

CHAPTER 1

Leonhard Euler (1707-1783)

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1.1. Introduction

No one can dispute the statement that Euler was the greatest mathematician and natural philosopher of the 18th century and one of the greatest of all time. He worked on all branches of mathematics, both pure and applied, known in his time. To the end of his life he carried in his mind the entire corpus of mathematics and physics of his epoch. He achieved universality in the most effortless manner, and many of the themes he worked on are still active today. He created new branches of mathematics, like combinatorial topology, graph theory, and the calculus of variations. He was the founder of modern differential and integral calculus as we know them today, and his books introduced algebra and calculus and their applications to enormous numbers of students. It could be said without exaggeration that he did to analysis what Euclid did to geometry, except that Euler himself created a huge part of what went into his books. With his lifelong interest and beautiful contributions, he started the process of establishing number theory as a major discipline in mathematics, a process that was completed by Gauss and the publication of his monumental *Disquisitiones Arithmeticae*. Ever since, number theory has attracted the attention and interest of the greatest mathematicians. As Laplace said, all the mathematicians of his time were his students.

Euler is easily the most prolific mathematician of all time. The range and volume of his output is simply staggering. He published over 850 papers, almost all of substantial length, and more than 25 books and treatises. In 1907 the Swiss Academy of Sciences established the **Euler Commission** with the charge of publishing the complete body of work consisting of all of his papers, manuscripts, and correspondence. This project, known as *Opera Omnia* [1], began in 1911 and is still in progress. His scientific publications, not counting his correspondence, run to over 70 volumes, each between approximately 300 and 600 pages. Thousands of pages of handwritten manuscripts are still not in print. He was in constant communication with all the great scientists of his day, and his correspondence covers several thousand pages, taking up the entire Series IV of *Opera Omnia*. The first volume

of this series is completely devoted just to a catalogue and brief summaries of letters written by him and to him. *Opera Omnia*, of which I shall give more details later, includes his books on differential and integral calculus, calculus of variations, his great treatises on algebra and on analysis, his unfinished treatise on number theory, and his justly famous *Letters to a German Princess on Different Subjects in Natural Philosophy*, which is one of the most popular books on science ever written, translated into eight languages and reprinted countless times [2]. When publication is completed, *Opera Omnia* will be a gigantic landmark in the history of science. Its pages will contain a substantial part of all mathematical research that was carried out in the middle of the 18th century. The 18th century can thus be truly said to be the age of Euler. Furthermore, large parts of the mathematics of the 19th century flowed out of his work. What is even more remarkable is that some of his themes have generated new interest and attention even today. In spite of the fact that he was almost totally blind in the last fifteen years of his life, he wrote over 400 memoirs during that time, about half of his entire output, many of which required colossal calculations that he carried out entirely in his head. The memoirs he submitted to the St. Petersburg Academy were still being published decades after his death; indeed, one of them was not published till 1862, almost 80 years after his death.

Here is a list, which is at best partial, of topics Euler worked on in his lifetime, many of which were founded by him and in almost all of which his work was pioneering:

- Differential and integral calculus
- Logarithmic, exponential, and trigonometric functions
- Differential equations, ordinary and partial
- Elliptic functions and integrals
- Hypergeometric integrals
- Classical geometry
- Number theory
- Algebra
- Continued fractions
- Zeta and other (Euler) products
- Infinite series and products
- Divergent series
- Mechanics of particles
- Mechanics of solid bodies
- Calculus of variations
- Optics (theory and practice)
- Hydrostatics
- Hydrodynamics
- Astronomy
- Lunar and planetary motion
- Topology
- Graph theory

Euler was awarded many honors in his life. He was a member of both the St. Petersburg and Berlin Academies of Sciences. He was elected a member of the Royal Society of London in 1749 and the Académie des Sciences of Paris in 1755. His teacher, Johann Bernoulli, not an easy man to get along with, called him *the incomparable Leonhard Euler* and *mathematicorum princeps*. Several lunar features have been named after him, and the list of mathematical and other scientific discoveries named after him is almost endless: Euler line of a triangle, Euler angles of a rotation, Euler-Lagrange equations of the calculus of variations, Eulerian integrals, Euler characteristic, Euler equations of motion of solid bodies, Euler equations of fluid mechanics, Euler function $\varphi(n)$, Euler-Maclaurin sum formula, Euler's constant, Euler products, and so on. The ten franc bill in Switzerland has his picture on it. Some of the problems he worked on are still open, and his work links with an astonishing amount of contemporary research in both pure and applied mathematics. His books *Introductio in Analysin Infinitorum* and *Mechanica*, which went through many editions, brought calculus and mechanics to the entire scientific world. There was no longer any necessity of reading the obscure papers of Leibniz or the works of Newton couched in an opaque geometrical language that was unsuitable for most problems.

To survey Euler's life and work in detail in a single book is impossible. However there are many excellent accounts which survey parts of his life and work, and I have made free use of these. Without even remotely attempting any sort of completeness, I have listed some of these in [3]. One of the most interesting is the account of A. Weil [3a]: it deals just with Euler's work on number theory, which occupies a bare(!) 4 volumes of the 70 plus volumes of his *Opera Omnia*, but still is over 120 pages long. In addition there are also the introductions to the various volumes of *Opera Omnia* written by experts which deal in depth with his work contained in those volumes. All of this suggests that a complete *scientific biography* of Euler, treating all of his work with historical accuracy and placing it in a modern perspective, will be so complex that its length will exceed any reasonable bound.

A major part of Series I of *Opera Omnia* is concerned with analysis. To Euler this very often meant working with infinite series and transformations of series and integrals. He was, without any doubt, the greatest master of infinite series and products of his, and indeed of any, time. Perhaps only Jacobi and Ramanujan from the modern era can evoke comparable wonder and admiration as formalists. Before Euler, infinite series and products made their appearance only in isolated works (Leibniz, Gregory, Wallis, etc.) and always in an auxiliary manner. Euler was the first to treat them systematically and in great depth, not only for their applications, but also for their own intrinsic interest. In this area he discovered some of the most beautiful formulae ever to be found in mathematics. His ideas on summing divergent series, which he used brilliantly, for instance, in discovering the functional equation of the zeta function at the integer points, led directly to the modern theory of summation of divergent series that was at the center of research in analysis in the 19th century and in the early part of the 20th (see [4], Chs. 1, 2). In recent years summation of divergent series has again become powerful in a wide range of problems, ranging from quantum field theory to dynamical systems.

With its emphasis on concepts and structures, modern mathematics has mostly shied away from formulae. Nevertheless, many of the greatest peaks of mathematics are described by formulae of one sort or another. Here is a sample of some of them that Euler discovered.

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi} = -1$$

$$\log(-1) = i\pi + 2ki\pi (k = 0, \pm 1, \pm 2, \dots)$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \quad \cos x = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2\pi^2}\right)$$

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \dots + \frac{1}{n^{2k}} + \dots = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}$$

$$\frac{\pi}{\sin s\pi} = \frac{1}{s} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n+s} - \frac{1}{n-s} \right)$$

$$\pi \cot s\pi = \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{n+s} - \frac{1}{n-s} \right)$$

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \dots$$

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots$$

$$\frac{\pi}{3} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} \dots$$

$$\frac{\pi^2}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \dots$$

$$\frac{\pi^2}{6\sqrt{3}} = 1 - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} - \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\frac{1 - 2^{m-1} + 3^{m-1} - \text{etc.}}{1 - 2^{-m} + 3^{-m} - \text{etc.}} = - \frac{1.2.3 \dots (m-1)(2^m - 1)}{(2^{m-1} - 1)\pi^m \cos \frac{m\pi}{2}}$$

$$\sum_{n>m>0} \frac{1}{n^2 m} = \sum_{n>0} \frac{1}{n^3}$$

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{n=1}^{\infty} (-1)^n \left(x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}}\right)$$

$$1 - 1!x + 2!x^2 - 3!x^3 + \dots = \frac{1}{1+} \frac{x}{1+} \frac{x}{1+} \frac{2x}{1+} \frac{2x}{1+} \frac{3x}{1+} \frac{3x}{1+} \text{ etc.}$$

$$1 - 1! + 2! - 3! + \dots = 0.596347362123 \dots$$

$$\sum_{k=0}^m f(k) = \int_0^m f(x)dx + \frac{1}{2}(f(0) + f(m))$$

$$+ \sum_{k \geq 1} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(m) - f^{(2k-1)}(0)\right)$$

$$\frac{\partial}{\partial y} F(x, y, y') = \frac{d}{dx} \left(\frac{\partial}{\partial y'} F(x, y, y')\right)$$

1.2. Early life

Leonhard Euler was born in Basel, Switzerland, on April 15, 1707. His father, Paul Euler, was a parish priest, and soon after Leonhard's birth, he settled in a small village nearby where Leonhard grew up. Paul Euler wanted his son to become a priest, and so Leonhard's early education emphasized theology and related subjects. But it soon became clear that Euler's interests and abilities were in mathematics. Paul Euler had attended the lectures in Basel of the great Jacob Bernoulli. Jacob Bernoulli held the mathematics chair at the University of Basel and was one of the most distinguished mathematicians of Europe in his time. The University of Basel was founded in 1460 and was a leading center of learning and research at the beginning of the 18th century. Because of his experience in attending the lectures of Jacob Bernoulli, Paul Euler was able to instruct his young son in mathematics in his early years. With this encouragement Euler studied for himself difficult works in algebra when he was in his early teens. But these studies did not fully satisfy Leonhard, who was discovering his passion for mathematics. By

the time Leonhard was ready to begin higher studies, Jacob Bernoulli had died, and his younger brother, Johann Bernoulli, had succeeded him at the University of Basel to the chair of mathematics. Johann Bernoulli was regarded as the foremost mathematician in Europe of his day. Paul Euler had roomed with him when they were both students at Basel. Leonhard was introduced to Johann Bernoulli and his sons, Nicolaus and Daniel. Daniel would go on to become a famous mathematician and scientist in his own right as well as a lifelong friend of Euler. Johann Bernoulli recognized Euler's genius very early and pushed him to study the masters, helping him in this endeavor. Euler recalls in his autobiographical writings how he was able to meet regularly with Johann Bernoulli and discuss with him the difficulties he had encountered during his mathematical studies and how he worked very hard so as not to bother his mentor with unnecessary questions [3b]:

I soon found the opportunity to gain introduction to the famous professor Johann Bernoulli, whose good pleasure it was to advance me further in the mathematical sciences. True, because of his business he flatly refused to give me private lessons, but he gave me much wiser advice, namely to get some more difficult mathematical books and work through them with all industry, and wherever I should find some check or difficulties, he gave me free access to him every Saturday afternoon and was so kind as to elucidate all difficulties, which happened with such greatly desired advantage that whenever he had obviated one check for me, because of that ten others disappeared right away, which is certainly the way to make a happy advance in the mathematical sciences.

This period of regular contact with a great mathematician like Johann Bernoulli was decisive in the development of Euler as a mathematician. Johann Bernoulli, together with his brother, Jacob, had studied the works of Leibniz on calculus carefully and had made many contributions to problems involving the properties of curves which are the solutions of many types of extremal problems, such as the isochrone and the brachistochrone. These problems must have captured Euler's imagination, because he went on later to formulate these questions in great generality and derived what we now call the Euler-Lagrange equations in the calculus of variations. Johann Bernoulli was a combative and irascible person, as evidenced by the many instances of quarrels with his elder brother and even his own sons on questions of priority, but Euler was his favorite disciple and remained so throughout his life. Perhaps this says more about Euler's personality than that of Johann Bernoulli.

Euler completed his university studies in 1726 and began his independent investigations immediately thereafter. He participated in a competition organized by the Paris Academy of Sciences on the most efficient way to arrange masts on a ship, and although he had never been on a ship, his entry received the second prize and a favorable mention and was published. The prizes of the Paris Academy were substantial in terms of money, and they were generally offered for the solution of some important problem. They were therefore influential in fostering research and for the most part were administered fairly. Euler was to win it twelve times in his career. He also participated in a competition for a professorship in physics

at Basel and wrote a monograph on sound, *Dissertatio physica de sono*, supporting his application. In this monograph he summarized the existing knowledge on acoustics, formulated mathematically with complete clarity the basic questions, and answered some of them. Although he did not succeed in getting the position, this work would remain a classic for years to come. He came in second, possibly because he was thought to be too young for that position. By this time his friends Daniel and Nicolaus Bernoulli had already gone to St. Petersburg and were members of the newly established Academy of Sciences there. This academy had been founded in 1725 by Catherine I, the widow of Peter the Great, following the plans of the late great czar. Through the recommendation of the Bernoulli brothers Euler was offered a position in the Academy in 1726. The position was in physiology, and Euler joined the University at Basel to study the subject and look for mathematical applications in it as a preparation for his move to St. Petersburg. He left Basel to go to St. Petersburg and joined the Academy in 1727, two years after the Academy was established. But by then Nicolaus Bernoulli had died, and Euler was invited to be an adjunct in mathematics, a position that was more suitable for his talents and interests. He became a professor of physics in 1730, and when Daniel Bernoulli left for Switzerland in 1733, Euler succeeded him to the chair in mathematics at the Academy. He stayed in St. Petersburg, for 14 years altogether, before he went to Berlin in 1741 and joined the Academy there. He worked 25 years in Berlin before returning to St. Petersburg, where he worked till his death in 1787. He never returned to Switzerland, although he retained his Swiss citizenship till the end.

Before we begin a sketch of Euler's academic life it may be useful to set down a time line that details the main events of his career.

- 1707 Born in Basel, Switzerland, April 15.
- 1725 Peter the Great and his widow Catherine establish the St. Petersburg Academy of Sciences in St. Petersburg, Russia.
- 1727 Euler moves to St. Petersburg and becomes an adjunct in mathematics.
- 1733 Euler takes over the chair in mathematics after Daniel Bernoulli returns to Basel. Gets married and buys a house.
- 1735 Solves the problem of finding the sum of $\sum_{n \geq 1} \frac{1}{n^2}$ and acquires an international reputation.
- 1738 Euler loses the vision in his right eye after a serious illness.
- 1741 Political turmoil in Russia after death of the czarina and the regency. Euler leaves Russia to join the Academy of Sciences in Berlin, Prussia.
- 1762 Catherine (the Great) II becomes the czarina in Russia and starts the efforts to get Euler back.
- 1766 Euler returns to St. Petersburg. His eyesight begins to deteriorate.
- 1771 Euler loses the vision in his left eye also.
- 1783 Dies in St. Petersburg on September 18.

It may probably be helpful to have a feeling for the chronology of Russian imperial succession to follow the events involving Euler. Peter I (the Great) died in 1725, and his widow, Catherine I, followed him on the throne. The St. Petersburg Academy was established in the same year by her, following closely Peter's plans. She died in 1727, and Peter II followed her as the czar. But the rule of Peter II lasted only till 1730, when Anna Ivanovna, niece of Peter I, succeeded him. Her reign lasted till 1740, when Ivan VI succeeded to the throne. Being too young, his mother, Anna Leopoldovna, acted as the regent. But the regency was very short lived, and Anna and Ivan VI were overthrown by Elizabeth Petrovna, daughter of Peter I, in a coup. She ruled from 1741 to 1762. Moscow University was founded in 1755 during her reign. In 1762 Peter III became the czar but was murdered almost immediately, and his widow, Catherine II (the Great Catherine), gained power and ruled Russia as the czarina till 1796 with great distinction. Apart from renewing the glory of the St. Petersburg Academy by getting Euler to come back, she created the beginnings of the great art collections of the Hermitage museum, expanded Russian influence politically as well as militarily, and made Russia a great European power. One may legitimately compare her reign to that of Elizabeth I of England.

1.3. The first stay in St. Petersburg: 1727-1741

The years 1727-1741 at St. Petersburg were very productive for Euler scientifically and very comfortable personally. It was during his first stay in St. Petersburg that his transformation into a mathematician of the foremost rank took place. His colleagues at the Academy were first rate scientists and the conditions offered by the Academy were very generous, a fact which he acknowledged in a letter written in 1749 [3c]:

... I and all others who had the good fortune to be for some time with the Russian Imperial Academy cannot but acknowledge that we owe everything which we are and possess to the favorable conditions we had there...

In return these scientists were expected to publish their research and add prestige to the Academy and advise the czar on whatever question for which their help was sought. This freedom to work on problems that interested him apart from the time he was assisting the government was priceless for Euler, and his mathematical personality bloomed under these conditions. Euler himself was in charge of projects involving cartography (which produced severe strain on his eyes, as he remarked once in a letter), shipbuilding, and general questions of military science.

Initially Euler had difficulties in his career mainly because of the interference of the administrator for the Academy, one Schumacher, whose main interest seemed to lie in the suppression of talent wherever it might rear its inconvenient head ([3b], p. xv). When Schumacher contrived to place Euler on the same level as a few mediocre colleagues, Euler protested strongly to him (see [3d], p. 23):

It seems to me that it is very disgraceful to me, that I, who up to now have had more salary than the others, shall now be set equal to them... I think that the number of those who have carried [mathematics] as far as I is pretty small in the whole of Europe, and none of them will come for 1000 rubles.

But the growing stature of Euler in the world of mathematics and physics eventually overcame all obstacles, and Euler rose rapidly in the ranks of the Academy. During the early years in St. Petersburg Daniel Bernoulli was a constant companion to Euler, both personally and scientifically. But gradually Daniel Bernoulli got tired of life in Russia and the constant intrigues of Schumacher and could not wait to go back to Switzerland. When an opportunity came he took it and left for Basel to become a professor of anatomy and botany there. However it must be said that he never again reached afterwards the level of scientific work that he had attained in Russia. This is understandable since in St. Petersburg there was the constant presence of and inspiring interaction with Euler. After Daniel Bernoulli's departure Euler was appointed to the chair of mathematics and paid handsomely, so that a measure of financial security was achieved. Euler got married in 1733 to Katherina Gsell, the daughter of a Swiss painter named Gsell who was working in St. Petersburg, and purchased a house on the banks of the Neva River close to the Academy. Euler's son, Johann Albrecht, who would become a high-level scientist and a member of the Academy and with whom Euler would collaborate much later in his life, was born in 1734.

