray transform [51].

In his paper (*C.R. Acad. Sci.* Paris **342** (2006), 1–6) Rouvière proved an inversion formula for the X-ray transform on the noncompact symmetric space $X = G/K$. His method is based on a skillful reduction to the hyperbolic plane $\mathbf{H}^2$. The method does not work for the compact space U/K because of the antipodal manifolds. While the X-ray transform on the compact space U/K is injective (as mentioned) one can still ask:

(a) Is there a general inversion formula?
(b) If one only integrates over minimal geodesics (this transform is called the *Funk transform*), is this transform injective?

(a) A global formula from [92] implies an inversion formula for the Funk transform of f provided f is supported in a ball of radius $\pi/4$ (with U/K of maximum sectional curvature 1; here U/K is assumed irreducible and simply connected). The proof is based on results from [21] according to which all the minimal geodesics are conjugate under U. In the absence of the support condition the global formula mentioned also involves integration over the antipodal manifolds.
(b) In a recent preprint by Klein, Thorbergsson and Verhóczki, the Funk transform is shown to be injective without the support condition.

The case of $X = G/K$ and $\Xi = G/MN$, the space of horocycles, is discussed in a later section.

One particularly simple instance of the dual setup is when Ξ is the unit disk, X its boundary, both having the group $G = \mathbf{SU}(1,1)$ acting transitively by

$$g : z \to \frac{az+b}{\bar{b}z+\bar{a}} \qquad |z| \le 1 \qquad |a|^2 - |b|^2 = 1\,.$$

Here the transform $f \to \widehat{f}$ from $C(X)$ to $C(\Xi)$ turns out to be just the usual Poisson integral and the inversion formula is just the classical Schwarz theorem for the boundary values of a harmonic function. Other classical theorems in Potential Theory furnish answers to the range question (iii). While the range here consists of the harmonic functions there is a striking analogy with the case $X = \mathbf{R}^3$, $\Xi =$ space of lines in X, where John (*Duke Math. J.* (1938)) gave a precise description of the range as solutions of Ásgeirsson's ultrahyperbolic equation in four variables.

4. Spherical Functions. Spherical Transforms.

Articles: *[22], *[26], *[28], *[37], [38], *[43], *[48], *[91]
Books: [B6], [B7]

Let G be a connected noncompact semisimple Lie group with finite center, K a maximal compact subgroup. As proved by Gelfand, the convolution algebra $C_c(K\backslash G/K)$ of K-bi-invariant functions on G is commutative under convolution. Given its standard topology, the continuous homomorphisms of the algebra $C_c(K\backslash G/K)$ onto $\mathbf{C}$ are functionals

$$f \to \int_G f(x)\varphi(x)\,dx$$

where the functions φ are the so-called *zonal spherical functions* on G. Harish–Chandra gave a formula for these φ in terms of the Iwasawa decomposition of G whereby the set of these $\varphi = \varphi_\lambda$ is parametrized by λ in $\mathfrak{a}_c^*/W$. On the other hand the space $L^1(K\backslash G/K)$ is a commutative Banach algebra under convolution and its maximal ideal space corresponds to those spherical functions φ_λ which are bounded. Johnson and I found in [26] that the bounded spherical functions are those φ_λ (in the parametrization) for which λ lies in a certain explicit tube in $\mathfrak{a}_c^*$ over a certain convex set in $\mathfrak{a}^*$. Flensted–Jensen has discovered an interesting relationship between the spherical functions on G and its complexification G^C (*J. Funct. Anal.* (1978)). This leads, for example, to a somewhat simplified proof of the boundedness criterion above.

The paper [22] is devoted to the problem of finding the range of the space $\mathcal{D}(K\backslash G/K)$ under the spherical transform. For this I used Harish–Chandra's expansion of the spherical function on the positive Weyl chamber $\mathfrak{a}^+$. The coefficients are certain meromorphic functions on $\mathfrak{a}^*$ and the problem is to prove uniform compact support of the integral

$$\int_{\mathfrak{a}^*} \varphi_\lambda(\exp H) F(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\,.$$

This is proved in [22] for each term $\psi_\mu(\lambda, H)$ in the series for φ_λ by shifting the contour $\mathfrak{a}^*$ into the complex space $\mathfrak{a}_c^*$ yet avoiding the singularities in the coefficients $\psi_\mu(\lambda, H)$ as well as $c(\lambda)$. Fortunately, there was a half-space in $\mathfrak{a}_c^*$ free of singularities. It remained to justify the term-by-term integration

$$\int_{\mathfrak{a}^*} \sum_\mu \psi_\mu(\lambda, H) F(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda = \sum_\mu \int_{\mathfrak{a}^*} \psi_\mu(\lambda, H) F(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda$$

and this was done by Gangolli (*Ann. of Math.* (1971) 150–165) by the device of multiplying the expansion of φ_λ by a suitable factor. A simplified justification is done in my paper [28], p. 38 using the same device.

