Introduction

This book is a gentle introduction to *continued fractions* by way of pattern recognition and applications.

What is a continued fraction?

Definition 1: Finite simple continued fractions. A *finite simple continued fraction* is a finite list of integers n_i, called *partial denominators*, with $n_i > 0$ for all $i > 0$, denoted by

$$[n_0; n_1, n_2, n_3, \ldots, n_k],$$

where integer i ranges from 0 to some integer k. Associated with this continued fraction are $k+1$ fractions, C_0, C_1, through C_k, referred to as *convergents*, whose values are

$$C_0 = n_0, \qquad C_1 = n_0 + \frac{1}{n_1}, \qquad C_2 = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2}},$$

and so on.

To illustrate, convergent 1 for the simple continued fraction $[1; 2, 3, 4, 5]$ is $C_1 = 1 + \frac{1}{2} = \frac{3}{2}$, whereas convergent 4 is

$$C_4 = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}} = \frac{225}{157}.$$

Since C_4 is the last convergent, we write $\frac{225}{157} = [1; 2, 3, 4, 5]$.

Continued fractions may be infinitely long.

Definition 2: Infinite simple continued fractions. An *infinite simple continued fraction* is a list of integer-valued partial denominators n_i with $n_i > 0$ for $i > 0$, denoted by

$$[n_0; n_1, n_2, n_3, ...],$$

where i is a nonnegative integer. As in the finite case, the infinite continued fraction has convergents C_i computed as given in Definition 1. This time there are infinitely many convergents. As the name suggests, the sequence of convergents often has a limit ω, in which case we write

$$\omega = [n_0; n_1, n_2, n_3, ...].$$

We will discuss questions of convergence in Chapter IX.

Each positive rational number can be written as a finite simple continued fraction. Each infinite simple continued fraction evaluates to some irrational number, and every positive irrational number can be written as a simple continued fraction, as we illustrate at various times through the text. For example, the simple continued fraction representation for $\sqrt{2}$ is

$$\sqrt{2} = [1; 2, 2, 2, ...] = [1; \overline{2}].$$

Its convergents (along with their decimal approximations) are

$$1, \frac{3}{2} = 1.5, \frac{7}{5} = 1.4, \frac{17}{12} \approx 1.41667, \frac{41}{29} \approx 1.41379, \frac{99}{70} \approx 1.41429.$$

As the term *simple continued fraction* implies, not all continued fractions are simple. In Chapter IX, numerators of the partial denominators will be permitted to be 1 or −1. The notation for these continued fractions is

$$[n_0; \epsilon_i n_1, \epsilon_2 n_2, \epsilon_3 n_3, ...]$$

where ϵ_i is either 1 or −1, with convergents

$$C_0 = n_0, \qquad C_1 = n_0 + \frac{\epsilon_1}{n_1}, \qquad C_2 = n_0 + \cfrac{\epsilon_1}{n_1 + \cfrac{\epsilon_2}{n_2}},$$

and so on. For example, π can be written as the continued fraction

$$\pi = [3; 7, 16, -294, ...] \rightarrow \left\{C_0 = 3, C_1 = \frac{22}{7}, C_2 = \frac{355}{113}, C_3 = \frac{104348}{33215}, ...\right\}. \quad (1)$$

Much more general continued fraction examples are given on page 35 of Strand II, in Exercise VI.10c, and in Example IX.43.

Of what use is a continued fraction? Suppose we wish to approximate the number π. The first six digits of π's decimal expansion are 3.14159. Therefore, one seemingly good approximation for π as a fraction in lowest terms should be $\frac{314159}{100000}$. However, from (1), π's second convergent, $C_2 = \frac{355}{113}$, is simpler than $\frac{314159}{100000}$ and is an order of magnitude closer to π:

$$\left|\pi - \frac{314159}{100000}\right| \approx 2.65 \times 10^{-6} \qquad \text{versus} \qquad \left|\pi - \frac{355}{113}\right| \approx 2.67 \times 10^{-7}.$$

Continued fractions are, in general, an elegant way to find good fractional approximations for irrational numbers.

In general, when measuring phenomena using given units, the numbers we get often appear to have no repeating pattern in their decimal expansions. The diagonal of the unit square has length $\sqrt{2}$. The old Greek puzzle about doubling the volume of a cube involves scaling its side length by $\sqrt[3]{2}$. The ratio of a circle's circumference to its diameter is π. Correct to five decimal places (using kilograms, meters, and seconds), the universal gravitational constant is $G \approx 6.67408 \times 10^{-11}$. Wherever we look we find what appear to be irrational numbers. Whenever we approximate, we use rational numbers. Strange as it might at first sound, oftentimes the rational approximations given to us by continued fractions enable us to see patterns more clearly in our universe.