In St. Petersburg Euler's genius flourished, and he worked on a variety of subjects. Papers streamed from his pen to fill the pages of the proceedings of the Academy. He produced fundamental memoirs on number theory, infinite series, calculus of variations, mechanics, as well as applications of these topics to various questions in music, cartography, shipbuilding, and so on. He prepared close to 100 memoirs for publication on a variety of topics. He acquired a European reputation as one of the greatest living mathematicians and mathematical physicists through the originality and prolific nature of his work, and the remarkable manner in which he could develop applications of mathematics to all sorts of situations. For a detailed account of Euler's first years in St. Petersburg, the reader is referred to [3d]. Here I shall discuss only the highlights of his work in mathematics during the stay in St. Petersburg.

Very early in his stay at St. Petersburg Euler became acquainted with Christian Goldbach. This soon developed into a lifelong friendship which played a great role in Euler's life, both personally and scientifically. Goldbach was "an energetic and intelligent Prussian for whom mathematics was a hobby, the entire realm of letters an occupation, and espionage a livelihood" (see [3b], p. xv). Goldbach was a catalyst for Euler, suggesting to him possible questions and acquainting him with what he had learned in his travels. They exchanged a huge number of letters, most of them on problems of number theory and analysis. The very first letter that Goldbach wrote to Euler in December 1729 contains a postscript asking whether Euler knew of Fermat's statement that all numbers of the form $2^{2^n} + 1$ were primes. Euler was initially cool to the suggestion of Goldbach but became very interested in number theory soon afterwards and disproved the assertion about Fermat primes. It was Goldbach who stimulated Euler to look deeply into Fermat's discoveries and supply the proofs of all of Fermat's statements. Eventually Euler would go far beyond Fermat, discovering towards the end of his life the law of quadratic reciprocity for which he could not supply a proof. It was proved by Gauss.

It was during this period that he solved one of the most famous open problems at that time, namely to find the sum of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots,$$

whose solution had eluded all the leading mathematicians of that era. Euler discovered that the sum is

$$\frac{\pi^2}{6}$$

and also that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}.$$

It is this work more than anything else that established him as the foremost mathematician of his time. Later on he would succeed in proving that

$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \dots = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}$$

where B_{2k} are rational numbers, named Bernoulli numbers by Euler, in view of the fact that they were originally introduced by Jacob Bernoulli in his *Ars Conjectandi*. It must be noted that the evaluation of the series

$$1 + \frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}} + \frac{1}{4^{2k+1}} + \dots$$

is open to this day and appears to be out of reach at present.

Closely related to the evaluations of the series above is the question of computing them accurately to several decimal places. For low values of k this is indeed a problem because the series converge slowly:

$$\frac{1}{(n+1)^k} + \frac{1}{(n+2)^k} + \dots \asymp \frac{1}{n^{k-1}}.$$

These questions of accurate numerical evaluation, which were very dear to Euler's heart, must have inspired him to make his remarkable discovery of what is now called the Euler-Maclaurin summation formula. Euler discovered it in 1734 and gave several applications; Maclaurin arrived at it apparently independently in 1738. With characteristic generosity Euler did not engage in any dispute about priority except to content himself to the statement that the result and its demonstration were publicly read before the Academy in 1734. The summation formula was a favorite of Euler. He used it in hundreds of problems and with many ingenious variations. The summation formula is

$$\begin{aligned} f(0) + f(1) + \dots + f(m) &= \int_0^m f(x) dx + \frac{1}{2}(f(0) + f(m)) \\ &\quad + \sum_{k \geq 1} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(m) - f^{(2k-1)}(0) \right). \end{aligned}$$

The point is that the right side of the formula is a series which is typically divergent and so has to be treated as an asymptotic series. This requires great skill in deciding up to what stage one should carry the sum on the right, and Euler was a past master in this.

It was during these years that Euler went deeply into problems of finding curves satisfying extremal conditions, what we now call the calculus of variations. I have already remarked on the fact that Euler's interest in these problems goes back to

his days of apprenticeship to Johann Bernoulli. But Euler was to take this subject from the realm of a few special problems to a far-reaching theory. This work would appear as a book in 1744, but much of the work must have been done during his St. Petersburg stay.

His two-volume work *Mechanica* also dates to this period. Euler was the first person to introduce analytical and algebraic methods in mechanics, thus breaking away from the tradition of Newton and his contemporaries, who insisted on geometric methods. In retrospect, Euler's methods revolutionized the treatment of problems by basing their solutions on differential equations. With hindsight we should not find this too surprising, since the mechanical trajectories are solutions of the Euler-Lagrange variational equations for a suitable functional (action) and so quite within the horizon of Euler at that time. It is fair to say that with the publication of *Mechanica*, Euler's reputation as a natural philosopher of the highest rank was solidified. He also published two volumes on *Scientia navalis* (hydrodynamics, shipbuilding and navigation).

Add to all of this his work on cartography, theory of music, and other duties executed in service of the government, and we get a picture of a man of prodigious energy and creativity, with his powers approaching a peak, working all the time. This enormous workload, especially the drudgery and close reading involved in his cartographic work, led to a severe illness, as a result of which he lost the sight in his right eye in 1738. He was reported to have said at that time that "this means less distraction" and continued to work at the same punishing rate.

1.4. The Berlin years: 1741-1766

The death of the czarina in 1740 and the resulting political turmoil and xenophobia in Russia created conditions that were very difficult, even dangerous, for foreigners in Russia. Euler got an offer from King Frederick II of Prussia to become the director of mathematics in the Prussian Academy of Sciences. He left St. Petersburg to go to Berlin in 1741 and joined the Academy as its top mathematician. He was to remain with the Academy for the next 25 years. He was compensated generously and was able to purchase a good house. But what was remarkable was that the St. Petersburg Academy still retained him as a member and continued to pay him a pension. In recompense, Euler sent almost half of his publications to the St. Petersburg Academy for publication, wrote several important books on its commission, advised the Academy on its manifold scientific activities, and served as its representative in the Western world. He was clearly relieved to be away from the danger and turmoil of Russia. In a letter to Johann Caspar Wettstein, Euler wrote (see [3d], p.159):

I can do just what I wish [in my research]. . . . The king calls me his professor, and I think I am the happiest man in the world

In Berlin Euler reached the peak of his career. More than 100 memoirs were sent to the St. Petersburg Academy, and about 125 memoirs were published in the Berlin Academy on all possible topics in mathematics and physics. This was an astonishing burst of creativity, unparalleled before or since.

However, over the years the relations between him and King Frederick slowly deteriorated. The king was essentially an ignoramus so far as mathematics was concerned and was much taken with superficial cleverness. He admired French literati and figures of science (like Voltaire and d'Alembert) but without any deep

understanding of their merits or their work. He certainly did not comprehend fully the monumental stature of Euler. When Euler first came to Berlin the king had appointed Maupertius as the president of the Academy. Maupertius of course was not at the level of Euler as a scientist, but Euler, as was typical of him, maintained cordial relations with Maupertius and was very influential in the affairs of the Academy. Maupertius died in 1759, and after that Euler functioned as the *de facto* president. The king never conferred on Euler the title of the president of the Academy. When it appeared around 1763 that Frederick was contemplating the appointment of d'Alembert as the president of the Berlin Academy (although d'Alembert eventually rejected the king's offer), Euler began to think about leaving Berlin. Eventually he wrote to the secretary of the St. Petersburg Academy about his wish to return. Meanwhile Catherine (the Great) II had become the czarina in Russia, and one of her most important priorities was to restore the St. Petersburg Academy to its former position of glory and importance. This meant going after Euler, and she ordered her ambassador in Prussia to make an offer to Euler on any terms that Euler would specify. At first the king refused Euler's requests to leave, but he could not prevail against the pressure of the formidable czarina. So he finally gave in and allowed Euler to leave with his family and several assistants, expressing his frustration and irritation in crude jokes to his companions. Euler returned to Russia in 1766 via Poland, where he was treated with great respect and warmth by the Polish king Stanislas, a former lover of Catherine II, thanks almost certainly to her suggestions. He returned to St. Petersburg virtually as a conquering hero, universally admired and respected.

While he was in Berlin Euler published several works. His great treatise on analysis, *Introductio in analysin infinitorum*, in which he gave a majestic exposition of his theory of circular functions and their applications, was published in 1748. This book was a landmark in analysis and dominated the field for the next century or so. It is still beautiful to read. All later books on calculus cover only parts of the material of this book and without the compelling force of the *Introductio*. The monograph on the calculus of variations that was begun while he was in St. Petersburg was published in 1744. The treatise on mechanics of a solid body, *Theoria motus corporum solidorum seu rigidorum*, was published in 1765.

It was during these years that Euler obtained his famous infinite product expansion for $\sin x$, which allowed him to establish with complete rigor his earlier results on the zeta values. It was also during this period that he discovered the relation between the solutions of the Pell's equation $x^2 - Ny^2 = 1$ and the continued fraction for \sqrt{N} . The Euler equations for the motion of a rigid body are still among the basic examples of dynamical systems on a Lie group. His infinite product for $\zeta(s)$ as well as its functional equation dates back to this time. It is thus clear that this was his greatest period, when everything went wonderfully well.

1.5. The second St. Petersburg stay and the last years: 1766-1783

Euler's prodigious scientific activity continued unabated after his return to St. Petersburg even though another serious illness resulted in his losing almost entirely the sight in his other eye. He thus became almost totally blind after 1771. Unbelievably, almost half of his work, about 400 memoirs, was written during this second St. Petersburg stay, a substantial part of which was carried out during this period of near total blindness. His book on algebra was published during this

period. He worked with assistants, many of whom were first-rate scientists in their own right, and one of whom was his own son, Johann Albrecht. He had a huge slate board fitted to his desk in his study on which he wrote in big letters so that he could dimly see what was being written. His memory was prodigious, and his ability to perform intricate calculations absolutely incredible. Still it must have been hard on him to rely on others in working out calculations. In a moving letter to Lagrange, answering a letter from him, he writes (see [3a], p. 168):

I have had all your calculations read to me, concerning the equation $101 = p^2 - 13q^2$, and I am fully persuaded of their validity; but as I am unable to read or write, I must confess that my imagination could not follow the reasons for all the steps you have had to take, nor keep in mind the meaning of all your symbols. It is true that such investigations have formerly been a delight to me and that I have spent much time on them; but now I can only undertake what I can carry out in my head, and often I have to depend on some friend to do the calculations which I have planned....

It was during this period that he completed some of his work on lunar motion that included colossal calculations, many of which were carried out by him entirely in his head. Other books during this period include *Dioptrica* (1769-1771); the three volumes of *Institutiones calculi integralis* (1768-1770); and his treatise on algebra, *Vollständige Anleitung zur Algebra* (1770), first published in Russian. These books were dictated to his assistants.

Euler's wife, Katherina, died in 1773, and Euler married her sister, Abigail Gsell, three years later. Euler died on 18 September 1783 as a result of a cerebral hemorrhage. There was no indication of any problem till his very death, and he worked on problems literally till his last breath. His death was quick and painless. On the day of his death he was working with his assistants on the orbit of the recently discovered planet Uranus,* and on his slate board was found a calculation of how high a hot air balloon could rise, perhaps stimulated by the news of the first ascents of such balloons.† Eulogies were delivered by the Marquis de Condorcet and by Euler's great-grandson, P.-H. Fuss. For English translations of these see [5].

1.6. Opera Omnia

The first thing that strikes anyone is the sheer amount and scope of Euler's scientific work. The definitive catalogue of Gustav Eneström lists over 850 titles of memoirs with very few real repetitions.

The Swiss Academy of Sciences established the **Euler Commission** in 1907 with the charge to publish all of Euler's papers, correspondence, manuscripts, notes, and diaries. This of course required not only financial assistance but also the cooperation of literally hundreds of leading mathematicians and Eulerian scholars of great distinction around the world. Publication of the collected works of Euler

*Uranus was discovered by the English astronomer William Herschel in 1781. The determination of its orbit, however, proved troublesome and was not completed till 1846.

†Brothers Joseph and Etienne Montgolfier, paper manufacturers, were pioneers in the flights of hot air balloons and had succeeded in making several unmanned flights in 1783. Their first flight to carry living things (animals actually) took place on September 19, 1783. However, news of the earlier flights must have reached St. Petersburg during the last days of Euler.

began in 1911 but was stopped while still incomplete. It was resumed a few years ago and is now essentially complete, except for a few volumes of Series IV A and some of Series IV B; these are in preparation and scheduled to appear soon. Many of the volumes have substantial introductions written by modern experts in various fields. Each volume is between 300 and 600 pages approximately so that one is looking at a collection of over 30,000 pages in print. It is the result of a gigantic collective scientific effort unlike any the scientific world has ever seen. The collection is called *Opera Omnia* and is in four series. I have listed at the end the titles and the editors of all the volumes so that the reader can get an idea of the monumental nature of the effort that has gone into this project, as well as the prodigious scope of the man whose works are being published. There is nothing that is even remotely like this in the whole history of science.

1.7. The personality of Euler

The portrait of Euler that emerges from his publications and letters is that of a genial man of simple tastes and conventional religious faith. He was even wealthy, at least in the second half of his life, but ostentation was not a part of his lifestyle. His memory was prodigious, and contemporary accounts have emphasized this. He would delight relatives, friends, and acquaintances with a literal recitation of any song from Virgil's *Aeneis*, and he would remember minutes of Academy meetings years after they were held. He was not given to envy, and when someone made an advance on his work his happiness was genuine. For example, when he learnt of Lagrange's improvements on his work on elliptic integrals, he wrote to him that his admiration knew no bounds and then proceeded to improve upon Lagrange! (See [3a], p. 284.)

But what is most characteristic of his work is its clarity and openness. He never tries to hide the difficulties from the reader. This is in stark contrast to Newton, who was prone to hide his methods in obscure anagrams, and even from his successor, Gauss, who very often erased his steps to present a monolithic proof that was seldom illuminating. In Euler's writings there are no comments on how profound his results are, and in his papers one can follow his ideas step by step with the greatest of ease. Nor was he chary of giving credit to others; his willingness to share his summation formula with Maclaurin, his proper citations to Fagnano when he started his work on algebraic integrals, his open admiration for Lagrange when the latter improved on his work in calculus of variations are all instances of his serene outlook. One can only contrast this with Gauss's reaction to Bolyai's discovery of non-Euclidean geometry. Euler was secure in his knowledge of what he had achieved but never insisted that he should be the only one on top of the mountain.

Perhaps the most astonishing aspect of his scientific opus is its universality. He worked on everything that had any bearing on mathematics. For instance his early training under Johann Bernoulli did not include number theory; nevertheless, within a couple of years after reaching St. Petersburg he was deeply immersed in it, recreating the entire corpus of Fermat's work in that area and then moving well beyond him. His founding of graph theory as a separate discipline, his excursions in what we call combinatorial topology, his intuition that suggested to him the idea of exploring multizeta values are all examples of a mind that did not have any artificial boundaries. He had no preferences about which branch of mathematics

was dear to him. To him, they were all filled with splendor, or *Herrlichkeit*, to use his own favorite word.

Hilbert and Poincaré were perhaps the last of the universalists of the modern era. Already von Neumann had remarked that it would be difficult even to have a general understanding of more than a third of the mathematics of his time. With the explosive growth of mathematics in the twentieth century we may never see again the great universalists. But who is to say what is and is not possible for the human mind?

It is impossible to read Euler and not fall under his spell. He is to mathematics what Shakespeare is to literature and Mozart to music: universal and *sui generis*.

Notes and references

[1] **Leonhardi Euleri Opera Omnia**, Edited by the Euler Commission of the Swiss Academy of Science in collaboration with numerous specialists, 1911-. Originally started by the publishing house of B. G. Teubner, Leipzig and Berlin. Birkhäuser, Boston and Basel, has continued publication and has made available all volumes of the *Opera Omnia*.

Series prima : *Opera mathematica*.

This is Series I and contains all his papers on what we may call “pure mathematics”. It has 29 volumes, 30 volume-parts.

Vollständige Anleitung zur Algebra.

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Commentationes arithmeticae. Contributions to number theory.

I-2 (611 pages, 1915), Ferdinand Rudio (Ed.).

I-3 (543 pages, 1917), Ferdinand Rudio (Ed.).

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Commentationes algebraicae ad theoriam aequationum pertinentes.

I-6 (509 pages, 1921), Ferdinand Rudio, Adolf Krazer, and Paul Stäckel (Eds.).

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Introductio in Analysin Infinitorum, parts 1, 2.

I-8 (392 pages, 1922), Adolf Krazer and Ferdinand Rudio (Eds.).

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I-11 (462 pages, 1913), Friedrich Engel and Ludwig Schlesinger (Eds.).

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Commentationes analyticae ad theoriam serierum infinitarum pertinentes.

His papers on infinite series, products, zeta values, and related matters are in I-14, I-15, and I-16:

I-14 (617 pages, 1925), Carl Boehm and Georg Faber (Eds.).

I-15 (722 pages, 1922), Georg Faber (Ed.).

I-16/1 (355 pages, 1933), Carl Boehm (Ed.).

I-16/2 (332 pages, 1935), Carl Boehm (Ed.).

Commentationes analyticae ad theoriam integralium pertinentes.

Related matters involving integrals are in I-17, I-18, and I-19:

I-17 (457 pages, 1914), August Gutzmer (Ed.).

I-18 (475 pages, 1920), August Gutzmer and Alexander Liapounoff (Eds.).

I-19 (494 pages, 1932), Alexander Liapounoff, Adolf Krazer, and Georg Faber (Eds.).