An exposition of Harish–Chandra's theory with some simplifications is given in [B6], incorporating also Rosenberg's method for the inversion formula (*Proc. Amer. Math. Soc.* (1977)). Another major simplification is Anker's proof (*J. Funct. Anal.* (1991)) of Harish-Chandra's determination of the image of the Schwartz space under the spherical transform.

The Abel transform mentioned earlier is considered again in [91] and some new operational properties established, for example that its adjoint $\mathcal{A}^*$ is a bijection of $\mathcal{C}_W^\infty(A)$ onto $\mathcal{C}^\infty(K\backslash G/K)$, the subscript denoting Weyl group invariance.

Harish–Chandra's Eisenstein integrals (generalized spherical functions) defined for each $\delta \in \widehat{K}$ are studied in [28] and [43]. Here I proved that they satisfy functional equations, one for each $s \in W$, and that they can be expressed in terms of the zonal spherical function φ_λ. This plays a role in the proof of the Paley–Wiener theorem for the Fourier transform.

5. Duality for Symmetric Spaces.

Articles: *[15], [17], *[18], *[19], *[20], *[27], *[28], *[43], *[48], *[74],
Book: [B7]

The duality referred to here is between the symmetric space $X = G/K$ and the space $\Xi = G/MN$ of horocycles in X. Here G has the Iwasawa decomposition $G = NAK$ and M is the centralizer of A in K. In the framework of an earlier section, we have the double fibration

$$\begin{array}{ccc} & G/M & \\ \swarrow & & \searrow \\ X = G/K & & \Xi = G/MN \end{array}$$

and the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ are

$$\widehat{f}(\xi) = \int_{\xi} f(x)\, dm(x) \quad \text{(the horocycle transform)}$$

$$\check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi)\, d\mu(\xi) \quad \text{(the dual transform).}$$

The injectivity of $f \to \widehat{f}$ on $L^1(X)$ can be deduced from the boundedness criterion of φ_λ mentioned earlier. For G a complex classical group Gelfand, Naimark and Graev had shown how a function $F \in \mathcal{C}_c^\infty(G)$ can be explicitly given in terms of its integrals over the conjugacy classes in G. Combined with the decomposition $G = NAK$ this proves an inversion formula for $f \to \widehat{f}$ for complex classical G. For general G the inversion and Plancherel formula for $f \to \widehat{f}$ were given in [15], [17], [19]. The kernel of the map $\varphi \to \check{\varphi}$ and the surjectivity $\mathcal{C}^\infty(\Xi)^\vee = \mathcal{C}^\infty(X)$ are established in [B7].

The main theme in [20], [28] is the pattern of analogies between G/K and G/MN. The algebra $\mathbf{D}(G/MN)$ is commutative as is the algebra $\mathbf{D}(G/K)$ and the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ intertwine the operators in $\mathbf{D}(G/K)$ and $\mathbf{D}(G/MN)$. The analogs for G/MN of spherical functions on G/K are the *conical distributions* on G/MN which by definition are

(i) MN-invariant.
(ii) Eigendistributions of each $D \in \mathbf{D}(G/MN)$.

While the set of spherical functions is parametrized by $\mathfrak{a}_c^*/W$ (corresponding to $K\backslash G/K \sim A/W$) I proved in [28] and [43] that the set of conical distributions is parametrized by $\mathfrak{a}_c^* \times W$, corresponding to the decomposition $\Xi = MN \cdot (A \times W) \cdot \xi_0$ via the Bruhat decomposition of G. The injectivity criterion for the Poisson transform $\mathcal{P}_\lambda$ mentioned below is of central importance for this problem. The parametrization required the exclusion of certain singular eigenvalues. For X of rank 1 this restriction was removed by Men Cheng Hu (MIT thesis of 1973; *Bull. Am. Math.*

Soc. (1975)), thus finishing the determination of all conical distributions for this case.