A brief outline of this book. This book is an exploration of continued fractions. It includes brief forays into ideas that are from outside elementary number theory, yet are part of the standard undergraduate mathematics curriculum. Besides number theory, the text uses elements of calculus (limits, integrals, and series), vector calculus, discrete mathematics, linear algebra, probability, mathematical statistics, combinatorics, graph theory, geometry, differential equations, and analysis, as well as allusions to abstract algebra. As a guide to the reader, those sections of the book that include such forays or include enrichment material on a particular idea are marked with an asterisk. Thus, for example, the reader will see the asterisk in the Chapter VIII section

Newton's case for a flattened Earth*.

This asterisk is a cue that this section is optional for understanding the book and may be a section to skip on a first reading.

This book contains twelve strands and twelve chapters. The strands are meant to be somewhat light-hearted introductions to the following chapter. They involve a single idea, puzzle, or personality related in some way to the material of the subsequent chapter. While successive chapters of the book are related to previous chapters and foreshadow later chapters, each chapter can be read more or less on its own. Although the book is structured to reach a climax in Chapter IX on continued fractions, each chapter is also an end in and of itself.

Here is an outline of the chapters.

- Chapter I starts with the set of positive integers. Rearranging heaps of n pebbles into arrays of p rows of q pebbles, where p and q are integers, soon leads to the discovery of the fundamental theorem of arithmetic, and gives a natural way to think of taking parts of a whole. That is, splitting pq into p equal parts leads to the idea of the unitary fraction $\frac{1}{p}$, a fundamental building block of continued fractions.

- Chapter II presents the well-ordering principle and mathematical induction, which, among other things, gives a division algorithm. With this tool, we show how to find the simple continued fraction of any fraction.

- Chapter III shows how a recursive application of the division algorithm leads to Euclid's method for finding the greatest common divisor of two positive integers and for solving Diophantine equations. As we will see, Euclid's method for the greatest common divisor of two positive integers is equivalent to finding the simple continued fraction representation for the ratio of the given two integers.

- Chapter IV shows how each positive (non-integer) fraction $\frac{p}{q}$, where p and q are integers, is a combination of two unique simpler fractions $\frac{a}{b}$ and $\frac{c}{d}$ that solve the Diophantine equation $px - qy = \pm 1$. This structure defines a tree of fractions and allows us to identify each fraction between 0 and 1 with a tree address. In fact, finite continued fractions allow us to find the tree address of any fraction, and to know the fraction at any address.

- In Chapter V, the idea of unitary fractions from earlier chapters extends naturally to the harmonic series whose terms H_n are the sums of the first n unitary fractions. Recall from calculus that Euler's constant γ is the limit of the difference between H_n and $\ln n$. We illustrate the generation of a non-simple continued fraction using γ, where the numerators of the partial denominators are allowed to be either 1 or -1 (rather than always being 1).

- In Chapter VI, we generate families of numbers recursively and, in doing so, explore series. Recall from calculus that the real natural number e is the sum of an infinite number of unitary fractions: $e = \sum_{i=0}^{\infty} \frac{1}{i!}$. We show how to find e's infinite simple continued fraction representation, $e = [n_0; n_1, n_2, \ldots]$, and determine an explicit formula that yields the partial denominator n_i for any desired positive integer i.

- Chapter VII is about simple harmonic motion—the approximate motion of some planets and satellites about their suns or planets, respectively. We show how simple harmonic motion gives a geometrical algorithm for finding a continued fraction equal to a given number. We explore more traditional algorithms in Chapter IX.

- Chapter VIII showcases a few classic ratios involving the ellipse from the seventeenth and eighteenth centuries, ratios which we then represent as continued fractions.

- In Chapter IX, we present a variety of continued fraction algorithms, and show that each positive irrational number ω has many convergent continued fraction representations. When restricting integer partial denominators n_i to be either only positive integers or to always have magnitude at least 2 (when $i \geq 1$), we show that every infinite continued fraction converges using the tree of fractions between successive integers presented in Chapter IV.

- The final three chapters are applications of continued fractions to the motion of the Moon, Earth, and Venus.

How could this book be used in the undergraduate mathematics classroom? As a minimal requirement for reading this book, the reader should be aware of mathematical induction, a topic often introduced in Discrete Mathematics (or any bridge course to writing proofs) or Calculus I. A subset of the chapters could serve as at least one of the texts for a course on number theory (Chapters I through IV, IX, and XII and some of the strands such as Strand VI), for a course on the history of mathematics (any subset works), or for a capstone course. A brave soul might use the book as a text for Discrete Mathematics.

Readers interested in following a minimal path to the applications of the later chapters should familiarize themselves with Diophantine equations in Chapter III; be able to compute the general mediant of two neighboring Farey fractions, and to find the mother and father fractions for any given fraction in the Stern-Brocot tree from Chapter IV; understand the continued fraction constructions of Example II.4, Example III.8, Puzzle V.6, and Example VI.31; and read the first half of Chapter IX.