Commentationes analyticae ad theoriam integralium ellipticorum pertinentes.

Papers on elliptic integrals are in I-20 and I-21:

I-20 (371 pages, 1912) Adolf Krazer (Ed.).

I-21 (380 pages, 1913), Adolf Krazer (Ed.).

Commentationes analyticae ad theoriam aequationum differentialium pertinentes.

Papers on differential equations are in I-22 and I-23:

I-22 (420 pages, 1936), Henri Dulac (Ed.).

I-23 (455 pages, 1938), Henri Dulac (Ed.).

Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solution problematis isoperimetrici latissimo sensu accepti.

I-24 (308 pages, 1952), Constantin Carathéodory (Ed.).

Commentationes analyticae ad calculum variationum pertinentes.

I-25 (343 pages, 1952): Constantin Carathéodory (Ed.).

I-24 and I-25 contain his works on the calculus of variations.

Commentationes geometricae.

I-26 (362 pages, 1952), Andreas Speiser (Ed.).

I-27 (400 pages, 1954), Andreas Speiser (Ed.).

I-28 (381 pages, 1955), Andreas Speiser (Ed.).

I-29 (488 pages, 1956), Andreas Speiser (Ed.).

Series secunda: *Opera mechanica et astronomica.*

This is Series II and contains works devoted to mechanics and astronomy and has 31 volumes, 32 volume-parts.

Mechanica sive motus scientia analytice exposita.

II-1 (417 pages, 1912) and II-2 (460 pages, 1912), Paul Stäckel (Ed.).

Theoria motus corporum solidorum seu rigidorum ex primis nostrae cognitionis principis stabilita et ad omnes motus qui in huiusmodi corpora cadere possunt accomodata.

II-3 (327 pages, 1948) and II-4 (359 pages, 1950), Charles Blanc (Ed.).

Commentationes mechanicae.

II-5 (326 pages, 1957), Joachim Otto Fleckenstein (Ed.).

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II-6 (302 pages, 1957), and II-7 (326 pages, 1958), Charles Blanc (Ed.).

Mechanica corporum solidorum.

II-8 (417 pages, 1965), and II-9 (441 pages, 1968), Charles Blanc (Ed.).

Commentationes mechanicae ad theoriam flexibilium et elasticorum pertinentes.

- II-10 (451 pages, 1947), Fritz Stüssi and Henri Favre (Eds.).
 II-11/1 (383 pages, 1957), Fritz Stüssi and Ernst Trost (Eds.).
 II-11/2 (435 pages, 1960), Clifford Ambrose Truesdell (Ed.).

Commentationes mechanicae ad theoriam fluidorum pertinentes.

- II-12 (288 pages, 1954), Clifford Ambrose Truesdell (Ed.).
 II-13 (375 pages, 1955), Clifford Ambrose Truesdell (Ed.).

Neue Grundsätze der Artillerie.

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 II-16 (327 pages, 1979), and II-17 (312 pages, 1982), Charles Blanc and Pierre de Haller (Eds.).

Scientia navalis.

- II-18 (427 pages, 1967), and II-19 (459 pages, 1972), Clifford Ambrose Truesdell (Ed.).

Commentationes mechanicae et astronomicae ad scientiam navalem pertinentes.

- II-20 (275 pages, 1974), and II-21 (241 pages, 1978), Walter Habicht (Ed.).

Theoria motuum lunae, nova methodo pettractata.

- II-22 (412 pages, 1958), Leo Courvoisier (Ed.).

Sol et luna.

- II-23 (336 pages, 1969), J. O. Fleckenstein (Ed.).
 II-24 (326 pages, 1991), Charles Blanc (Ed.).

Commentationes astronomicae ad theoriam perturbationum pertinentes.

- II-25 (331 pages, 1960), Max Schürer (Ed.).
 II-26, and II-27 (In preparation).

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- II-28 (332 pages, 1959), Leo Courvoisier (Ed.).

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- II-29 (420 pages, 1961), Leo Courvoisier (Ed.).

Sphärische Astronomie und Parallaxe.

- II-30 (351 pages, 1964), Leo Courvoisier (Ed.).

Kosmische Physik (In preparation).**Series tertia.** *Opera physica, Miscellanea.*

This is Series III, dedicated to physics and other miscellaneous contributions.

Commentationes physicae ad physicam generalem et ad theoriam soni pertinentes.

- III-1 (591 pages, 1926), Eduard Bernoulli, Rudolf Bernoulli, Ferdinand Rudio, and Andreas Speiser (Eds.).

Rechenkunst. Accesserunt commentationes ad physicam generalem pertinentes et miscellanea.

- III-2 (431 pages, 1942), Edmund Hoppe, Karl Matter, and Johann Jakob Burckhardt (Eds.). *Dioptrica.*
 III-3 (510 pages, 1911), III-4 (543 pages, 1912), Emil Cherbuliez (Ed.).

Commentationes opticae.

- III-5 (395 pages, 1963), David Speiser (Ed.).
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Commercium cum Johanne (I) Bernoulli et Nicolao (I) Bernoulli.

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CHAPTER 3

Zeta Values

- 3.1. Summary
 - 3.2. Some remarks on infinite series and products and their values
 - 3.3. Evaluation of $\zeta(2)$ and $\zeta(4)$
 - 3.4. Infinite products for circular and hyperbolic functions
 - 3.5. The infinite partial fractions for $(\sin x)^{-1}$ and $\cot x$. Evaluation of $\zeta(2k)$ and $L(2k + 1)$
 - 3.6. Partial fraction expansions as integrals
 - 3.7. Multizeta values
- Notes and references

3.1. Summary

Before Euler's work, infinite series and products had occurred only sporadically in mathematics and nearly always in an isolated manner. With Euler this situation changed dramatically. He was the first and greatest master of infinite series and products and created the first general theory dealing with them. Certainly the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{R^n}$$

for special values such as $R = 2, 4$ goes back to the Greeks. For example, the quadrature of the parabola due to Archimedes depends on the formula

$$\frac{4}{3} = 1 + \frac{1}{4} + \frac{1}{4^2} + \dots$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

appears in the work of Nicholas of Orseme (1323-1382), and Pietro Mengoli (1625-1686) seems to have posed [1] the problem of finding the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

The series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

was discovered by Leibniz, although it appears to have been known to Gregory. We shall follow current universal usage and write

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

following Riemann's notation in his famous paper on the distribution of primes. Many famous mathematicians of the 17th and 18th centuries, including the Bernoulli brothers Jacob and Johann, worked on the problem of determining $\zeta(2)$, which came to be known as the Basel problem. However its solution eluded all these mathematicians. Even the approximate computation of its sum, accurate to several decimal places, proved troublesome because of the slow convergence of the series.

Enter Euler. First in 1731 he discovered a brilliant transformation of the series for $\zeta(2)$. He proved that [2]

$$\zeta(2) = (\log 2)^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 2^n}$$

which is rapidly convergent because of the factors 2^n in the denominator. Now, from the Taylor series for $\log(1-x)$ we have

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

and so, taking $x = 1/2$,

$$\log 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}.$$

The geometric nature of the terms again allows an accurate computation of the series from a small number of terms. Euler did this and found that

$$\zeta(2) = 1.644934 \dots$$

The problem of numerical evaluation of $\zeta(n)$ for higher values of n probably led Euler to his discovery of his famous *Euler-Maclaurin summation formula*. He announced it in 1732 [3a] and discussed it more elaborately in 1736 in [3b] and in his book [3c] on differential calculus. From then on this was his preferred tool for very accurate numerical summation of series. It was by using it that he was able to show that [3b], [3d]

$$\zeta(2) = 1.64493406684822643647 \dots$$

$$\zeta(3) = 1.202056903159594 \dots$$

$$\zeta(4) = 1.08232323371113819 \dots$$

So matters stood till suddenly and unexpectedly, around 1735, Euler had a stroke of inspiration that led him to the exact value of $\zeta(2)$, $\zeta(4)$, etc. [4]. Here is the translation by A. Weil of the opening lines of Euler's paper ([5], p. 261): *So much work has been done on the series $\zeta(n)$ that it seems hardly likely that anything new about them may still turn up. . . . I, too, in spite of repeated efforts, could achieve nothing more than approximate values for their sums. . . . Now, however, quite unexpectedly, I have found an elegant formula for $\zeta(2)$, depending on the quadrature of a circle [i.e., upon π].* Across a gulf of centuries, this passage, and indeed the whole paper, still conveys the excitement Euler must have felt on his discovery. Euler's method was an audacious, one might say, even reckless, generalization of Newton's theorem on the symmetric functions of the roots of a polynomial to a power series. By using it Euler succeeded in determining the exact value of $\zeta(2)$

and $\zeta(4)$. In a letter to Daniel Bernoulli he communicated his formulae

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \quad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}.$$

Although questions were raised by the Bernoulli brothers and others regarding the validity of Euler's method in deriving these results, the numerical calculations essentially confirmed his results, and Euler's reputation as a mathematician of the first rank was established. Euler was aware that the objections to his derivation were grave and legitimate and so he continued to work on meeting these objections and making his arguments more solid, at least by the standards of his era (actually, as we shall see later, even by modern standards once we have the notion of uniform convergence). He succeeded in doing this around 1742 [6] when he proved the infinite product expansion

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

from which, by a legitimate application of Newton's theorem generalized to the context of power series, he could justify completely the values calculated earlier in [4]:

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ \zeta(8) &= \frac{\pi^8}{9450} \\ \zeta(10) &= \frac{\pi^{10}}{93555} \\ \zeta(12) &= \frac{691\pi^{12}}{6825 \times 93555} \end{aligned}$$

It is here that he remarks that the product expansions show that the equations of infinite degree that he had used earlier had no roots other than the obvious real ones (see p. 146 in [6] and Weil [5], p. 271). The appearance of 691 might perhaps have suggested to him that there was a connection with the *Bernoulli numbers*, which he had already encountered in his work on the summation formula. He then went on to prove the beautiful formula [7]

$$\zeta(2k) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \cdots = \frac{(-1)^{k-1} B_{2k} 2^{2k}}{2(2k)!} \pi^{2k}$$

where the B_m are the *Bernoulli numbers* given as the coefficients of the expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}.$$

These would follow from the striking partial fraction expansion

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right)$$

which itself is a consequence of his infinite product for $\sin x$, as was pointed out to him by Nicolaus Bernoulli.* Euler knew this and had in fact two different proofs of this, and in any case it already appeared implicitly in his paper.

In addition to the zeta series Euler also considered the L -series

$$L(2k+1) = 1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \dots$$

For these he obtained the formulae

$$L(2k+1) = \frac{1}{2^{2k+2}(2k)!} E_{2k} \pi^{2k+1}$$

where the E_{2k} are the *Euler numbers* which are defined by

$$\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k} z^{2k}}{(2k)!}.$$

These would follow from the equally striking partial fraction expansion

$$\frac{\pi}{\sin \pi x} = \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n+x} - \frac{1}{n-x} \right).$$

All these results were given a majestic exposition in his great book [3d].

These infinite product and partial fraction expansions were the beginning of a glorious era in the global theory of complex functions. On the one hand, in the hands of Weierstrass, Mittag-Leffler, Hadamard and others it would lead to the golden age of entire function theory. Hadamard used his work to get a proof of the prime number theorem (at the same time as de la Vallée Poussin). On the other hand, at the hands of Riemann, whose work was preceded by that of Eisenstein and Gauss, it would lead to function theory on compact complex manifolds and dominate much of twentieth century geometry. In a third direction it would lead, at the hands of Dirichlet, Kronecker, and their modern successors (not to mention Eisenstein and Gauss again), to the arithmetic theory of abelian extensions of algebraic number fields. It is thus extraordinary that Euler's work could be thought of as the springboard for his successors. In our own time the themes initiated by Riemann have perhaps obscured the possibility that the theory of entire functions has still many deep secrets waiting to be discovered.

Let us return to the zeta values. Euler gave another proof, perfectly unobjectionable by the standards of his (or our) day and completely elementary, for his formula $\zeta(2) = \pi^2/6$. He started from the formula

$$\frac{1}{2}(\arcsin x)^2 = \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}} dt.$$

Taking $x = 1$ gives $\pi^2/8$ for the left-hand side. On the other hand, the integral on the right can be evaluated by expanding $\arcsin t$ as a power series and integrating term by term. The result is

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$$

which leads to

$$\zeta(2) = \frac{\pi^2}{6}$$

*See [1] of Ch. 2, Notes and References.

since

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \zeta(2) - \frac{1}{2^2}\zeta(2) = \frac{3}{4}\zeta(2).$$

However, in spite of all his efforts, Euler could not make any progress in the problem of finding the sum

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots$$

nor the more general sums

$$1 + \frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}} + \dots$$

He gave some wonderful formulae for $\zeta(3)$ such as

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \dots = \frac{\pi^2}{\log 2} + 2 \int_0^{\pi/2} x \log \sin x dx$$

(see [1] for a detailed discussion). Amazingly, the problem of evaluating the $\zeta(2k+1)$ has remained intractable to this day.

It is remarkable, but fitting, that the next step in the story of zeta values was taken by Euler himself. In a beautiful paper [9] written in 1775 more than 30 years after his great discoveries on zeta values, he introduced the analogues of the zeta series in two variables, more precisely the series

$$\zeta(a, b) = \sum_{n>m>0} \frac{1}{n^a m^b}.$$

Here a and b are positive integers and $a \geq 2$ so that the series is convergent. He obtained several striking formulae for these double zeta values among which

$$\zeta(2, 1) = \zeta(3)$$

stands out. It must be noted that the above definition of $\zeta(a, b)$ differs from Euler's, who summed over $n \geq m$; thus

$$\zeta_{\text{Euler}}(a, b) = \zeta(a, b) + \zeta(a + b).$$

The previous formula then appears in [8] as

$$\zeta_{\text{Euler}}(2, 1) = 2\zeta(3).$$

It is very clear from his paper that his goal was to express, for each given positive integer $r \geq 3$, the $\zeta(a, b)$ with $a + b = r$ as a rational linear combination of $\zeta(r)$ and the $\zeta(p)\zeta(r-p)$ ($2 \leq p \leq r-2$). He succeeded in many cases but not all.

It is obvious that Euler's idea can be generalized to higher dimensions. The *multizeta values* (MZV) are defined by

$$\zeta(a_1, \dots, a_r) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{n_1^{a_1} \dots n_r^{a_r}}$$

where the a_i are positive integers and $a_1 \geq 2$ to ensure convergence. Actually one can use a summation process to define the MZV even when there is divergence.

The first new result after Euler was Roger Apéry's stunning proof in 1978 that $\zeta(3)$ is irrational ([28] of Ch. 2.). Going beyond Apéry, T. Rivoal has now proved that an infinite number of the $\zeta(2k+1)$ must be irrational, although no specific $\zeta(2k+1)$ is known to be irrational other than $\zeta(3)$ ([29] of Ch. 2).

In recent years the MZV have generated tremendous interest, and people have shown them to be connected to some very deep aspects of algebraic geometry and

arithmetic. Some remarkable conjectures have been proposed, but the evaluations of the MZV appear out of reach today. If the conjectures are true, they would imply that

$$\pi, \zeta(3), \zeta(5), \zeta(7), \dots,$$

are algebraically independent over \mathbf{Q} ; even this is a far cry from any exact evaluation of the $\zeta(2k+1)$.

3.2. Some remarks on infinite series and products and their values

In Euler's time there was no habit of precise definitions and this led to enormous and intense discussions regarding infinite series and their sums in which nonmathematicians, occasionally even clergymen, participated. For instance such a great mathematician as Leibniz could not decide whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + \dots$$

had a sum. He thought that as the partial sum s_n is 1 for odd n and 0 for even n , the sum should be $1/2$. But since there was no precise definition of what a sum ought to be, there was confusion and disagreement. Euler had a much clearer idea than his contemporaries on what one means when one says that such and such a series has the sum s . In fact precisely because his ideas were close to ideas in our own time he was able to make a substantial theory of *divergent series* and make serious use of them. His 1760 paper on divergent series signals the birth of the modern theory of divergent series (see Chapter 5).

The problems of dealing with infinite series fall into several categories.

- (1) To decide if the series is convergent.
- (2) If convergent, to determine the exact value of the sum.
- (3) In case evaluation of the sum proves impossible, to try to evaluate it to a high degree of accuracy by summing up to a certain limit; here if the terms a_n decrease to 0 slowly, as is often the case, transformations are needed before approximate calculation can be attempted. Euler discovered many of these transformations.
- (4) To try to associate a "sum" to a divergent series. Euler was the first mathematician to do this systematically.

In 1674 Leibniz had calculated that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

where the terms are the reciprocals of odd numbers with signs that alternate. There were other series of this type, such as

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots$$

which was attributed to Newton by Euler, and

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

which follows from the Taylor series for $\log(1+x)$, namely

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

by taking $x = 1$.

For a series of *positive* terms there are only two possibilities: either the series is convergent or divergent to ∞ . If

$$0 \leq a_n \leq b_n,$$

then the series $\sum_n a_n$ converges if the series $\sum_n b_n$ converges, while the series $\sum b_n$ diverges if the series $\sum a_n$ diverges. This principle leads to the various tests of convergence that were in vogue in the 19th century. The choice of $b_n = r^n$ leads to the ratio and root tests which are not conclusive; if we choose $b_n = n^{-s}$, it leads to a conclusive test at least for series where $a_n = f(n)$ for a rational function f . This is the Gauss test. All this became clear only in the 19th century. For a while people believed that one could write down a *universal test* of convergence, but the work of *Abel*, *Dini*, *Pringsheim*, and others showed that this is impossible. For instance, given any series $\sum a_n$ of positive terms which is convergent, there is another series $\sum b_n$ which is convergent, but such that

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0,$$

or, as one would say, the series $\sum_n b_n$ *converges more slowly than the series* $\sum_n a_n$. Similarly, given any divergent series of positive terms one can find a divergent series which diverges more slowly. For a beautiful account of the classical work on series, see Knopp's famous book [10].