A representation of G is *spherical* if there is a fixed vector under K and *conical* if there is a fixed vector under MN. It is proved in [17] and [28] that an irreducible finite-dimensional representation is *spherical* if and only if it is *conical*. This leads to an explicit characterization of their highest weights. It also leads to simultaneous imbeddings of X and Ξ into the same finite-dimensional vector space. The horocycles then appear as certain plane sections with X ([B7], Ch. II, §4).

With π an irreducible finite-dimensional spherical representation of G on V let v_K be a $\pi(K)$ fixed vector and v'_K a K fixed vector under the contragredient representation $\check{\pi}$. Then π is equivalent to the natural representation of G on the space of functions

$$g \to \langle \pi(g^{-1})v, v'_K \rangle \qquad v \in V$$

and in this model the spherical and conical vectors are respectively given by

$$\varphi_K(gK) = \langle \pi(g^{-1})v_K, v'_K \rangle \,,$$
$$\psi_{MN}(gK) = e^{-\Lambda(A(g))} \,,$$

Λ being the highest restricted weight of π and $A(g)$ as in this Introduction §1. In a lecture in Iceland, 2007, David Vogan interpreted the identity above of the spherical and conical representation in terms of a "deformation" of K into MN. Also Adam Korányi has shown that the MN-fixed vector is a suitable limit of an orbit of the K-fixed vector.

The continuous decompositions of $L^2(X)$ and $L^2(\Xi)$ into irreducibles correspond under the maps $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$. The map $\varphi \to \check{\varphi}$ induces for each $\lambda \in \mathfrak{a}_c^*$ a "Poisson transform" $\mathcal{P}_\lambda$ which maps functions (and functionals) on $B = K/M$ into a joint eigenspace $\mathcal{E}_\lambda(X)$ of the operators in $\mathbf{D}(G/K)$. In [27], [28] and [43] it is stated and proved that $\mathcal{P}_\lambda$ is injective if and only if $\Gamma_X^+(\lambda)^{-1} \neq 0$ where Γ_X^+, the Gamma function of X, is the denominator in the Gindikin–Karpelevich formula for Harish–Chandra's $\mathbf{c}(\lambda)$ function. For X of rank one this was done in [28] by analyzing the Poisson transform at ∞. For X of higher rank this was done in [43] by studying its behavior at the origin, using methods from Kostant's work on the spherical principal series. The surjectivity of $\mathcal{P}_\lambda$ for hyperbolic spaces X (with analytic functionals on B) and the surjectivity of $\mathcal{P}_\lambda$ for general X on the K-finite level was proved in [40] and [43]. The surjectivity without the K-finiteness assumption was proved by Kashiwara, Kowata, Minemura, Okamoto, Oshima and Tanaka (*Ann. of Math.* (1978)). Professor Okamoto has kindly explained how this remarkable collaboration came about.

"I still remember vividly how we reached our proof of your great conjecture. We tried very hard in vain to prove your conjecture in the higher rank case, using the concept of analytical functionals. To understand your conjecture in a different way, we decided to apply directly the original idea of Prof. Sato about hyperfunctions to the new approach.

I took advantage of the fact that Prof. Sato is an uncle of Minemura so that Minemura asked his uncle (Prof. Sato) to spare some time on Saturdays or Sundays for discussions about the possibility of applying directly the original idea of Sato's hyperfunctions to prove your conjecture. We traveled to Kyoto University every weekend. During this time Kashiwara (Kyoto University) and Oshima (Tokyo University) came to join in our discussions.

To carry out our idea in the higher rank case, we realized that it is important to extend the notion of the ordinary differential equation with regular singularity to the notion of "the system of partial differential equations with regular singularity".

The splendid aspect of your conjecture is to have given birth to a new branch of Mathematics: Kashiwara–Oshima: Systems of differential equations with regular singularities and their boundary value problems, *Ann. of Math.* (1977)."

An engaging account of this work is contained in a book by Schlichtkrull (Birkhäuser (1984)).

Schmid has indicated how the rank 1 method (which relied on hypergeometric function estimates) might extend to the general rank.

The Poisson transform of distributions on B was investigated by Lewis and by Oshima–Sekiguchi. The latter authors also treat the case of affine symmetric G/H. Generalization to vector bundles was done by An Yang, (Thesis MIT, (1994)) and by Martin Olbrich (Dissertation, Berlin (1994)).

