

# Contents

<b>Preface by the Editors</b> .....	ix
<b>Acknowledgments</b> .....	xiii
<b>I Introduction</b> .....	<b>1</b>
<b>II Lectures on the number theory of quaternions</b> .....	<b>17</b>
<b>Preface</b> .....	19
<b>1 The quaternions and how to compute with them</b> .....	21
Box A: Cayley's definition of quaternions using matrices .....	27
<b>2 The field of quaternions and their permutations and inversions</b> .....	33
Box B: Structure! About fields, skew fields, rings, algebras, and their automorphisms .....	37
<b>3 The field <math>R</math> and its permutations</b> .....	45
Box C: Algebraic numbers and quadratic fields .....	48
Box D: How to introduce algebraic integers? Some thoughts about number theory .....	53
<b>4 The integer quaternions</b> .....	59
Box E: Integral closure and orders in number fields .....	67
<b>5 The permutations of integer quaternions</b> .....	73
Box F: Commutative divisibility .....	76
<b>6 Greatest common divisor and quaternion ideals</b> .....	77
Box G: Euclidean rings: From geometry to arithmetic .....	81
<b>7 Even and odd quaternions. Associated and primary quaternions</b> .....	91
Box H: Modular arithmetic and the Chinese remainder theorem .....	95

<b>8</b>	<b>The integer quaternions modulo an odd number</b> . . . . .	97
	Box I: Arithmetical functions, Dirichlet series and Euler products . . . . .	105
<b>9</b>	<b>The prime quaternions</b> . . . . .	109
	Box J: Prime quaternions and prime numbers . . . . .	113
<b>10</b>	<b>The factorization theorem</b> . . . . .	119
	Box K: Almost unique factorization of quaternion integers and consequences . . . . .	124
<b>11</b>	<b>The representations of a positive integer as a sum of four squares</b> . . . . .	129
	Box L: The number of representations as a sum of squares . . . . .	131
<b>12</b>	<b>A problem due to Euler</b> . . . . .	139
	Box M: Magic squares of squares . . . . .	150
	<b>Notes and addendum</b> . . . . .	157
<b>III</b>	<b>Two Arithmetics</b> . . . . .	<b>161</b>
	III.1 Rudolf Lipschitz and Adolf Hurwitz . . . . .	161
	III.2 Differences and references . . . . .	168
	Box N: Another proof of the four-square theorem following Lipschitz and Dickson . . . . .	176
	III.3 The notion of integers in the theory of algebraic numbers . . . . .	178
	III.4 Research in <i>Vorlesungen</i> . . . . .	184
<b>IV</b>	<b>A view into Hurwitz's <i>Mathematische Tagebücher</i></b> . . . . .	<b>191</b>
	IV.1 The legacy . . . . .	191
	IV.2 Diaries No. 14, 15 and 25 . . . . .	195
	IV.3 Quaternions in the later diaries . . . . .	200
	Box O: Dickson's generalized maximal orders . . . . .	207
<b>V</b>	<b>On the composition of quadratic forms</b> . . . . .	<b>211</b>
	I. . . . .	214
	II. . . . .	219
	III. . . . .	221
	IV. . . . .	228
	Box P: Octonions and the eight-square identity . . . . .	237

<b>VI Abstraction and generalization</b>	<b>241</b>
VI.1 Octonions and algebras . . . . .	241
Box Q: A letter from Hades . . . . .	243
VI.2 Sums of squares, quadratic forms, and more . . . . .	253
<b>VII Epilogue</b>	<b>257</b>
Box R: Fermat’s method of infinite descent and Euler’s proof of the four-square theorem . . . . .	259
<b>Appendices</b>	<b>267</b>
Appendix A: Theses . . . . .	267
Appendix B: Elementary number theory in a nutshell . . . . .	268
<b>References</b> . . . . .	<b>273</b>
<b>List of protagonists</b> . . . . .	<b>287</b>
<b>Index</b> . . . . .	<b>289</b>