Let us look at the series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

Since x^{-s} is decreasing for $x \geq 1$ we get

$$\frac{1}{(n+1)^s} < \int_n^{n+1} \frac{1}{x^s} dx < \frac{1}{n^s}.$$

For $s > 1$ we have convergence since

$$1 + \frac{1}{2^s} + \dots + \frac{1}{n^s} < 1 + \int_1^\infty \frac{dx}{x^s} = 1 + \frac{s}{s-1},$$

while for $s = 1$ we have logarithmic divergence because

$$\frac{1}{2} + \dots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \log n < 1 + \frac{1}{2} + \dots + \frac{1}{n},$$

hence divergence for $s \leq 1$. In the convergent case if

$$R_N = \sum_{n=N+1}^\infty \frac{1}{n^s}.$$

Then

$$\int_{N+1}^\infty \frac{1}{x^s} dx < R_N < \int_N^\infty \frac{1}{x^s} dx$$

which gives the error estimate

$$\frac{1}{s-1} \frac{1}{(N+1)^{s-1}} < R_N < \frac{1}{s-1} \frac{1}{N^{s-1}}.$$

In other words, *the error* R_N *committed in stopping after* N *terms lies between*

$$\frac{1}{s-1} \frac{1}{(N+1)^{s-1}} \text{ and } \frac{1}{s-1} \frac{1}{N^{s-1}}.$$

Thus, when $s = 2$, the error R_N lies between $\frac{1}{N+1}$ and $\frac{1}{N}$. To get accuracy up to 6 decimals we thus have to take a million terms. This should give an idea of the difficulty of numerical computation of these series, difficulties that Euler often overcame with his famous summation formula.

Examples of series whose sums could be evaluated were already known when Euler started working on these things, and he added many more to this list. We mention a few of these:

$$\begin{aligned}\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots \\ \log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots \\ \frac{\pi}{2\sqrt{2}} &= 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots \\ \frac{1}{3} \left(\log 2 + \frac{\pi}{\sqrt{3}} \right) &= 1 - \frac{1}{4} + \frac{1}{7} - \dots\end{aligned}$$

For the first two we integrate $(1+x^2)^{-1}$ and $(1+x)^{-1}$ from 0 to 1. The last two summations, obtained by integrating $(1+x^2)/(1+x^4)$ and $1/(1+x^3)$ respectively from 0 to 1, are less elementary and follow from Euler's general expression for such integrals in terms of circular functions (cf. infra). The series for $\log 2$ is not good for numerical evaluation. If we take $x = -1/2$ in the series for $\log x$ we get the series for $\log 2$ which Euler used for his very accurate computation of $\log 2$ correct to several decimal places:

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}.$$

The geometric nature of the series ensures its rapid convergence.

Unlike series of positive terms the series of terms of arbitrary signs do not behave well under various transformations. It was Riemann who really clarified the true nature of series of terms with arbitrary signs. For the series

$$a_1 + a_2 + \dots$$

let us define

$$a_n^+ = \frac{|a_n| + a_n}{2}, \quad a_n^- = \frac{|a_n| - a_n}{2}.$$

Let us consider the partial sums of the three series

$$\sum_n a_n, \quad \sum_n a_n^\pm.$$

Suppose that the series $\sum_n a_n$ is convergent to the sum s . Then for absolute convergence it is necessary and sufficient that s_n^\pm have finite limits s^\pm and $s = s^+ - s^-$. For nonabsolute or conditional convergence it is necessary and sufficient that $s_n^\pm \rightarrow \infty$ but $s_n^+ - s_n^- \rightarrow s$. This shows that conditional convergence is very delicate and that it depends on the difference of two sequences going to ∞ in such a way that their difference goes to a finite limit. Riemann used this to show that an absolutely convergent series can be *rearranged* in any manner without affecting its convergence or its sum, but a conditionally convergent series can be rearranged to have any real number as its sum or to diverge to $\pm\infty$ or to oscillate with prescribed upper and lower limits. This is the famous *Riemann rearrangement theorem*. If the a_n are complex or more generally vectors in a finite dimensional vector space, the

set of sums obtained by rearrangement is an affine subspace; in the complex case it is either a point or a line or the whole complex plane.

The evaluations above would not be completely understood till Abel and Weierstrass laid the foundations of the theory of convergent series of functions, especially power series. For instance, the series for $(1+x^2)^{-1}$ converges only in $(0, 1)$, and so to get the Leibniz result one has to proceed in two stages. In the first stage we integrate from 0 to y and get

$$\arctan y = y - \frac{y^3}{3} + \frac{y^5}{5} - \dots \quad (0 < y < 1).$$

This already requires the use of the uniform convergence of the power series. Then we let y approach 1. The left side goes to $\pi/4$, but in the right side we have to pass to the limit termwise, and this time it is more delicate as there is no uniform convergence to fall back on. This was done first by Abel. In any case these things were beyond what was understood and available in the 18th century.

For completeness let us sketch the proofs of both steps mentioned above. Consider a power series

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \quad (0 < x < 1)$$

convergent in $(0, 1)$. If $0 < x \leq r < r' < 1$, we have $|a_n r'^n| \leq C$ for a constant C and hence

$$|a_n x^n| \leq C \left(\frac{r}{r'}\right)^n$$

showing uniform convergence in $[0, r]$. Term by term integration and differentiation are thus permitted in any interval $[0, y]$ with $y < 1$. Abel's result is more delicate and depends on *Abel's partial summation*, the discrete version of integration by parts, which was probably developed by Abel to deal with series of terms whose signs are not fixed. It asserts that if the series $\sum_n a_n$ is convergent, then

$$\lim_{x \rightarrow 1^-} f(x) = \sum_n a_n.$$

Here the existence of the limit on the left side is part of the assertion. The partial summation formula is

$$\sum_{0 \leq r \leq n} a_r b_r = \sum_{0 \leq r \leq n-1} s_r (b_r - b_{r+1}) + s_n b_n, \quad s_r = \sum_{0 \leq j \leq r} a_j.$$

This leads to

$$\sum_{0 \leq r \leq n} a_r x^r = (1-x) \sum_{0 \leq r \leq n-1} s_r x^r + s_n x^n.$$

Since s_n is bounded we may let $n \rightarrow \infty$ to get, for $0 < x < 1$,

$$f(x) = (1-x)g(x), \quad g(x) = \sum_{r \geq 0} s_r x^r = \frac{s}{1-x} + \sum_{r \geq 0} (s_r - s)x^r.$$

Hence, writing $c_r = s_r - s$ we get

$$f(x) = s + (1-x)h(x), \quad h(x) = \sum_{r \geq 0} c_r x^r.$$

Now $c_r \rightarrow 0$ and so, for any $\delta > 0$ we can find $k > 1$ so that $|c_r| \leq \delta$ for $r > k$. Then

$$|(1-x)h(x)| \leq (1-x) \sum_{1 \leq r \leq k} |c_r| x^k + \delta$$

so that

$$\limsup_{x \rightarrow 1^-} |(1-x)h(x)| \leq \delta.$$

Letting $\delta \rightarrow 0$ we see that $(1-x)h(x) \rightarrow 0$ as $x \rightarrow 1^-$. This is Abel's theorem.

When the series $\sum_n a_n$ does not converge but $\lim_{x \rightarrow 1^-} f(x)$ exists and has the value s , we may *define* the sum of the series $\sum_n a_n$ as s . This is the definition of *Abel summability*, but it goes back to Euler, for whom it was a favorite method of summation. He used it extensively.

Infinite products. We have already seen that infinite product expansions of the circular functions form the center piece of Euler's discoveries on the zeta values. With this in mind let us review a few elementary facts about infinite products. We shall consider only infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n), \quad (a_n \in \mathbf{C}, a_n \rightarrow 0 \text{ as } n \rightarrow \infty).$$

Then $|a_n| \leq 1/2$ for all n sufficiently large so that we may assume that $|a_n| \leq 1/2$ for all n . The product is said to be *convergent* with the value p if

$$p_n = \prod_{k=1}^n (1 + a_k) \rightarrow p \neq 0.$$

We write

$$\prod_{n=1}^{\infty} (1 + a_n) = p.$$

It is easy to see that

$$\prod_{n=1}^{\infty} (1 + a_n) \quad \text{and} \quad \sum_{n=1}^{\infty} \log(1 + a_n)$$

have the same behavior, \log being the principal branch at 1, and the sum s of the series is related to p by $p = e^s$. Moreover,

$$\sum_{n=1}^{\infty} |a_n| < \infty \iff \sum_{n=1}^{\infty} |\log(1 + a_n)| < \infty.$$

To see this, note that for $|x| \leq 1/2$,

$$|\log(1+x) - x| \leq \frac{|x|^2}{2}(1 + |x| + |x|^2 + \dots) \leq \frac{|x|^2}{1 - |x|} \leq \frac{1}{2}|x|$$

so that

$$\frac{1}{2}|x| \leq |\log(1+x)| \leq \frac{3}{2}|x|.$$

The assertion is then clear.

3.3. Evaluation of $\zeta(2)$ and $\zeta(4)$

We have seen that

$$\frac{1}{n+1} < R_n = \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} + \dots < \frac{1}{n}.$$

So if one wants to calculate $\zeta(2)$ accurately up to 6 decimals we must take $n > 10^6$; i.e., we must take a million terms. This must have been daunting even to Euler, who was an indefatigable calculator. In his very first contribution to the subject in

1731, entitled *De Summatione Innumerabilium Progressionum*, I-14, 25-41, Euler discovered a transformation of this series that allowed him to calculate $\zeta(2)$ to 6 decimal places, and even more accurately had he wanted to do so. Euler proved that

$$\zeta(2) = \log u \log(1-u) + \sum_{n=1}^{\infty} \frac{u^n}{n^2} + \sum_{n=1}^{\infty} \frac{(1-u)^n}{n^2} \quad (0 < u < 1).$$

In particular, taking $u = 1/2$, we get

$$\zeta(2) = (\log 2)^2 + \sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 2^{n-1}}.$$

Euler's argument is the following. Start with

$$-\frac{\log(1-x)}{x} = 1 + \frac{x}{2} + \frac{x^2}{3} + \dots$$

Integrating from 0 to 1 we have,

$$\zeta(2) = \int_0^1 \frac{-\log(1-x)}{x} dx.$$

We split the integration from 0 to u and u to 1. So for any u with $0 < u < 1$,

$$\zeta(2) = \int_0^u \frac{-\log(1-x)}{x} dx + \int_u^1 \frac{-\log(1-x)}{x} dx.$$

In the second integral we change x to $1-x$ to get

$$\zeta(2) = \int_0^u \frac{-\log(1-x)}{x} dx + \int_0^{1-u} \frac{-\log x}{1-x} dx = I_1 + I_2.$$

For I_1 we expand $\log(1-x)$ as a series and integrate term by term from 0 to u . So

$$I_1 = \sum_{n=1}^{\infty} \frac{u^n}{n^2}.$$

To evaluate I_2 we integrate by parts first to get

$$I_2 = \log x \log(1-x) \Big|_0^{1-u} + \int_0^{1-u} \frac{-\log(1-x)}{x} dx.$$

The second term is like I_1 . For the first term note that

$$\log(1-x) = -x - \frac{x^2}{2} - \dots \sim -x \quad (x \rightarrow 0)$$

so that

$$\log x \log(1-x) \sim -x \log x \rightarrow 0 \quad (x \rightarrow 0).$$

Hence we obtain

$$\zeta(2) = \log u \log(1-u) + \sum_{n=1}^{\infty} \frac{u^n}{n^2} + \sum_{n=1}^{\infty} \frac{(1-u)^n}{n^2},$$

which is Euler's result. For numerical evaluation Euler used the formula for $\log 2$ obtained earlier,

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}.$$

Using this Euler computes

$$(\log 2)^2 = 0.480453\dots, \quad \sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 2^{n-1}} = 1.164481\dots$$

to obtain

$$\zeta(2) = 1.644934\dots$$

The above transformation used by Euler is the first instance of the appearance of the *dilogarithm*. Let

$$\operatorname{Li}_2(x) := \int_0^x \frac{-\log(1-t)}{t} dt = \int_{x>t_1>t_2>0} \frac{dt_1 dt_2}{t_1(1-t_2)}.$$

Then

$$\operatorname{Li}_2(x) = \sum_n \frac{x^n}{n^2}, \quad \operatorname{Li}_2(1) = \zeta(2).$$

The discussion above is essentially the proof of the functional equation

$$\operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = -\log x \log(1-x) + \operatorname{Li}_2(1) \quad (0 < x < 1)$$

which also is in Euler (loc. cit.). The dilogarithm has obvious generalizations to several variables, and these play an important role in the theory of MZV.

In addition to $\zeta(s)$ Euler worked with several other functions N, M, L given by

$$N(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}$$

$$M(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} \quad (s > 0)$$

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n+1)^s} \quad (s > 0).$$

Unlike $\zeta(s)$ and $N(s)$ which are defined only for $s > 1$, the functions M and L are defined for $s > 0$. Moreover

$$N(s) = (1 - 2^{-s})\zeta(s)$$

$$M(s) = (1 - 2^{1-s})\zeta(s).$$

The other function $L(s)$ is the first instance of what Dirichlet would later introduce as an “ L -function associated with a character”. In this case the character is essentially the function ω defined on the odd integers by

$$\omega(n) = (-1)^{(n-1)/2}.$$

It is an easy verification that ω is *multiplicative*, i.e.,

$$\omega(mn) = \omega(m)\omega(n) \quad (m, n \text{ odd numbers}).$$

One can then write

$$L(s) = \sum_{m=1}^{\infty} \frac{\omega(2m+1)}{(2m+1)^s} \quad (s > 0).$$

Notice that ω has period 4, using which we can extend it to all integers by requiring that

$$\omega(2m) = 0, \quad \omega(n) = \omega(n+4).$$

Then

$$L(s) = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s}.$$

Euler preferred to work with $L(s)$ and $M(s)$ because of their convergence for $s > 0$, while $N(s)$ and $\zeta(s)$ converged only for $s > 1$.

The accurate evaluation of $\zeta(2)$ must have motivated Euler to look for a general method of summing series. He succeeded in this attempt when he discovered what is now known as the *Euler Maclaurin summation formula*. This was around 1732, and Maclaurin discovered it in 1742. While Euler's priority was unquestioned, he made no attempt to deny Maclaurin his share of the discovery, which was quite typical of him. With this formula he computed the values $\zeta(k)$ for even k up to 26. These evaluations would serve him well when objections were raised about his first proof of the exact even zeta values.

Euler's first proof that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. Around 1735 Euler suddenly found a method to evaluate $\zeta(2)$ exactly. He found that

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}.$$

He was quite excited about his discovery and communicated it to his friend Daniel Bernoulli and to other mathematicians. We shall now take a look at his method which was given in the paper entitled *De Summis Serierum Reciprocarum*, I-14, 73-86.

Euler's method, as we mentioned in §3.1, was based on a generalization of Newton's theorem on symmetric functions of the roots of a polynomial, to the case when the polynomial is replaced by a power series (he called this an *infinite equation*). Let us first recall Newton's theorem. Let $f(s)$ be a polynomial of degree k in the variable s and let a, b, \dots, q, r be its roots. Then

$$f(s) = L(s - a)(s - b) \dots (s - q)(s - r)$$

where L is the coefficient of s^k in $f(s)$. Newton's theorem asserts that any symmetric polynomial of the roots can be expressed as a polynomial of the coefficients of f . To prepare us for Euler's generalization of this to polynomials of infinite degree, let us rewrite f starting from the constant term (normalized to be 1) as

$$1 - \alpha s + \beta s^2 - \dots \pm \rho s^k = \left(1 - \frac{s}{a}\right) \left(1 - \frac{s}{b}\right) \dots \left(1 - \frac{s}{q}\right) \left(1 - \frac{s}{r}\right).$$

From this, expanding the right side and equating coefficients of the powers of x on both sides, we get

$$\alpha = a + b + \dots + r, \quad \beta = ab + ac + \dots + qr, \quad \gamma = abc + \dots + pqr$$

and so on. Here β, γ, \dots are the sums of products of the roots taken 2, 3, ... at a time. If

$$S_k = a^k + b^k + \dots + r^k,$$

then Newton's formulae give

$$S_2 = \alpha^2 - 2\beta, \quad S_3 = \alpha^3 - 3\alpha\beta + 3\gamma, \quad S_4 = \alpha^4 - 4\alpha^2\beta + 4\alpha\gamma + 2\beta^2 - 4\delta$$

and so on. When the polynomial is replaced by a power series

$$f(s) = 1 - \alpha s + \beta s^2 - \gamma s^3 + \delta s^4 - \varepsilon s^5 + \dots$$

one has to now allow for the fact that the roots will be infinite in number. Let us write them as A, B, C, D, \dots . Then, imitating the case of the polynomials, Euler wrote

$$1 - \alpha s + \beta s^2 - \gamma s^3 + \dots = \left(1 - \frac{s}{A}\right) \left(1 - \frac{s}{B}\right) \left(1 - \frac{s}{C}\right) \left(1 - \frac{s}{D}\right) \dots$$

If now one takes over (as Euler did) the Newton formulae for sums of powers of the roots, one obtains

$$\alpha = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} + \dots, \quad \beta = \frac{1}{AB} + \frac{1}{AC} + \frac{1}{BC} + \dots$$

and so on. The formulae for S_1, S_2, S_3, \dots remain unchanged. Thus

$$S_2 = \frac{1}{A^2} + \frac{1}{B^2} + \dots = \alpha^2 - 2\beta.$$

There is no obstacle to writing down expressions for the sums of higher powers of the roots. Following Euler let us write

$$\begin{aligned} P &= \frac{1}{A} + \frac{1}{B} + \dots, & Q &= \frac{1}{A^2} + \frac{1}{B^2} + \dots, & R &= \frac{1}{A^3} + \frac{1}{B^3} + \dots \\ S &= \frac{1}{A^4} + \frac{1}{B^4} + \dots, & T &= \frac{1}{A^5} + \frac{1}{B^5} + \dots, & V &= \frac{1}{A^6} + \frac{1}{B^6} + \dots \end{aligned}$$

and so on. To make the calculations easier Euler wrote down expressions for these that are *recursive*. Thus

$$\begin{aligned} P &= \alpha, & Q &= P\alpha - 2\beta, & R &= Q\alpha - P\beta + 3\gamma \\ S &= R\alpha - Q\beta + P\gamma - 4\delta, & T &= S\alpha - R\beta + Q\gamma - P\delta + 5\varepsilon \end{aligned}$$

and so on.