The flat analog of the double fibration above, replacing horocycles by their tangent planes, is investigated in [48] and [69]. The results are analogous but the expected tools like spherical function theory become degenerate so the results are obtained by looking at the flat case as a limiting case of the curved case.

The complexification of the double fibration above, replacing each group L by L^c has been actively studied in recent years (Gindikin, Krötz, Ólafsson, Stanton), and the extension of the joint eigenfunctions of $\mathbf{D}(G/K)$ to a tube around G/K in G^c/K^c plays a central role.

6. Representation Theory.

Articles: [17], *[19], *[20], *[27], *[28], [29], *[30], [31], *[43], [47], *[48], [49]

Book: [B7]

Given a coset space L/H with $\mathbf{D}(L/H)$ commutative the group L acts naturally on each joint eigenspace of the operators in $\mathbf{D}(L/H)$. I have called these representations *eigenspace representations*. If U/K is a symmetric space of the compact type the eigenspace representations are precisely the irreducible spherical representations π of U. *Spherical* means that there is a fixed vector under $\pi(K)$. If U is simply connected the highest weights of these π are given by the characterization mentioned before. This was further generalized by Schlichtkrull (1984) and by Johnson (1987).

In the papers listed above the irreducibility question for the eigenspace representations is considered for the cases $\mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)$, the noncompact symmetric space G/K, the dual space G/MN and G/N for G complex. For G/K the joint eigenspaces are parametrized by $\mathfrak{a}_c^*/W$ (since each joint eigenspace contains a unique spherical function); the irreducibility turns out to be equivalent to $1/\Gamma_X(\lambda) \neq 0$ where $\Gamma_X(\lambda)$ is the denominator of $\mathbf{c}(\lambda)\mathbf{c}(-\lambda)$.

Using the structure of $\mathbf{D}(G/MN)$ the eigenspace representations for G/MN turn out to be just the members of the spherical principal series of G; the intertwining operators, realizing the equivalence between two equivalent representations from this series, consist of convolutions with conical distributions ([27], Theorem 4.6). Another parallel determination of these operators was done by Schiffmann and by Kunze, Knapp and Stein. The irreducibility of these representations is equivalent to $1/\Gamma_X(\lambda) \neq 0$; this gives another proof and interpretation of the algebraic irreducibility criterion of Wallach, Parthasarathy–Rao–Varadarajan and Kostant though stated in quite different terms. For G complex, $\mathbf{D}(G/N)$ is still commutative, and the eigenspace representations form the full principal series.

For a non-Riemannian symmetric space G/H the eigenspace representations have been studied by Huang (*Ann. of Math.* (2001)). For the hyperbolic spaces over various fields this had been done very completely by Schlichtkrull (1987). For G nilpotent or solvable they have been effectively studied by Hole, Jacobsen and Stetkær (*Math. Scand.* (1975, 1983, 1981)).

7. Fourier Transform on G/K.

Articles: *[19], *[20], [24], *[28], *[35], [36], *[37], *[43], [54], [57], [79], *[81], *[82] *[91]

The spherical function φ_λ on G is K-bi-invariant, that is K-invariant on $X = G/K$ and the theory of the spherical transform $\widetilde{f}(\lambda) = \int_X f(x)\varphi_\lambda(x)\,dx$ only concerns functions f on X which are K-invariant. The analog of φ_λ for $\mathbf{R}^2$ is the Bessel function $J_0(\lambda|x|)$ which is a radial eigenfunction of the Laplacian L on $\mathbf{R}^2$ and the Bessel transform on $\mathbf{R}^+$ becomes the analog of the spherical transform. In [19] I proposed a definition of a *Fourier transform* on $X = G/K$, applying to all functions on X but reducing to the spherical transform for f K-invariant in the same way as the Fourier transform on $\mathbf{R}^2$ when used on radial functions reduces to the Bessel transform. Given the Iwasawa decomposition $G = NAK$ $g = n \exp A(g)k$, $A(g) \in \mathfrak{a}$; we define the vector-valued inner product $A(x,b)$ on $X \times B$ by

$$A(gK, kM) = A(k^{-1}g)\,.$$

Putting $e_{\lambda,b}(x) = e^{(i\lambda+\rho)(A(x,b))}$ we define the Fourier transform of a function f on X by

