

# Contents

<b>Preface</b> . . . . .	vii
<b>1 Introduction</b> . . . . .	1
1.1 Problem (KW-1) . . . . .	2
1.2 Problem (KW-2) . . . . .	3
1.3 Variational analysis of (KW-2) . . . . .	4
1.4 Sub-critical approximation and blow-up analysis . . . . .	6
1.5 Some general existence results . . . . .	8
<b>2 Preliminaries</b> . . . . .	11
2.1 A quick review of differential geometry . . . . .	11
2.2 Conformal changes of metric . . . . .	16
2.3 Hölder and Sobolev spaces on closed manifolds . . . . .	17
2.4 Regularity and existence theory for linear elliptic operators . . . . .	18
2.5 Regularity of weak solutions to some non-linear equations . . . . .	19
<b>3 The Yamabe quotient and resolution of (KW-1)</b> . . . . .	23
3.1 On the first eigenvalue of $L_g$ and the Yamabe quotient . . . . .	23
3.2 Spherical decreasing rearrangement . . . . .	27
3.3 The Yamabe quotient of the standard sphere . . . . .	31
3.4 Existence of Yamabe metrics . . . . .	33
3.5 Proof of Theorem A . . . . .	38
<b>4 Variational structure of (KW-2)</b> . . . . .	41
4.1 Existence and uniqueness for non-positive curvatures . . . . .	41
4.2 Variational approach in positive curvature . . . . .	43
4.3 Proof of Theorem B . . . . .	47
<b>5 Classification of entire solutions in the Euclidean space</b> . . . . .	51
5.1 Moving planes and moving spheres . . . . .	51
5.2 Liouville theorem in the sub-critical case . . . . .	56
<b>6 Blow-up analysis</b> . . . . .	61
6.1 Isolated and isolated-simple blow-ups . . . . .	62
6.2 Blow-up analysis in dimensions 3 and 4 . . . . .	70

<b>7</b>	<b>Finite-dimensional reduction of <math>(E_{K,\tau})</math></b>	77
7.1	Lyapunov–Schmidt reduction	78
7.2	Existence of blowing-up solutions to $(E_{K,\tau})$	83
7.3	Blow-up analysis in arbitrary dimension	86
<b>8</b>	<b>General existence results for <math>(E_K)</math></b>	91
8.1	Proof of Theorem C	91
8.2	Existence results via min-max theory	96
8.3	Proof of Theorem D	101
<b>9</b>	<b>Forbidden curvatures</b>	107
9.1	First non-existence results	107
9.2	Forbidden functions with a given Morse structure	109
9.3	Blow-up analysis via study of singular solutions	113
<b>10</b>	<b>Related directions and perspectives</b>	119
10.1	Blow-up analysis in general dimensions	119
10.2	Morse theory and Morse homology	120
10.3	Higher-order and fully non-linear versions	122
10.4	Manifolds with boundary	124
10.5	The complete case	126
10.6	The Einstein constraint equations	127
<b>A</b>	<b>Some useful results</b>	131
A.1	Pohozaev’s identity	131
A.2	Kazdan–Warner identity	133
A.3	A rigidity result	135
	<b>References</b>	139