As aids to the reader, the appendices include the following items.

- A list of symbols used throughout the text.

- An introduction to vectors and matrices with respect to the matrix multiplication of Chapters IV and X.

- Algorithmic code for a score of algorithms introduced in the text. In addition to a presentation of the *Mathematica* code for many of the algorithms used in this text, we provide access to them via an AMS website `www.ams.org/bookpages/dol-53` as both a pdf file and a *Mathematica* notebook. For most of these selections, the code is easily adaptable to any computer algebra system (CAS).

- Comments on selected exercises.

Snippets of this book have appeared in print over the years. Strand II is a version of A. Simoson, Life lessons from Leibniz, *Math Horizons* **22**:4 (2015) 5–7, 29 © Mathematical Association of America, 2015, all rights reserved. Strand V is an adaptation of B. Linderman and A. Simoson, A Bach diesel canon, *Math Horizons* **25**:4 (2018) 5–7 © Mathematical Association of America, 2018, all rights reserved. Strand VIII is an expanded version of A. Simoson, Minimizing Utopia, *Math Horizons* **23**:3 (2016) 18–21 © Mathematical Association of America, 2016, all rights reserved, a version of which, The size and shape of Utopia, also appeared in the *Proceedings of the Bridges Jyväskylä 2016 Conference* [**139**]. A portion of R. Fillers, B. Linderman, and A. Simoson, Mancala as nim, *Coll. Math. J.* **45**:5 (2014) 350–356 © Mathematical Association of America, 2014, all rights reserved, appears as a case study in Chapter II. Strand VI is an expanded version of A. Simoson, Extrapolating Plimpton 322, *Coll. Math. J.*, **50**:3, © Mathematical Association of America, 2019, all rights reserved. A condensed version of J. Dodge and A. Simoson, Ben-Hur staircase climbs, *Coll. Math. J.* **43**:4 (2012) 274–284 © Mathe- matical Association of America, 2012, all rights reserved, appears as an example in Chapter VI. Adaptations and combinations of A. Simoson, Newton's radii, Maupertuis' arc length, and Voltaire's giant, *Coll. Math. J.* **42**:3 (2011) 274–284 © Mathematical Association of America, 2011, all rights reserved, and A. Sim- oson, Newton's 501 jeans, *The Mathematical Scientist* **43**:1 (2018) 1–9 © Applied Probability Trust, 2018, appear as a case study in Chapter VIII. Chapter X is a version of A. Simoson, Periodicity domains and the transit of Venus, *Amer. Math. Monthly* **121**:4 (2014) 283–298 © Mathematical Association of America, 2011, all rights reserved. Chapter XI is an expanded version of A. Simoson, Lunar rhythms and strange signatures, *The Mathematical Scientist* **41**:1 (2016) 25–39 © Applied Probability Trust, 2016. Chapter XII is a version of A. Simoson, Diophantine eclipses, *The Mathematical Scientist* **42**:2 (2017) 74–89 © Applied Probability Trust, 2017.

This book contains some whimsy. Musings on the Ishango bone are pushed to the limit in Chapter I. Mancala of Chapter II analyzed as nim is probably intractable for most configurations. We translate the firing sequence of a twelve-cylinder engine into a musical score, even though a typical diesel train engine makes 500 to 1500 rotations per minute. We consider the problem of dropping a small black hole at Earth's surface, and we make conjectures about the longevity of the 17-year cicada.

Numerous illustrations appear in the text. Where noted in a figure's caption, permission use has been granted. Figures appearing without acknowledgment are in the public domain. Some of the figures are my sketches. The flower figure on the cover is meant to be a visual characterization for the optimal continued fraction convergents to the natural number e, as explained fully in [**143**].

Finally, I wish to thank a number of people.

- A colleague Bill Linderman who rendered the diesel canon for the diesel engine firing of Chapter I and Strand V as a musical score using the software *Sibelius*.
- King University students Rhianna Fillers, Shuler Hopkins, and Sam Barker for joint work on nim of Chapter II and the synchronization example of the 17-year cicada in Strand X.
- Keri-Lynn Paulson, a King University librarian, who helped secure various high-resolution images from a number of libraries.
- Dan Kalman for directing my attention to strange Strang figures upon seeing my analysis of the signatures of strange harmonic motion as presented in Chapter VII.
- King University astronomer Raymond Bloomer for fielding many questions about celestial mechanics.
- The trustees of King University for a 2016 spring semester sabbatical to focus on this manuscript.
- Senior Acquisitions Editor Stephen Kennedy for encouraging me to submit a book manuscript featuring continued fractions.
- The Dolciani Committee chaired by Harriet Pollatsek for careful reading of multiple drafts of the manuscript.
- Senior Production Editor Lauren Foster for rendering the manuscript into its published form, and allowing me to make last minute revisions.

25 March 2019