$$\widetilde{f}(\lambda, b) = \int\limits_X f(x) e_{-\lambda,b}(x)\, dx$$

for all $(\lambda, b) \in \mathfrak{a}_c^* \times B$ for which the integral converges. The definition is suggested by the geometric analogy between horocycles and hyperplanes; $A(x, b)$ is a "vector-valued distance" from o to the horocycle through x with normal b. A representation-theoretic interpretation was pointed out by Sitaram (*Pacific J. Math.* (1988)) and Ólafsson–Schlichtkrull (Tribute to Mackey Volume, AMS (2008)). All the principal theorems from classical Fourier transform theory have their analogs for this Fourier transform. Thus [19] and [28] contain the Plancherel theorem (with range) and the inversion formula which can be written

$$\delta_o = \int\limits_{\mathfrak{a}^* \times B} e_{\lambda,b}\, d\mu(\lambda, b)\,,$$

in analogy with the Euclidean version

$$\delta_o = \int\limits_{\mathbf{R}^+ \times \mathbf{S}^{n-1}} e_{\lambda,\omega} \lambda^{n-1}\, d\lambda\, d\omega\,, \qquad e_{\lambda,\omega}(x) = e^{i\lambda(x,\omega)}\,.$$

The Paley–Wiener theorem (description of $\mathcal{D}(X)^{\sim}$) is proved in [19] and [28]. This theorem was used in [43] to prove the necessary and sufficient condition for the bijectivity of the Poisson transform for K-finite functions on B. In turn this was used to prove the irreducibility criterion for the eigenspace representations mentioned earlier. Another proof of the Paley–Wiener theorem was given by Torasso (1977). The paper [91] contains a strong form of the Riemann–Lebesgue lemma, proved jointly with Sengupta and Sitaram. These two and several others, Mohanty, Ray, Cowling, Bagci, Sarkar, Sundari, Rawat, have made great progress in topics such as the analog of the Wiener–Tauberian theorem and Hardy's theorem concerning simultaneous exponential decay of a function and its Fourier transform. Okamoto and especially Eguchi have proved deep theorems for the Fourier transform on the Schwartz spaces $\mathcal{S}(X)$ and their L^p analogs (Eguchi, *J. Funct. Anal.* (1979)).

The analog of the above Fourier transform for the dual compact symmetric space U/K was developed by Sherman (*Acta Math.* (1990)), and for vector bundles over X by Camporesi (*Pacific J. Math* (1997)). A Paley–Wiener type theorem for the compact case has been obtained (2000) by Gonzalez (on U) and more generally (2008) by Ólafsson and Schlichtkrull (on U/K).

In more recent years Fourier analysis theory for the non-Riemannian symmetric spaces G/H has been vigorously developed by van den Ban, Carmona, Delorme, van Dijk, Faraut, Flensted–Jensen, Matsuki, Molchanov, Ólafsson, Oshima, Rossmann, Schlichtkrull and many others.

8. Multipliers.

[1], [2], *[3], [4], [5]

Multipliers appear naturally in classical Fourier analysis on groups, where they correspond to operators on function spaces commuting with translations. In the papers listed the setup is a semisimple Banach algebra A; a subalgebra A_0, *the derived algebra*, is introduced as the algebra of elements $x \in A$ for which the multiplication $y \to xy$ is continuous from the topology of the *spectral norm* $\|y\|_{sp} = \|\widehat{y}\|_\infty$ to the original topology. Several properties of A_0 are proved in [3], for example the determination of the representative algebra $\widehat{A}_0$ as the intersection of certain L^1 spaces.

For various function algebras on a compact group G the derived algebra A_0 is explicitly determined. For G the circle group the results specialize to classical results of Littlewood, Sidon and Zygmund in Fourier series theory; for G the Bohr compactification of $\mathbf{R}$ they solve the multiplier problem for almost periodic functions on $\mathbf{R}$. My Ph.D. thesis consisted of [2] and [3].

For $A = L^1(G)$, where G is a compact group (abelian or not) A_0 turns out to be $L^2(G)$ and in fact,

$$\|f\|_2 \leq \sqrt{2} \sup_g \{\|f * g\|_1 : \|g\|_{sp} \leq 1\}.$$

For G abelian it follows from a combination of the papers Edwards and Ross (*Helgason's number and lacunarity constants*, *Bull. Austr. Math. Soc.* (1973)) and Sawa (*Studia Math.* (1985)) that $\sqrt{2}$ can be replaced by $2/\sqrt{\pi}$ and that this is the best possible constant. For G nonabelian or G/K symmetric it is not known whether the best constant is independent of G.