Euler applied this theory to the case

$$f(s) = 1 - \frac{\sin s}{y}$$

where y is a parameter in the sense that various values can be given to it later. Since

$$\sin s = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \dots$$

we have

$$1 - \frac{s}{y} + \frac{s^3}{3!y} - \frac{s^5}{5!y} + \dots = \left(1 - \frac{s}{A}\right) \left(1 - \frac{s}{B}\right) \left(1 - \frac{s}{C}\right) \left(1 - \frac{s}{D}\right) \dots$$

In this case the earlier formulae reduce to

$$\alpha = \frac{1}{y}, \quad \beta = 0, \quad \gamma = -\frac{1}{6y}, \quad \delta = 0$$

and so on. Thus

$$P = \frac{1}{y}, \quad Q = \frac{1}{y^2}, \quad R = \frac{1}{y^3} - \frac{1}{2y}, \quad S = \frac{1}{y^4} - \frac{2}{3y^2}$$

and so on.

$y = 1$: The function $1 - \sin s$ has the roots

$$\frac{\pi}{2}, \frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, \frac{5\pi}{2}, -\frac{7\pi}{2}, -\frac{7\pi}{2}, \dots$$

where the repetitions express the fact that these are all *double roots*. Writing $q = \pi/2$, the reciprocals of the roots are

$$\frac{1}{q}, \frac{1}{q}, -\frac{1}{3q}, -\frac{1}{3q}, \frac{1}{5q}, \frac{1}{5q}, -\frac{1}{7q}, -\frac{1}{7q}, \frac{1}{9q}, \frac{1}{9q}, \dots$$

The formula for the sum of roots gives

$$1 = 2 \times \left(\frac{1}{q} - \frac{1}{3q} + \frac{1}{5q} - \frac{1}{7q} + \dots \right)$$

which leads to Leibniz's result

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

The formula for the sums of squares of roots gives

$$1 = 2 \times \left(\frac{1}{q^2} + \frac{1}{9q^2} + \frac{1}{25q^2} + \frac{1}{49q^2} + \dots \right)$$

leading to

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

If we now observe that the right side is

$$N(2) = (1 - 2^{-2})\zeta(2)$$

we get

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

This was Euler's proof.

One need not stop here of course. Going to the third and fourth powers we get

$$\begin{aligned} \frac{2}{q^3} \times \left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots \right) &= \frac{1}{2} \\ \frac{2}{q^4} \times \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right) &= \frac{1}{3}. \end{aligned}$$

These lead to

$$\begin{aligned} L(3) &= \left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots \right) = \frac{\pi^3}{32} \\ N(4) &= \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right) = \frac{\pi^4}{96}. \end{aligned}$$

Using the relation

$$N(4) = (15/16)\zeta(4)$$

we get

$$\zeta(4) = \frac{\pi^4}{90}.$$

Euler actually did not stop with these in his 1734/35 paper but went on to calculate

$$L(5) = \frac{5\pi^5}{1536}, \quad N(6) = \frac{\pi^6}{960}, \quad L(7) = \frac{61\pi^7}{184320}, \quad N(8) = \frac{17\pi^8}{161280}$$

and hence also

$$\zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}.$$

In every case he found that the value is a rational multiple of an appropriate power of π . We shall return to this point later.

Further calculations. It was natural for Euler to ask what additional formulae could be obtained by this method if one takes other values for y . Euler did this and found that although nothing new was obtained for sums of higher powers, the formulae for the sum of the roots led to new results.

$y = 1/\sqrt{2}$: The roots of the equation $\sin s = 1/\sqrt{2}$ are $2k\pi + \pi/4$, i.e.,

$$\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{5\pi}{4}, -\frac{7\pi}{4}, +\frac{9\pi}{4}, \frac{11\pi}{4}, \dots$$

giving the formula

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots$$

which Euler attributed to Newton.

$y = \sqrt{3}/2$: The roots of the equation $\sin s = \sqrt{3}/2$ are $2k\pi + \pi/3$, i.e.,

$$\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{4\pi}{3}, -\frac{5\pi}{3}, +\frac{7\pi}{3}, \frac{8\pi}{3}, -\frac{10\pi}{3}, -\frac{11\pi}{3}, \dots$$

giving the formula

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \dots$$

The calculation of the sums of higher powers of the reciprocal integers by taking these various values of y does not yield additional formulae but provides confirmation of the method and the results obtained before.

This method works for all y . Write $y = \sin \sigma$. The roots of

$$1 - \frac{\sin s}{\sin \sigma} = 0$$

are

$$\sigma, \sigma \pm 2\pi, \sigma \pm 4\pi, \dots$$

and

$$\pi - \sigma, \pi - \sigma \pm 2\pi, \pi - \sigma \pm 4\pi, \dots$$

Euler's argument then gives

$$1 - \frac{\sin s}{\sin \sigma} = \prod_{n=-\infty}^{\infty} \left(1 - \frac{s}{2n\pi + \sigma}\right) \left(1 - \frac{s}{2n\pi + \pi - \sigma}\right).$$

For convergence purposes this should be rewritten as

$$\begin{aligned} 1 - \frac{\sin s}{\sin \sigma} &= \left(1 - \frac{s}{\sigma}\right) \prod_{n=1}^{\infty} \left(1 - \frac{s}{2n\pi + \sigma}\right) \left(1 + \frac{s}{2n\pi - \sigma}\right) \\ &\times \prod_{n=1}^{\infty} \left(1 - \frac{s}{(2n-1)\pi - \sigma}\right) \left(1 + \frac{s}{(2n-1)\pi + \sigma}\right). \end{aligned}$$

We shall give a rigorous demonstration later, due to Euler himself. The formula for the sum of roots then gives

$$\begin{aligned} \frac{1}{\sin \sigma} &= \frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{1}{2n\pi + \sigma} - \frac{1}{2n\pi - \sigma} \right) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)\pi - \sigma} - \frac{1}{(2n-1)\pi + \sigma} \right) \end{aligned}$$

which can be written as

$$\frac{1}{\sin \sigma} = \frac{1}{\sigma} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n\pi + \sigma} - \frac{1}{n\pi - \sigma} \right).$$

Writing πs for σ leads to

$$\frac{\pi}{\sin \pi s} = \frac{1}{s} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n+s} - \frac{1}{n-s} \right).$$

Euler writes this as

$$\frac{\pi}{\sin \pi s} = \frac{1}{s} - \frac{1}{1+s} + \frac{1}{1-s} + \frac{1}{2+s} - \frac{1}{2-s} - \frac{1}{3+s} + \frac{1}{3-s} + \text{etc.}$$

Euler would return to this infinite partial fraction later and prove it by another method, based on integral calculus and using what Legendre would later call the Eulerian integrals. The sums evaluated earlier correspond to the cases $s = 1/2, 1/4, 1/6$. Going back to the infinite product and writing the expression for the sum of squares and cubes of the roots, we get

$$\begin{aligned} \frac{1}{s^2} + \sum_{n=1}^{\infty} \left(\frac{1}{(n+s)^2} + \frac{1}{(n-s)^2} \right) &= \frac{\pi^2}{\sin^2 \pi s} \\ \frac{1}{s^3} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{(n+s)^3} - \frac{1}{(n-s)^3} \right) &= \frac{\pi^3}{\sin^3 \pi s} - \frac{\pi^3}{2 \sin \pi s}. \end{aligned}$$

The last case Euler considered was the case $y = 0$. The equation $1 - \sin s/y = 0$ will now be written as $\sin s = 0$ and on removing the trivial root $s = 0$ becomes

$$\frac{\sin s}{s} = 0.$$

We get the equation

$$\frac{\sin s}{s} = 1 - \frac{s^2}{3!} + \frac{s^4}{5!} - \frac{s^6}{7!} + \dots = \left(1 - \frac{s}{A}\right) \left(1 - \frac{s}{B}\right) \left(1 - \frac{s}{C}\right) \left(1 - \frac{s}{D}\right) \dots$$

The roots are now

$$\pm\pi, \pm 2\pi, \pm 3\pi, \dots$$

and so the product formula becomes

$$\frac{\sin s}{s} = \left(1 - \frac{s^2}{\pi^2}\right) \left(1 - \frac{s^2}{4\pi^2}\right) \left(1 - \frac{s^2}{9\pi^2}\right) \dots = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2\pi^2}\right).$$

The method used above then gives the sums of *even* powers of the reciprocal integers, and Euler calculated them up to $\zeta(12)$. Thus he obtained

$$\begin{aligned}\zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ \zeta(8) &= \frac{\pi^8}{9450} \\ \zeta(10) &= \frac{\pi^{10}}{93555} \\ \zeta(12) &= \frac{691\pi^{12}}{6825 \times 93555}.\end{aligned}$$

He noticed that the values of $L(2k+1)$ and $N(2k)$ differ by a rational multiple of π , giving him the value of π as the ratio of two infinite series in many different ways, for example

$$\begin{aligned}\pi &= \frac{25}{8} \frac{1 + \frac{1}{3^6} + \frac{1}{5^6} + \dots}{1 - \frac{1}{3^5} + \frac{1}{5^5} - \dots} \\ \pi &= \frac{192}{61} \frac{1 - \frac{1}{3^7} + \frac{1}{5^7} - \dots}{1 + \frac{1}{3^6} + \frac{1}{5^6} + \dots}.\end{aligned}$$

Objections to Euler's method. Euler lost no time in communicating his discoveries to his friends and other mathematicians. Some of them, Daniel and Nicholas Bernoulli for instance, raised objections to his method. The application of Newton's theorem to a context far more general than the original one was a troublesome point. A more fundamental objection was that the function $\sin s$ could have complex roots, apart from the obvious ones that Euler wrote down. Moreover, while a polynomial can be easily seen to be factorizable as a product of linear factors corresponding to its roots, the extension of such factorizations to transcendental functions such as $\sin s$ seemed too big a step and demanded justification. For instance, $\sin s/s$ and $e^s \sin s/s$ have the same real roots, but clearly both cannot have the same product formula. It was pointed out to him that if one uses an ellipse instead of a circle to define the function $\sin s$, the zeros would be the same but the function would be different and the same series would appear to have a different value, now depending on the lengths of the axes of the ellipse, which is absurd.* This objection could really be answered only after Weierstrass's work on functions with product representations and specified zeros. To these one can also add a more modern objection, one that would become clear after Riemann constructed his theory of rearrangements of a series that is not absolutely convergent, namely that in the case of such a series, the sum of the infinite series depends on the order of summation and so is really not a symmetric function of the roots. Thus the order in which the roots are taken, as well as the terms of the infinite products that enter Euler's proofs, becomes critical in a way that is not clear in his treatment. The

*See [1], Ch. 2., Vol. II, 433-435 (Daniel Bernoulli to Euler), and 681-689 (Nicolaus Bernoulli to Euler).

same objection could be raised against the product formula for $1 - \frac{\sin s}{\sin \sigma}$ and hence also the partial fraction for $\frac{\pi}{\sin \pi s}$.

Euler himself was aware of these shortcomings of his proof but was encouraged by two facts. First, in the argument starting with $y = 1$ the first step gave Leibniz's series for $\pi/4$, and the one with $y = 1/2\sqrt{2}$ gave Newton's formula, both of which had been proved by other methods. So even if $\sin s$ had other roots, the sum of their reciprocals must be 0. Second, the value $\pi^2/6$ for $\zeta(2)$ was clearly confirmed by numerical calculation so that the other roots, if any, do not contribute to the calculations. These were the reasons that persuaded him to publish and publicize his method. Nevertheless he recognized the validity of the objections and devoted the next few years to making the arguments as solid as possible.

The objection that the sum of the roots depends on the order in which the roots are enumerated was of course not one his contemporaries could have made. Nevertheless it disappears in Euler's proof based on the infinite product expansion

$$\frac{\sin s}{s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2\pi^2}\right)$$

that he wrote down in [6]. The absolute convergence of the product makes the justification of the application of Newton's theorem more routine.

The calculations that he did for the values of $\zeta(2k)$ and $L(2k+1)$ for the first few values of k must have convinced Euler that he should try to determine these values for all k . This he would do later.

3.4. Infinite products for circular and hyperbolic functions

After his first evaluation of $\zeta(2)$ and $\zeta(4)$ and $\zeta(2k)$ for some small values of k , Euler began his long campaign to consolidate his proofs. This took him the better part of the next few years. Finally, around 1742, he obtained the infinite product expansion for $\frac{\sin x}{x}$ that conclusively demonstrated that his results were correct. In this section we shall discuss Euler's proof of the infinite product expansions of the circular functions (I-14, 138-155). Although Euler worked only with real x , he considered both the hyperbolic and the circular functions so that in essence he was working over the complex plane. Eventually he gave a beautiful exposition of these results in his book I-8.

The starting point for Euler was the limit formula

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

He would write this as

$$e^x = \left(1 + \frac{x}{n}\right)^n$$

where n is an infinite magnitude, sometimes denoted by ι . He used the limit formula also for imaginary values of the exponent expressed in current notation as

$$e^{ix} = \lim_{n \rightarrow \infty} \left(1 + \frac{ix}{n}\right)^n.$$

From this one has

$$\sin x = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{ix}{n}\right)^n - \left(1 - \frac{ix}{n}\right)^n}{2i}$$

$$\cos x = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{ix}{n}\right)^n + \left(1 - \frac{ix}{n}\right)^n}{2}$$

as well as the companion formulae

$$\frac{e^x - e^{-x}}{2} = \sinh x = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{x}{n}\right)^n - \left(1 - \frac{x}{n}\right)^n}{2}$$

$$\frac{e^x + e^{-x}}{2} = \cosh x = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{x}{n}\right)^n + \left(1 - \frac{x}{n}\right)^n}{2}.$$

Euler's brilliant idea was to factorize the polynomials

$$\frac{\left(1 + \frac{ix}{n}\right)^n - \left(1 - \frac{ix}{n}\right)^n}{2i} \quad \text{and} \quad \frac{\left(1 + \frac{x}{n}\right)^n - \left(1 - \frac{x}{n}\right)^n}{2}$$

into quadratic factors and pass to the limit as $n \rightarrow \infty$; the infinite product expansions for $\sin x$ and $\sinh x$ follow immediately.

Factorization using roots of unity. The roots of the equation

$$T^n - 1 = 0$$

are

$$e^{\frac{2r\pi i}{n}} \quad (r = 0, 1, 2, \dots, n-1).$$

However this equation has real coefficients and so the roots can be arranged in pairs where each pair consists of conjugate quantities. Thus, if $n = 2p+1$, then the roots can be written as

$$e^{\frac{2r\pi i}{n}} \quad (r = 0, \pm 1, \pm 2, \dots, \pm p).$$

On the other hand

$$(T - e^{\frac{2r\pi i}{n}})(T - e^{-\frac{2r\pi i}{n}}) = T^2 - 2T \cos \frac{2r\pi}{n} + 1$$

from which we obtain the factorization

$$T^n - 1 = (T - 1) \prod_{k=1}^p \left(T^2 - 2T \cos \frac{2k\pi}{n} + 1 \right) \quad (n = 2p + 1).$$

Writing $T = X/Y$ and clearing of fractions we get

$$X^n - Y^n = (X - Y) \prod_{k=1}^p \left(X^2 - 2XY \cos \frac{2k\pi}{n} + Y^2 \right).$$

Let us now take

$$X = 1 + \frac{ix}{n}, \quad Y = 1 - \frac{ix}{n}$$

and write

$$q_n(x) = \frac{\left(1 + \frac{ix}{n}\right)^n - \left(1 - \frac{ix}{n}\right)^n}{2ix}.$$

Then we get, since $X - Y = \frac{2ix}{n}$, after a little calculation,

$$q_n(x) = C_n \prod_{k=1}^p \left(1 - \frac{x^2}{n^2} \frac{1 + \cos \frac{2k\pi}{n}}{1 - \cos \frac{2k\pi}{n}} \right)$$

where C_n is a numerical constant. Since

$$q_n(x) = 1 + \dots$$

while the product on the right side of the equation above takes the value C_n at $x = 0$, we must have $C_n = 1$. It is an elementary exercise to verify this directly also. Thus we finally obtain

$$q_n(x) = \prod_{k=1}^p \left(1 - \frac{x^2}{n^2} \frac{1 + \cos \frac{2k\pi}{n}}{1 - \cos \frac{2k\pi}{n}} \right).$$

It remains only to let n go to ∞ . We know that

$$\lim_{n \rightarrow \infty} q_n(x) = \frac{e^{ix} - e^{-ix}}{2ix} = \frac{\sin x}{x}.$$

On the other hand let us examine the k^{th} term of the product on the right side, for a fixed k . From the power series for $\cos u$ it follows that

$$\lim_{u \rightarrow 0} \cos u = 1, \quad \lim_{u \rightarrow 0} \frac{1 - \cos u}{u^2} = \frac{1}{2}.$$

Hence

$$\lim_{n \rightarrow \infty} 1 + \cos \frac{2k\pi}{n} = 2, \quad \lim_{n \rightarrow \infty} n^2 \left(1 - \cos \frac{2k\pi}{n} \right) = \frac{2k^2}{\pi^2}.$$

Thus

$$\lim_{n \rightarrow \infty} 1 - \frac{x^2}{n^2} \frac{1 + \cos \frac{2k\pi}{n}}{1 - \cos \frac{2k\pi}{n}} = 1 - \frac{x^2}{k^2\pi^2}.$$

From this Euler deduced finally his famous infinite product expression:

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2} \right).$$

Clearly the same method will work when x is replaced by ix so that one obtains with virtually no change in the argument the formula

$$\frac{\sinh x}{x} = \frac{e^x - e^{-x}}{2x} = \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2} \right).$$

The product formulae for $\cos x$ and $\cosh x$ are derived in almost identical fashion.

