

# Contents

<b>1</b>	<b>Introduction</b> . . . . .	1
1.1	Introduction . . . . .	1
1.2	Overview of the results . . . . .	8
<b>2</b>	<b>Laplacians on graphs</b> . . . . .	15
2.1	Combinatorial and metric graphs . . . . .	15
2.2	Discrete Laplacians on graphs . . . . .	19
2.3	Function spaces on metric graphs . . . . .	23
2.4	Laplacians on weighted metric graphs . . . . .	24
<b>3</b>	<b>Connections via boundary triplets</b> . . . . .	33
3.1	Spectral properties: Graph Laplacians vs. Kirchhoff Laplacians . . . . .	33
3.2	Graph Laplacians as boundary operators . . . . .	36
3.3	Spectral properties: Metric graphs and models . . . . .	48
<b>4</b>	<b>Connections between parabolic properties</b> . . . . .	53
4.1	Markovian extensions . . . . .	53
4.2	Brownian motion and random walks . . . . .	55
4.3	Correspondence between quadratic forms . . . . .	56
4.4	Correspondence between Markovian extensions . . . . .	63
4.5	Recurrence/transience . . . . .	68
4.6	Stochastic completeness . . . . .	70
4.7	Spectral estimates . . . . .	73
4.8	Ultracontractivity estimates . . . . .	74
<b>5</b>	<b>One-dimensional Schrödinger operators with point interactions</b> . . . . .	81
5.1	The case $\alpha \equiv 0$ and Krein strings . . . . .	82
5.2	Connection via boundary triplets . . . . .	88
5.3	Jacobi matrices and Krein–Stieltjes strings as boundary operators . . . . .	91
<b>6</b>	<b>Graph Laplacians as boundary operators</b> . . . . .	97
6.1	Examples . . . . .	98
6.2	Life without loops I: Graph Laplacians . . . . .	101
6.3	Life with loops . . . . .	104
6.4	Intrinsic metrics . . . . .	105
6.5	Harmonic functions on graphs . . . . .	118

6.6	Life without loops II: Jacobi matrices on graphs . . . . .	124
6.7	Further comments and open problems . . . . .	130
<b>7</b>	<b>From continuous to discrete and back</b> . . . . .	<b>133</b>
7.1	Self-adjointness . . . . .	133
7.2	Markovian uniqueness and finite energy extensions . . . . .	143
7.3	Spectral estimates . . . . .	151
7.4	Recurrence and transience . . . . .	159
7.5	Stochastic completeness . . . . .	163
<b>8</b>	<b>Examples</b> . . . . .	<b>165</b>
8.1	Antitrees . . . . .	165
8.2	Cayley graphs . . . . .	174
8.3	Tessellations . . . . .	188
<b>A</b>	<b>Boundary triplets and Weyl functions</b> . . . . .	<b>199</b>
A.1	Linear relations . . . . .	199
A.2	Boundary triplets and proper extensions . . . . .	201
A.3	Weyl functions and extensions of semibounded operators . . . . .	202
A.4	Direct sums of boundary triplets . . . . .	204
<b>B</b>	<b>Dirichlet forms</b> . . . . .	<b>207</b>
B.1	Basic notions . . . . .	207
B.2	Transience, recurrence and stochastic completeness . . . . .	208
B.3	Extended Dirichlet spaces . . . . .	209
<b>C</b>	<b>Heat kernel bounds</b> . . . . .	<b>211</b>
<b>D</b>	<b>Glossary of notation</b> . . . . .	<b>213</b>
	<b>Index</b> . . . . .	<b>217</b>
	<b>References</b> . . . . .	<b>221</b>