For $\cos x$ we can also proceed by first noting that

$$\sin 2x = 2 \sin x \cos x$$

so that

$$\cos x = \frac{\sin 2x/2x}{\sin x/x}$$

from which we get at once the infinite product for $\cos x$:

$$\cos x = \prod_{k=1}^{\infty} \left(1 - \frac{4x^2}{(2k-1)^2\pi^2} \right).$$

Similarly we get

$$\cosh x = \frac{e^x + e^{-x}}{2} = \prod_{k=1}^{\infty} \left(1 + \frac{4x^2}{(2k-1)^2\pi^2} \right).$$

The direct derivation is equally straightforward. We factorize first $T^n + 1$. The roots of the equation

$$T^n + 1 = 0 \quad (n = 2p)$$

are

$$e^{\frac{\pi i}{2p}}, e^{\frac{3\pi i}{2p}}, e^{\frac{5\pi i}{2p}}, \dots, e^{\frac{(4p-1)\pi i}{2p}}$$

which can be arranged in p mutually conjugate pairs

$$e^{\pm \frac{\pi i}{2p}}, e^{\pm \frac{3\pi i}{2p}}, \dots, e^{\pm \frac{(2p-1)\pi i}{2p}}.$$

We thus obtain

$$T^n + 1 = \prod_{k=1}^p \left(T^2 - 2T \cos \frac{(2k-1)\pi}{2p} + 1 \right).$$

As before we deduce from this the factorization

$$X^n + Y^n = \prod_{k=1}^p \left(X^2 - 2XY \cos \frac{(2k-1)\pi}{2p} + Y^2 \right).$$

Take now

$$X = 1 + \frac{ix}{n}, \quad Y = 1 - \frac{ix}{n}$$

and proceed exactly as we do for the case of the sine to get for

$$p_n(x) = \frac{\left(1 + \frac{ix}{n}\right)^n + \left(1 - \frac{ix}{n}\right)^n}{2}$$

the factorization

$$p_n(x) = \prod_{k=1}^p \left(1 - \frac{x^2}{n^2} \frac{1 + \cos \frac{(2k-1)\pi}{n}}{1 - \cos \frac{(2k-1)\pi}{n}} \right) \quad (n = 2p).$$

Letting $n \rightarrow \infty$, we get

$$\cos x = \prod_{k=1}^{\infty} \left(1 - \frac{4x^2}{(2k-1)^2\pi^2} \right).$$

The factorization of $\cosh x$ is obtained in the same way with x replacing ix throughout.

The product formula for the function

$$1 - \frac{\sin s}{\sin \sigma}$$

can be obtained immediately from the above. Indeed, we write

$$1 - \frac{\sin s}{\sin \sigma} = \frac{2 \cos \frac{\sigma+s}{2} \sin \frac{\sigma-s}{2}}{\sin \sigma}$$

and use the products for \sin and \cos to get for it the expression

$$\left(1 - \frac{s}{\sigma}\right) \frac{\prod_{n=1}^{\infty} \left(1 - \frac{(\sigma+s)^2}{((2n-1)\pi)^2}\right) \prod_{n=1}^{\infty} \left(1 - \frac{(\sigma-s)^2}{(2n\pi)^2}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{\sigma^2}{((2n-1)\pi)^2}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\sigma^2}{(2n\pi)^2}\right)}.$$

Factorizing and rearranging a little we get

$$1 - \frac{\sin s}{\sin \sigma} = \left(1 - \frac{s}{\sigma}\right) \prod_{n=1}^{\infty} \left(1 - \frac{s}{2n\pi + \sigma}\right) \left(1 + \frac{s}{2n\pi - \sigma}\right) \\ \times \prod_{n=1}^{\infty} \left(1 - \frac{s}{(2n-1)\pi - \sigma}\right) \left(1 + \frac{s}{(2n-1)\pi + \sigma}\right).$$

Notice that the individual products separately converge because the terms are $1 + O(n^{-2})$; this justifies the rearrangement that was done.

A similar method can be used to obtain infinite product formulae for

$$\frac{\cos s \pm \cos \sigma}{1 \pm \cos \sigma}, \quad \frac{\sin s \pm \sin \sigma}{\sin \sigma}.$$

Thus, for example,

$$\frac{\cos s \pm \cos \sigma}{1 \pm \cos \sigma} = \prod_{n=1}^{\infty} \left(1 + \frac{s}{(2n-1)\pi - \sigma}\right) \left(1 - \frac{s}{(2n-1)\pi + \sigma}\right) \\ \times \prod_{n=1}^{\infty} \left(1 + \frac{s}{(2n-1)\pi + \sigma}\right) \left(1 - \frac{s}{(2n-1)\pi - \sigma}\right).$$

See pp. 132-134 in [3d] (where Euler writes v, g for our s, σ).

In this manner, all the results of Euler's initial paper [4] were put on a firm basis. The infinite product for $\frac{\sin x}{x}$, from which all other products and partial fractions can be deduced, is thus the cornerstone of Euler's entire theory.

Justification of Euler's proof from a later point of view. Actually Euler's derivation is almost unexceptionable. The only point that we would cavil at is the actual passage to the limit. For instance, in the case of $\sin x$, Euler has the product expression

$$q_n(x) = \prod_{k=1}^n (1 - a_k(n)x^2) \quad (n = 2p + 1)$$

where

$$a_k(n) = \frac{1 + \cos \frac{2k\pi}{n}}{1 - \cos \frac{2k\pi}{n}}$$

and simply passes to the limit *term by term*. Since however the number of factors is also going to infinity, it is clear from our vantage point that an additional argument is needed. We shall see that this is taken care of by the principle of *normal convergence*, which did not really become clear till the work of Weierstrass in the 19th century on uniform convergence. Certainly in Euler's time this proof was accepted without question; in any case it is fundamentally sound except for the technical point of uniform convergence.

This phenomenon actually occurs already in Euler's proof of the limit formula

$$e^{ix} = \lim_{n \rightarrow \infty} \left(1 + \frac{ix}{n}\right)^n.$$

Indeed, expanding the right side by the binomial theorem we get

$$1 + ix + (1 - 1/n) \frac{(ix)^2}{2!} + \dots (1 - 1/n)(1 - 2/n) \dots (1 - (k-1)/n) \frac{(ix)^k}{k!} + \dots$$

where the sum extends to n terms. Euler simply passed to the limit term by term.

We thus have the following situation. We have a sequence

$$a_1(n), a_2(n), \dots, a_{k_n}(n) \quad (k_n \rightarrow \infty)$$

and the corresponding sum and product

$$s_n = \sum_{k=1}^{k_n} a_k(n) \quad p_n = \prod_{k=1}^{k_n} (1 + a_k(n)),$$

and we know that for *each fixed* k ,

$$\lim_{n \rightarrow \infty} a_k(n) = a_k$$

exists. Under what circumstances are we then allowed to conclude that

$$\lim_{n \rightarrow \infty} s_n = \sum_{k=1}^{\infty} a_k, \quad \lim_{n \rightarrow \infty} p_n = \prod_{k=1}^{\infty} (1 + a_k)?$$

We can actually simplify a little by defining $a_k(n)$ for all k by saying that it is 0 if $k > k_n$. Then the problem is to find out when we can interchange summation and passing to the limit, i.e., when we can conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_k(n) &= \sum_{k=1}^{\infty} \lim_{n \rightarrow \infty} a_k(n) \\ \lim_{n \rightarrow \infty} \prod_{k=1}^{\infty} (1 + a_k(n)) &= \prod_{k=1}^{\infty} (1 + \lim_{n \rightarrow \infty} a_k(n)). \end{aligned}$$

The notion of *normal convergence*, which was introduced and exploited by Weierstrass, allows us to take care of this point. We shall say that the series

$$\sum_{k=1}^{\infty} a_k(n)$$

of complex terms is *normally convergent* if there is a sequence (M_k) such that

- (i) $|a_k(n)| \leq M_k$ for all k, n ;
- (ii) $\sum_{k=1}^{\infty} M_k$ is convergent.

Note that

$$|a_k| \leq M_k$$

for all k . Actually it is enough if we have (i) above for all $k \geq k_0$ where k_0 is *independent* of n . In the case of the series, we have, for any N ,

$$\left| \sum_k a_k(n) - \sum_k a_k \right| \leq \sum_{1 \leq k \leq N} |a_k(n) - a_k| + 2 \sum_{k > N} M_k$$

so that

$$\limsup_{n \rightarrow \infty} \left| \sum_k a_k(n) - \sum_k a_k \right| \leq 2 \sum_{k > N} M_k.$$

Letting $N \rightarrow \infty$ we get the result

$$\lim_{n \rightarrow \infty} \sum_k a_k(n) = \sum_k a_k.$$

For the product we work with the series

$$\sum_k \log(1 + a_k(n))$$

and use the estimate

$$|\log(1+x)| \leq 2|x| \quad (|x| \leq 1/2).$$

Then

$$|\log(1+a_k(n))| \leq 2M_k$$

and so we have normal convergence.

As a first application let us consider the limit formula

$$\lim_{n \rightarrow \infty} \left(1 + \frac{z_n}{n}\right)^n = e^z \quad (z_n \in \mathbf{C}, z_n \rightarrow z).$$

Then

$$\left(1 + \frac{z_n}{n}\right)^n = 1 + z_n + \frac{n(n-1)}{2!} \frac{z_n^2}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{z_n^3}{n^3} + \dots$$

which can be written as

$$\sum_{k=1}^{\infty} c_k(n) \frac{z_n^k}{k!}$$

where

$$c_k(n) = \begin{cases} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k}{n}\right) & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}.$$

If $|z_n| \leq A$ for all n and we take $a_k(n) = c_k(n)z_n^k/k!$, the obvious estimate

$$|a_k(n)| \leq \frac{A^k}{k!}$$

gives normal convergence and hence the limit formula. This argument is valid even when z_n is a complex *matrix* and is the basic fact in the theory of Lie groups.

We are now ready to justify Euler's passage to the limit when $n \rightarrow \infty$ term by term in the infinite products. We have a product of the form

$$\prod_{k=1}^{(n-1)/2} \left(1 \pm \frac{x^2}{n^2} \frac{1 + \cos \frac{2k\pi}{n}}{1 - \cos \frac{2k\pi}{n}}\right)$$

so that

$$a_k(n) = \pm \frac{x^2}{n^2} \frac{1 + \cos \frac{2k\pi}{n}}{1 - \cos \frac{2k\pi}{n}} \quad (2k < n).$$

We now observe that for some constant $\alpha > 0$ we have

$$1 - \cos u \geq \alpha u^2 \quad (0 < u \leq \pi).$$

Indeed, we note that $(1 - \cos u)/u^2 \rightarrow 1/2$ as $u \rightarrow 0$ and is never zero in $(0, \pi]$. Hence it must have a positive minimum α in $[0, \pi]$. Returning to $a_k(n)$ we then have, using the above lemma and remembering that $2k\pi/n \leq \pi$,

$$\left| \frac{1 + \cos \frac{2k\pi}{n}}{1 - \cos \frac{2k\pi}{n}} \right| \leq \frac{n^2}{\alpha} \frac{1}{2k^2\pi^2}$$

and hence

$$|a_k(n)| \leq \frac{x^2}{2\alpha\pi^2} \frac{1}{k^2}$$

giving normal convergence. The other cases may be treated in an entirely similar fashion.

There is no difficulty in carrying this entire discussion over to complex values of the argument. Nowadays these results are indeed established for complex values of

the argument, using periodicity arguments and Liouville's theorem that a bounded entire function is a constant. While this is admirable in itself, it is regrettable from the historical and pedagogical points of view. Euler's proof is beautiful, it displays the problem and its solution in the proper historical context, and moreover, the justification will take the student through the fine points of real analysis and show that the theorems that have been learned do not exist in a vacuum but serve a crucial purpose. There is no better demonstration of Euler's power and originality than this proof.

Symmetric functions of infinitely many variables. Once the product formulae have been established, it remains only to examine the Euler method of generalizing Newton's theorem. I follow an exposition of Cartier [11]. Write

$$P(t) = \prod_n (1 - tt_n) = 1 - \lambda_1 t + \lambda_2 t^2 - \cdots + (-1)^r \lambda_r t^r + \dots$$

where the t_n are complex quantities and t is also a complex variable. Formally we have

$$\lambda_r = \sum_{n_1 > n_2 > \cdots > n_r > 0} t_{n_1} t_{n_2} \cdots t_{n_r}$$

which are reminiscent of the multiple zeta series. Let

$$S_r = \sum_{n > 0} t_n^r.$$

Then logarithmic differentiation gives the relation

$$P'(t) + P(t) \sum_n \frac{t_n}{1 - tt_n} = 0.$$

Expanding both sides as power series in t we get the Newton formulae in the present context:

$$(*) \quad S_r = \lambda_1 S_{r-1} - \lambda_2 S_{r-2} + \cdots + (-1)^{r-2} \lambda_{r-1} S_1 + (-1)^{r-1} r \lambda_r.$$

Changing t to $-t$ we have

$$\sum_r \lambda_r t^r = \exp \left(\sum_n \log(1 + tt_n) \right)$$

which leads to

$$\sum_r \lambda_r t^r = \exp \left(\sum_r (-1)^r \frac{S_r}{r} t^r \right).$$

This is an alternative way to summarize the relations (*).

We now take

$$t_n = \frac{1}{n^2 \pi^2}, \quad P(t) = \frac{\sin t}{t}.$$

Then

$$\pi^{2r} \lambda_r = \zeta(2, 2, \dots, 2) (r \text{ arguments}), \quad \pi^{2r} S_r = \zeta(2r).$$

Since

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \cdots + (-1)^r \frac{t^{2r}}{(2r+1)!} + \dots$$

we have

$$\zeta(2, 2, \dots, 2) = \frac{\pi^{2r}}{(2r+1)!}, \quad \lambda_r = \frac{1}{(r+1)!}$$

and so the $\zeta(2r)$ can be calculated recursively. Indeed, if

$$\zeta(2r) = f_r \pi^{2r},$$

then

$$f_r = \sum_{p=1}^{r-1} (-1)^{p-1} \frac{f_{r-p}}{(2p+1)!} + (-1)^{r-1} \frac{r}{(2r+1)!}$$

from which it is immediate that the f_r are rational numbers.

The justification of these remarks is quite simple if we assume that

$$\sum_n |t_n| < \infty.$$

It is then a question of working with the product of N terms and letting $N \rightarrow \infty$.

Let

$$P_N(t) = \prod_{n \leq N} (1 - tt_n) = \sum_{n \geq 0} (-1)^n \lambda_r(N) t^n$$

where

$$\lambda_r(N) = \sum_{n_1 > n_2 > \dots > n_r > 0} t_{n_1}(N) \dots t_{n_r}(N), \quad t_n(N) = \begin{cases} t_n & \text{if } n \leq N \\ 0 & \text{if } n > N. \end{cases}$$

If

$$S_r(N) = \sum_{n > 0} t_n(N)^r$$

we have, by the classical theory,

$$S_r(N) = \lambda_1(N) S_{r-1}(N) + \dots + (-1)^r \lambda_{r-1}(N) S_1(N) + (-1)^{r+1} r \lambda_r(N).$$

To obtain (*) it is enough to show that as $N \rightarrow \infty$,

$$\lambda_r(N) \rightarrow \lambda_r, \quad S_r(N) \rightarrow S_r.$$

This is immediate using the criterion of normal convergence since

$$|t_{n_1}(N) \dots t_{n_r}(N)| \leq |t_{n_1}| \dots |t_{n_r}|, \quad |t_n(N)^r| \leq |t_n|^r$$

and

$$\sum_{n_1 > \dots > n_r > 0} |t_{n_1}| \dots |t_{n_r}| < \infty, \quad \sum_{n > 0} |t_n|^r < \infty.$$

On the other hand, it is clear that $P(t)$ exists for all $t \in \mathbf{C}$ and $P_N(t) \rightarrow P(t)$ for all t . Moreover

$$|\lambda_r(N)| \leq |\lambda_r| \leq \frac{1}{r!} \sum_{n_1, n_2, \dots, n_r > 0} |t_{n_1}| \dots |t_{n_r}| = \frac{(\sum_n |t_n|)^r}{r!}.$$

Hence

$$\sum_r |\lambda_r| |t|^r < \infty, \quad \sum_r (-1)^r \lambda_r(N) t^r \rightarrow \sum_r (-1)^r \lambda_r t^r$$

for all t , completing the proof that

$$P(t) = \sum_r (-1)^r \lambda_r t^r.$$

Other proofs. Euler himself had another very simple proof of the formula $\zeta(2) = \pi^2/6$. In a memoir which was forgotten but resurrected by Paul Stäckel in 1907

[12], Euler had given a direct proof of this. Euler's starting point is the power series for $\arcsin x$ which is obtained from the relation

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

by expanding in powers of t and integrating term by term. The result is

$$\arcsin x = x + \sum_{k \geq 1} \frac{1.3 \dots 2k-1}{2.4 \dots 2k} x^{2k+1}.$$

An integration by parts now leads to the relation

$$\int_0^x \frac{\arcsin t}{\sqrt{1-t^2}} dt = (\arcsin x)^2 - \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}} dt$$

so that

$$\frac{1}{2}(\arcsin x)^2 = \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}} dt.$$

We now replace $\arcsin x$ in the integrand by its power series expansion and integrate term by term from 0 to 1. An integration by parts shows that if

$$I_n = \int_0^1 \frac{t^n}{\sqrt{1-t^2}} dt,$$

then

$$I_{n+2} = \frac{n+1}{n+2} I_n$$

from which the I_n can be calculated for all n . The result is

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \dots + \frac{1}{(2k+1)^2} + \dots$$

which leads to Euler's formula. Euler tried to push this method for $\zeta(2k)$ for $k \geq 2$ but did not succeed. He even determined the power series for $(\arcsin x)^2$ in this attempt. Indeed, he wrote $y = (1/2)(\arcsin x)^2$ and from the integral representation obtained the differential equation

$$(1-x^2)y'' - xy' = 1$$

from which he determined the coefficients of the expansion of y recursively and obtained

$$\frac{1}{2}(\arcsin x)^2 = \sum_{r \geq 1} \frac{(2.4 \dots 2r)^2}{(2r)!} x^{2r}.$$

After mentioning his failure to push this method for the higher zeta values, he lists a few of these zeta values for the benefit of those who would try to work on this question.

Over the years there have been many proofs of Euler's formulae, but none that capture us with the imagination and originality that inform his proofs.

3.5. The infinite partial fractions for $(\sin x)^{-1}$ and $\cot x$. Evaluation of $\zeta(2k)$ and $L(2k+1)$

Already in 1734 when he determined the values of $\zeta(2k)$ for some small values of k , it must have become clear to Euler that

$$\zeta(2k) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \dots$$

should be a rational multiple of π^{2k} for all k and that similarly the series

$$L(2k+1) = 1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \dots$$

(which he had summed for small values of k) should have sums which are rational multiples of π^{2k+1} . In his papers [6], [7] Euler succeeded in evaluating all these series in closed form in terms of what would later be called *Bernoulli numbers* and *Euler numbers*. He did this starting from infinite partial fraction expansions for $\frac{1}{\sin x}$ and $\cot x$.

The expansions in question are

$$\begin{aligned} \frac{\pi}{\sin s\pi} &= \frac{1}{s} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n+s} - \frac{1}{n-s} \right) \\ \pi \cot s\pi &= \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{n+s} - \frac{1}{n-s} \right) \end{aligned}$$

where the brackets are introduced to ensure absolute convergence of the series involved. The expressions are valid for $0 < s < 1$ and indeed whenever s is not an integer. The absolute convergence is easy to check since

$$\frac{1}{n-s} - \frac{1}{n+s} = \frac{2s}{n^2 - s^2} = c_n(s) \frac{1}{n^2}$$

where $c_n = 2s/(1 - s^2/n^2)$ is bounded as $n \rightarrow \infty$; the convergence is even normal as long as s varies in a closed interval which does not contain any integer. Euler would write these as

$$\begin{aligned} \frac{\pi}{\sin s\pi} &= \frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} + \frac{1}{3-s} - \frac{1}{3+s} - \dots \\ \pi \cot s\pi &= \frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} + \frac{1}{2-s} - \frac{1}{2+s} + \frac{1}{3-s} - \frac{1}{3+s} - \dots \end{aligned}$$

not caring about the fine point of absolute convergence.

To obtain these the simplest proof is to use logarithmic differentiation of the infinite product for $\frac{\sin s}{s}$, as was noticed by Nicolaus Bernoulli* in a letter to Euler. From

$$\frac{\sin s}{s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2\pi^2} \right)$$

we have, taking logarithms,

$$\log \sin s - \log s = \sum_{n=1}^{\infty} \log(1 - s/n\pi) + \log(1 + s/n\pi) \quad (0 < s < \pi)$$

*See [1], Ch. 2, Vol. II, 681-689.

from which differentiation gives

$$\cot s = \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{n\pi + s} - \frac{1}{n\pi - s} \right) \quad (0 < s < \pi).$$

Replacing s by πs we get

$$\pi \cot s\pi = \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{n + s} - \frac{1}{n - s} \right) \quad (0 < s < 1).$$

The justification of this process requires normal convergence. For the partial fraction for $\frac{1}{\sin s}$ we follow the same procedure and logarithmically differentiate the infinite product for $1 - \sin s/\sin \sigma$ obtained earlier, with respect to s at $s = 0$. Once again we use normal convergence to justify this process. We get

$$\begin{aligned} \frac{1}{\sin \sigma} &= \frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{1}{2n\pi + \sigma} - \frac{1}{2n\pi - \sigma} \right) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)\pi - \sigma} - \frac{1}{(2n-1)\pi + \sigma} \right) \end{aligned}$$

which, on setting $\sigma = s\pi$, leads to

$$\frac{\pi}{\sin s\pi} = \frac{1}{s} + \sum (-1)^n \left(\frac{1}{n + s} - \frac{1}{n - s} \right).$$

The periodicity properties are more transparent if we write these as sums over the full set of integers. Thus

$$\pi \cot \pi s = PV \sum_{n \in \mathbf{Z}} \frac{1}{n + s}, \quad \frac{\pi}{\sin \pi s} = PV \sum_{n \in \mathbf{Z}} (-1)^n \frac{1}{n + s},$$

where the notation PV indicates that the series in question are understood as

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^{n=+N}.$$

Of course, as soon as we differentiate, the PV can be dropped as the series converge absolutely. The series for $\frac{\pi}{\sin \pi s}$ can also be obtained in another way. We logarithmically differentiate the product for $\cos \pi s$ or change s to $\frac{1}{2} - s$ in $\pi \cot \pi s$ to get the series

$$\pi \tan \pi s = PV \sum_{n \in \mathbf{Z}} \frac{1}{n + \frac{1}{2} - s}.$$

Adding the two series for $\pi \cot \pi s$ and $\pi \tan \pi s$, we get, after changing s to $\frac{1}{2}s$, the formula

$$\frac{\pi}{\sin \pi s} = PV \sum_{n \in \mathbf{Z}} (-1)^n \frac{1}{n + s}.$$

There are at least two other proofs for these formulae. One of these was given by Euler himself in [13] based on integrating certain rational functions that generalize the ones occurring in the Leibniz series. We shall give this in the next section. The other depends on the theory of what are now called the gamma and beta functions, or as Legendre called them, *Eulerian integrals of the first and second kind*. These integrals were first studied by Euler, and he knew their principal properties. We

shall give this proof also in the next section. It uses nothing that Euler did not know.

Bernoulli numbers and evaluation of $\zeta(2k)$. Let

$$Q(s) = \pi \cot \pi s.$$

Then

$$Q(s) - \frac{1}{s} = \sum_{n=1}^{\infty} \left(\frac{1}{n+s} - \frac{1}{n-s} \right).$$

Now $Q(s) - s^{-1}$ is smooth at $s = 0$ and

$$\frac{1}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \left(Q(s) - \frac{1}{s} \right) = \sum_{n=1}^{\infty} \left((-1)^{r-1} \frac{1}{(n+s)^r} - (r-1)! \frac{1}{(n-s)^r} \right).$$

Now if $r = 2k$ is even, we get

$$\frac{1}{(2k-1)!} \frac{d^{2k-1}}{ds^{2k-1}} \left(Q(s) - \frac{1}{s} \right) \Big|_{s=0} = -2\zeta(2k)$$

which was Euler's explicit formula. If we do the same with $r = 2k + 1$, we get 0.

Let us now take a closer look at the function Q . We have

$$Q(s) - \frac{1}{s} = \pi \cot \pi s - \frac{1}{s} = \frac{2i\pi}{e^{2i\pi s} - 1} + \pi i - \frac{1}{s}.$$

We now introduce the function

$$B(z) = \frac{z}{e^z - 1} - 1 + \frac{1}{2}z.$$

It is clear that $B(z)$ is analytic at $z = 0$ and its power series begins with the z^2 term. Moreover it is easy to check that

$$B(-z) = B(z)$$

so that only even powers of z occur in the expansion of $B(z)$. We may therefore write

$$B(z) = \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k}.$$

The coefficients B_{2k} are the so-called *Bernoulli numbers* because they appeared first in the work of Jacob Bernoulli in his book *Ars Conjectandi*. There are many different conventions about writing them, and we have chosen one of them. They have extraordinary properties and occur in the most diverse contexts. They may be calculated in succession using the recursion formulae obtained from the identity

$$\left(1 - \frac{z}{2} + \frac{B_2 z^2}{2!} + \dots + \frac{B_{2k} z^{2k}}{(2k)!} + \dots \right) \left(1 + \frac{z}{2} + \dots + \frac{z^k}{(k+1)!} + \dots \right) = 1.$$

The recursion formulae are

$$(2k+1)B_{2k} = - \sum_{r=1}^{r=k-1} \binom{2k+1}{2r} B_{2r} + k - \frac{1}{2}.$$

It follows easily from this by induction on k that the B_{2k} are all rational numbers. The first few of them are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

Notice the presence of 691 in the expression for B_{12} which occurred in Euler's first evaluation of $\zeta(12)$.

Returning to the evaluation of $\zeta(2k)$ we find

$$Q(s) - \frac{1}{s} = \frac{2\pi i}{2\pi i s} B(2\pi i s) = 2\pi i \sum_{k=1}^{\infty} B_{2k} \frac{(2\pi i s)^{2k-1}}{(2k)!}.$$

This formula shows that $Q(s) - 1/s$ is smooth and its derivatives at 0 are just the coefficients of the power series that appears on the right side of this formula. Hence

$$-2\zeta(2k) = \frac{1}{(2k-1)!} \frac{d^{2k-1}}{ds^{2k-1}} \left(Q(s) - \frac{1}{s} \right) \Big|_{s=0} = \frac{(2\pi i)^{2k} B_{2k}}{(2k)!}$$

leading to

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}$$

which is Euler's celebrated formula.

One can also calculate the derivatives at $s = \pi/2$ instead of at $s = 0$. Changing s to $s + 1/2$ we have

$$R(s) = -\pi \tan \pi s = \frac{1}{s + (1/2)} + \sum_{n=1}^{\infty} \left(\frac{1}{n + (1/2) + s} - \frac{1}{n - (1/2) - s} \right).$$

As before we can write

$$R(s) = \pi i \left(\frac{e^{2\pi i s} - 1}{e^{2\pi i s} + 1} \right) = \pi i \left(1 - \frac{2}{e^{2\pi i s} + 1} \right).$$

Now

$$\frac{1}{e^z + 1} = \frac{1}{e^z - 1} - \frac{2}{e^{2z} - 1} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(1 - 2^{2k}) B_{2k}}{(2k)!} z^{2k-1}$$

from which it follows easily that

$$R(s) = -2\pi i \sum_{k=1}^{\infty} \frac{(1 - 2^{2k}) B_{2k}}{(2k)!} (2\pi i s)^{2k-1}.$$

So

$$\frac{1}{(2k-1)!} \frac{d^{2k-1} R(s)}{ds^{2k-1}} \Big|_{s=0} = (-1)^{k-1} (1 - 2^{2k}) \frac{2^{2k} B_{2k}}{(2k)!} \pi^{2k}.$$

On the other hand we also have

$$\begin{aligned} \frac{1}{(2k-1)!} \frac{d^{2k-1} R(s)}{ds^{2k-1}} \Big|_{s=0} &= -2^{2k} + \sum_{n=1}^{\infty} \frac{-2^{2k}}{(2n+1)^{2k}} - \sum_{n=1}^{\infty} \frac{2^{2k}}{(2n-1)^{2k}} \\ &= -2^{2k+1} M(2k). \end{aligned}$$

Hence

$$M(2k) = (1 - 2^{-2k}) (-1)^{k-1} \frac{2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}$$

which agrees with the formula

$$M(2k) = (1 - 2^{-2k}) \zeta(2k).$$

Euler numbers and the evaluation of $L(2k+1)$. For the $L(2k+1)$ we start with the partial fraction expansion

$$\frac{\pi}{\sin \pi s} = \frac{1}{s} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n+s} - \frac{1}{n-s} \right).$$

Let

$$P(s) = \frac{\pi}{\sin \pi s}.$$

Then

$$\frac{1}{r!} \frac{d^r P(s)}{ds^r} = \frac{(-1)^r}{s^{r+1}} + (-1)^r \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{(n+s)^{r+1}} - (-1)^r \frac{1}{(n-s)^{r+1}} \right).$$

We now evaluate the derivatives at $s = \frac{1}{2}$. Now $P(s + \frac{1}{2}) = \pi \sec \pi s$. So

$$\frac{1}{2^{2k+1}(2k)!} \left. \frac{d^{2k} \pi \sec \pi s}{ds^{2k}} \right|_{s=0}$$

is equal to

$$1 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{(2n+1)^{2k+1}} - \frac{1}{(2n-1)^{2k+1}} \right).$$

Let us write

$$\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k} z^{2k}}{(2k)!}.$$

The E_{2k} are the so-called *Euler numbers*. Then taking $z = \pi s$ in this expansion we get

$$\frac{1}{2^{2k+1}(2k)!} E_{2k} \pi^{2k+1} = 2L(2k+1).$$

Thus

$$L(2k+1) = \frac{1}{2^{2k+2}(2k)!} E_{2k} \pi^{2k+1}.$$

Let $E_0 = 1$. The Euler numbers satisfy the recursion formulae

$$E_{2k} = \binom{2k}{2} E_{2k-2} - \binom{2k}{4} E_{2k-4} + \cdots + (-1)^{k-2} \binom{2k}{2k-2} E_2 + (-1)^{k-1} E_0$$

and hence are all integers. Therefore $L(2k+1)$ is a rational multiple of π^{2k+1} for all k . The first few Euler numbers are

$$E_2 = 1, E_4 = 5, E_6 = 61, E_8 = 1385$$

which lead to the formulae*

$$L(3) = \frac{\pi^3}{32}, L(5) = \frac{5\pi^5}{1536}, L(7) = \frac{61\pi^7}{184320}, L(9) = \frac{1385\pi^9}{1024 \times 40320}.$$

Sums of series and cyclotomic extensions. Let P, Q be as before:

$$P(s) = \frac{\pi}{\sin \pi s}, \quad Q(s) = \pi \cot \pi s,$$

and let

$$x = \sin \pi s, \quad y = \cos \pi s.$$

*See [1], Ch. 2, Vol. I, 131-132, 200-205 (Euler to Goldbach), in which Euler discusses the partial fractions and writes down the approximate values (up to a huge number of decimal places) of $\zeta(n)$ for $2 \leq n \leq 16$.

Then

$$\frac{dx}{ds} = \pi y, \quad \frac{dy}{ds} = -\pi x$$

while

$$P = \frac{\pi}{x}, \quad Q = \frac{\pi}{x}y.$$

Then we can obtain the derivatives of P and Q successively and get the following:

$$-\frac{dP}{ds} = \frac{\pi^2}{x^2}y, \quad \frac{d^2P}{ds^2} = \frac{\pi^3}{x^3}(y^2 + 1) \quad -\frac{d^3P}{ds^3} = \frac{\pi^4}{x^4}(y^3 + 5y) \text{ etc.}$$

and

$$-\frac{dQ}{ds} = \frac{\pi^2}{x^2}, \quad \frac{d^2Q}{ds^2} = \frac{\pi^3}{x^3}2y \quad -\frac{d^3Q}{ds^3} = \frac{\pi^4}{x^4}(4y^2 + 2) \text{ etc.}$$

Taking for s a rational value

$$s = \frac{p}{q}$$

we then obtain

$$\begin{aligned} \frac{(-1)^k}{k!q^{k+1}}P^{(k)}\left(\frac{p}{q}\right) &= \sum_{n \in \mathbf{Z}} \frac{(-1)^n}{(qn+p)^{k+1}}, \\ \frac{(-1)^k}{k!q^{k+1}}Q^{(k)}\left(\frac{p}{q}\right) &= \sum_{n \in \mathbf{Z}} \frac{1}{(qn+p)^{k+1}}. \end{aligned}$$

In particular

$$\begin{aligned} \frac{\pi^2 \cos \frac{p}{q}\pi}{q^2 \sin^2 \frac{p}{q}\pi} &= \frac{1}{p^2} - \frac{1}{(q-p)^2} - \frac{1}{(q+p)^2} + \frac{1}{(2q-p)^2} + \frac{1}{(2q+p)^2} - \dots \\ \frac{\pi^2}{q^2 \sin^2 \frac{p}{q}\pi} &= \frac{1}{p^2} + \frac{1}{(q-p)^2} + \frac{1}{(q+p)^2} + \frac{1}{(2q-p)^2} + \frac{1}{(2q+p)^2} + \dots \end{aligned}$$

Also

$$\begin{aligned} \frac{\pi}{q \sin \frac{p}{q}\pi} &= \frac{1}{p} + \left(\frac{1}{q-p} - \frac{1}{q+p} \right) - \left(\frac{1}{2q-p} - \frac{1}{2q+p} \right) + \dots \\ \frac{\pi \cos \frac{p}{q}\pi}{q \sin \frac{p}{q}\pi} &= \frac{1}{p} - \left(\frac{1}{q-p} - \frac{1}{q+p} \right) - \left(\frac{1}{2q-p} - \frac{1}{2q+p} \right) - \dots \end{aligned}$$

In [6] Euler takes various special values for p and q and evaluates the series explicitly for $k = 0, 1$. The series he obtains are actually Dirichlet series corresponding to various characters mod q and their variants. Thus, for $q = 3, p = 1, k = 0$ he gets

$$\frac{2\pi}{3\sqrt{3}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\chi(n)}{n}, \quad \frac{\pi}{3\sqrt{3}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

where

$$\chi(n) = \begin{cases} +1 & \text{if } n \equiv 1 \pmod{3}, \\ -1 & \text{if } n \equiv -1 \pmod{3}, \\ 0 & \text{if otherwise.} \end{cases}$$

For $q = 3, p = 1, k = 1$ we get

$$\frac{2\pi^2}{27} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\chi(n)}{n^2}, \quad \frac{4\pi^2}{27} = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^2}$$

where

$$\chi_0(n) = \begin{cases} +1 & \text{if } n \equiv \pm 1 \pmod{n}, \\ 0 & \text{if otherwise.} \end{cases}$$

For $q = 4, p = 1$ we have

$$\frac{\pi^2}{8\sqrt{2}} = \sum_{n=1}^{\infty} \frac{\chi_8(n)}{n^2}$$

where

$$\chi_8(n) = \begin{cases} +1 & \text{if } n \equiv \pm 1 \pmod{8} \\ -1 & \text{if } n \equiv \pm 3 \pmod{8} \\ 0 & \text{if otherwise.} \end{cases}$$

For $q = 6, p = 1$ we get

$$\frac{\pi^2}{6\sqrt{3}} = \sum_{n=1}^{\infty} \frac{\chi_{12}(n)}{n^2}$$

where

$$\chi_{12}(n) = \begin{cases} +1 & \text{if } n \equiv \pm 1 \pmod{12} \\ -1 & \text{if } n \equiv \pm 5 \pmod{12} \\ 0 & \text{if otherwise.} \end{cases}$$

These cases can be continued indefinitely. It is easy to take this discussion to the point where we conclude that

$$\sum_{n \in \mathbf{Z}} \frac{1}{(qn + p)^r} = f_r \pi^r, \quad \sum_{n \in \mathbf{Z}} \frac{(-1)^n}{(qn + p)^r} = g_r \pi^r$$

where f_r and g_r are elements in the field R_q generated by all the q^{th} roots of unity; they are thus certainly *cyclotomic numbers*. If we now take h to be a periodic function on the integers of period q , or, for odd q , $(-1)^n$ times such a periodic function, whose values are cyclotomic numbers, then

$$\sum_{n \in \mathbf{Z}} \frac{h(n)}{n^r} = g \pi^r$$

where g is a cyclotomic number. It would be almost a hundred years before Dirichlet would systematically introduce such series in his path-breaking memoirs on primes in residue classes and reveal the arithmetical meaning of some of these sums (class numbers), and even later than that when Kronecker would discover the true arithmetical significance of the cyclotomic extensions R_q (see Chapter 6).

The work of Jacobi and Weierstrass on elliptic functions. The trigonometric functions are essentially the only periodic functions of a real variable. Indeed, any period can be changed to period 1 by a change of scale, and any polynomial or rational expression in $\cos \pi s$ and $\sin \pi s$ is essentially the only periodic function with period 1. The partial fraction formula can be written as

$$(*) \quad \pi \cot \pi s = PV \sum_{-\infty}^{\infty} \frac{1}{n + s}.$$

This way (*) of writing the expansion as a principal value makes it almost obvious that the infinite series represents a function of period 1.

In the 19th century, when Jacobi and later Weierstrass began studying functions of a *complex* variable which are *doubly periodic*, for instance with periods 1 and τ

($\tau = i, \tau = 1 + i\sqrt{2}$ are classical examples, the first of which goes back to Gauss), it was natural to think of partial fractions and infinite product expansions for them. These series and products would now be taken over the lattice of all periods in the complex plane. The Weierstrass \wp -function and the infinite product expansions of Weierstrass and Jacobi may thus be viewed as natural descendants of Euler's work.

3.6. Partial fraction expansions as integrals

In view of the pivotal role of the partial fractions for $\frac{\pi}{\sin \pi s}$ and $\pi \cot \pi s$ in summing the zeta and the L -series, it was natural for Euler to try to derive these expansions by other more direct methods. His method had two stages. In the first, already given in [6], one obtains the partial fraction expansions as integrals from 0 to 1 of differential forms that are generalizations of the form $(1 + x^2)^{-1} dx$ which was used in evaluating the Leibniz series. The second stage is then dedicated to the evaluation of the integrals.

In generalization of the Leibniz series for $\arctan x$ Euler starts with the series obtained by evaluating the integrals

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1 + x^q} dx, \quad \int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1 - x^q} dx.$$

Here p, q are integers and we assume that

$$q > p > 0.$$

Note that the integrand in the first integral is continuous in the range $[0, 1]$. On the other hand, both the numerator and denominator of the integrand in the second integral are polynomials divisible by $1 - x$, and on removing this factor we obtain an integrand which is a ratio of two polynomials, the denominator being

$$(1 + x + x^2 + \dots + x^{q-1})$$

which is ≥ 1 on $[0, 1]$ and so the integrand is again continuous on $[0, 1]$.

For the first integral we start from the series

$$\frac{1}{1 + x^q} = 1 - x^q + x^{2q} - x^{3q} + x^{4q} + \dots$$

and multiply by x^{p-1} and x^{q-p-1} and integrate from 0 to 1 to get

$$\begin{aligned} \int_0^1 \frac{x^{p-1}}{1 + x^q} dx &= \frac{1}{p} - \frac{1}{q+p} + \frac{1}{2q+p} - \frac{1}{3q+p} + \dots \\ \int_0^1 \frac{x^{q-p-1}}{1 + x^q} dx &= \frac{1}{q-p} - \frac{1}{2q-p} + \frac{1}{3q-p} - \dots \end{aligned}$$

The integrals obviously exist separately since p, q are integers with $q > p > 0$ and the two series are convergent. We add the two series but group them in the following manner to get

$$\frac{1}{p} + \left(\frac{1}{q-p} - \frac{1}{q+p} \right) - \left(\frac{1}{2q-p} - \frac{1}{2q+p} \right) - \dots$$

This gives the formula

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1 + x^q} dx = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{nq-p} - \frac{1}{nq+p} \right).$$

We shall discuss later the proof that

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \frac{\pi}{q \sin(p/q)\pi}.$$

Assuming this we obtain

$$\frac{\pi}{q \sin(p/q)\pi} = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{nq-p} - \frac{1}{nq+p} \right).$$

Multiplying by q and replacing p/q by s we get

$$\frac{\pi}{\sin s\pi} = \frac{1}{s} + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n-s} - \frac{1}{n+s} \right) \quad (0 < s < 1).$$

To be sure, s is a rational number, but both sides are continuous functions and so the above formula is valid for *all* s in the range $0 < s < 1$. The continuity of the infinite series is a consequence of normal convergence when s varies in any interval of the form $[a, 1-a]$ where $0 < a < 1$. To verify normal convergence observe that $1 - s^2/n^2 = (1-s/n)(1+s/n) \geq a$ and hence

$$\left| \frac{1}{n-s} - \frac{1}{n+s} \right| = \frac{2s}{(1-s^2/n^2)} \frac{1}{n^2} \leq \frac{2}{a} \frac{1}{n^2}.$$

For the second integral the calculations are the same, but the integral cannot be split into two terms since the individual integrals do not converge. So we first write the series for $(x^{p-1} - x^{q-p-1})/1 - x^q$ in the form

$$\frac{x^{p-1} - x^{q-p-1}}{1-x^q} = x^{p-1} - \sum_{n=1}^{\infty} x^{nq-p-1} + \sum_{n=1}^{\infty} x^{nq+p-1}.$$

Hence integrating from 0 to y where $0 < y < 1$ we have

$$\int_0^y \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \frac{y^p}{p} - \sum_{n=1}^{\infty} \left(\frac{y^{nq-p}}{nq-p} - \frac{y^{nq+p}}{nq+p} \right).$$

It is now a question of taking the limit as $y \rightarrow 1$ and passing to the limit term by term on the right side. For this we must verify normal convergence of the series on the right when y varies in $[0, 1]$. Write

$$a_n(y) = \frac{y^{nq-p}}{nq-p} - \frac{y^{nq+p}}{nq+p}.$$

Then, for $0 \leq y \leq 1$,

$$\frac{da_n}{dy} = y^{nq-p} - y^{nq+p} = y^{nq-p}(1-y^{2p}) \geq 0$$

showing that a_n is increasing in y and hence reaches its maximum at $y = 1$ and minimum at $y = 0$. So

$$0 \leq a_n(y) \leq a_n(1) = \frac{1}{nq-p} - \frac{1}{nq+p} \leq \frac{2p}{q^2 - p^2} \frac{1}{n^2}.$$

So we have proved normal convergence. Thus

$$\int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \frac{1}{p} - \sum_{n=1}^{\infty} \left(\frac{1}{nq-p} - \frac{1}{nq+p} \right).$$

We shall later give a direct argument that

$$\int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1 - x^q} dx = \frac{\pi \cot(p/q)\pi}{q}.$$

Hence we obtain, writing $s = p/q$,

$$\pi \cot s\pi = \frac{1}{s} \sum_{n=1}^{\infty} \left(\frac{1}{n-s} - \frac{1}{n+s} \right) \quad (0 < s < 1).$$

Again, this is initially true for s rational, but we argue as before that it remains valid for all s with $0 < s < 1$.

Euler's evaluation of the integrals. In [13] Euler evaluated the integrals above by a direct method. We shall now discuss his derivation.

The starting point of Euler's derivation is the calculation of the indefinite integral (primitive) of

$$\frac{x^{m-1}}{1+x^{2n}} dx \quad (m, n \text{ integers, } 2m > n > 0)$$

which is a beautiful generalization of the fact that $\arctan x$ is the primitive of $(1+x^2)^{-1} dx$. Writing $\theta = \frac{\pi}{2n}$, Euler begins with the formula

$$\frac{x^{m-1}}{1+x^{2n}} = \frac{(-1)^{m-1}}{n} \sum_{k=1}^n \frac{x \cos(2k-1)m\theta + \cos(2k-1)(m-1)\theta}{1+2x \cos(2k-1)\theta + x^2}.$$

To prove this we recall that if R is a polynomial with distinct roots a_1, a_2, \dots, a_N (which can be complex) and g is any polynomial of degree $< N$,

$$\frac{g(x)}{R(x)} = \sum_{j=1}^N \frac{g(a_j)R'(a_j)^{-1}}{x-a_j}.$$

In fact, if we assume a representation $\sum_j c_j(x-a_j)^{-1}$ for the right side, we get

$$g(a_j) = c_j \prod_{k \neq j} (a_j - a_k) = c_j R'(a_j).$$

If g, P are real, then the complex roots are even in number and can be grouped in conjugate pairs. Let $R(x) = 1+x^{2n}$, $g(x) = x^{m-1}$ and let us write the roots of R as

$$-e^{\pm i(2k-1)\theta} \quad \left(k = 1, 2, \dots, n, \theta = \frac{\pi}{2n} \right).$$

Combining the fractions corresponding to the conjugate pairs we get the partial fraction expansion above for

$$\frac{x^{m-1}}{1+x^{2n}}.$$

Now we know that

$$\begin{aligned} \int \frac{Ax+B}{1+2x \cos \gamma + x^2} dx &= \frac{A}{2} \log(1+2x \cos \gamma + x^2) \\ &\quad + \frac{B-A \cos \gamma}{\sin \gamma} \arctan \frac{x \sin \gamma}{1+x \cos \gamma}. \end{aligned}$$

Substituting the appropriate values for A, B, γ , we then obtain for

$$\int_0^x \frac{x^{m-1}}{1+x^{2n}} dx$$

the formula

$$\begin{aligned} & \frac{(-1)^{m-1}}{2n} \sum_{k=1}^n \cos(2k-1)m\theta \log(1+2x\cos(2k-1)\theta+x^2) \\ & + \frac{(-1)^{m-1}}{n} \sum_{k=1}^n \sin(2k-1)m\theta \arctan \frac{x\sin(2k-1)\theta}{1+x\cos(2k-1)\theta} \end{aligned}$$

where $\theta = \frac{\pi}{2n}$. We now let $x \rightarrow \infty$ in this formula. The logarithmic terms become

$$\frac{(-1)^{m-1}}{n} \sum_{k=1}^n \cos \frac{(2k-1)m\pi}{2n} \log x + o(1)$$

while the arctan terms converge to

$$\frac{(-1)^{m-1}\pi}{2n^2} \sum_{k=1}^n (2k-1) \sin \frac{(2k-1)m\pi}{2n}.$$

We now have

$$\begin{aligned} & \sum_{k=1}^n \cos \frac{(2k-1)m\pi}{2n} = 0 \\ \text{(T)} \quad & \sum_{k=1}^n (2k-1) \sin \frac{(2k-1)m\pi}{2n} = \frac{(-1)^{m-1}n}{\sin \frac{m\pi}{2n}}. \end{aligned}$$

Assuming these for the moment we get

$$\int_0^\infty \frac{x^{m-1}}{1+x^{2n}} dx = \frac{\pi}{2n \sin \frac{m\pi}{2n}}.$$

We put $p = m, q = 2n$ and rewrite this as

$$\int_0^\infty \frac{x^{p-1}}{1+x^q} dx = \frac{\pi}{q \sin \frac{p\pi}{q}} \quad (q > p > 0).$$

To be sure, we assumed that q is even; but if q is odd, the substitution $x = y^2$ changes the integral to one with the even integer $2q$, and we obtain the above formula for odd q also.

It remains only to prove the trigonometric formulae (T). In [13] Euler evaluates a whole class of such sums. It is not difficult to prove (T) using complex variables. Write $u = e^{\frac{im\pi}{2n}}$. Then $u^{2n} = (-1)^m$. The sum of cosines is then

$$\Re(u + u^3 + \dots + u^{2n-1}) = \Re \frac{u(u^{2n} - 1)}{u^2 - 1} = ((-1)^m - 1) \Re \frac{u}{u^2 - 1}.$$

As $\bar{u} = u^{-1}$ an easy calculation shows that $u(u^2 - 1)^{-1}$ changes into its negative under conjugation and so is purely imaginary, thus having zero real part. This proves the first of the relations in (T). The sum of sines is

$$\Im(u + 3u^3 + \dots + (2n-1)u^{2n-1}) = \Im \left(z \left(\frac{z^{2n+1} - z}{z^2 - 1} \right) \Big|_{z=u} \right).$$

Now

$$\frac{z^{2n+1} - z}{z^2 - 1} = \frac{1}{2} \left(\frac{z^{2n}}{z+1} + \frac{z^{2n}}{z-1} \right) - \frac{1}{2} \left(\frac{1}{z+1} + \frac{1}{z-1} \right).$$

If we remember that $u^{2n} = (-1)^m$ and note that $z(z \pm 1)^{-2}$ is invariant under conjugation and hence real, we find that the required imaginary part is

$$(-1)^m n \Im \left(\frac{1}{u+1} + \frac{1}{u-1} \right) = \frac{(-1)^{m-1} n}{\sin \frac{m\pi}{2n}}.$$

We have thus proved (T) and hence finished the evaluation of our integral.

Since

$$\int_1^\infty \frac{x^{p-1}}{1+x^q} dx = \int_0^1 \frac{y^{q-p-1}}{1+y^q} dy$$

we then obtain

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \frac{\pi}{q \sin \frac{p}{\pi q}}.$$

This finishes the evaluation of the original integral of the first type.

Before we go on to the second integral we remark that Euler does not stop with the above calculation but goes on to study the more general integrals

$$J_k = \int_0^\infty \frac{x^{p-1}}{(1+x^q)^k} dx.$$

Indeed, an integration by parts gives the relation

$$\int_0^x \frac{x^{p-1}}{(1+x^q)^k} dx = \frac{x^p}{(k-1)q(1+x^q)^{k-1}} + \frac{(k-1)q-p}{(k-1)q} \int_0^x \frac{x^{p-1}}{(1+x^q)^{k-1}} dx.$$

(One starts with the integral on the right.) Letting $x \rightarrow \infty$ we get

$$J_k = \frac{(k-1)q-p}{(k-1)q} J_{k-1}$$

leading to

$$\int_0^\infty \frac{x^{p-1}}{(1+x^q)^k} dx = \frac{(q-p)(2q-p)\dots((k-1)q-p)}{q \cdot 2q \cdot \dots \cdot (k-1)q} \frac{\pi}{q \sin \frac{p\pi}{q}}.$$

Following the same method he also finds that

$$\int_0^\infty \frac{x^{m-1}}{1-2x^n \cos \omega + x^{2n}} dx = \frac{\pi \sin \frac{n-m}{n}(\pi-\omega)}{n \sin \omega \sin \frac{(n-m)\pi}{n}}.$$

For $\omega = \frac{\pi}{2}$ this reduces to the previous formula.

We now come to the second type of integrals. The first step is to obtain an expression for the indefinite integral

$$\int_0^x \frac{x^{m-1}}{1-x^{2n}} dx.$$

The method is exactly the same as the previous one. The roots of $x^{2n} - 1 = 0$ are written as

$$\pm 1, -e^{\pm \frac{ik\pi}{n}} \quad (k = 1, 2, \dots, n-1).$$

The partial fraction expansion for

$$\frac{x^{m-1}}{1-x^{2n}}$$

is now, with $\tau = \frac{\pi}{n}$,

$$\frac{1}{2n(1-x)} + \frac{(-1)^{m-1}}{2n(1+x)} + \frac{(-1)^{m-1}}{n} \sum_{k=1}^{n-1} \frac{x \cos km\tau + \cos k(m-1)\tau}{1 + 2x \cos k\tau + x^2}.$$

We integrate as before to get,* for $0 < x < 1$,

$$\begin{aligned} \int_0^x \frac{x^{m-1}}{1-x^{2n}} dx &= \frac{-1}{2n} \log(1-x) + \frac{(-1)^{m-1}}{2n} \log(1+x) \\ &\quad + \frac{(-1)^{m-1}}{2n} \sum_{k=1}^{n-1} \cos km\tau \log(1 + 2x \cos \tau + x^2) \\ &\quad + \frac{(-1)^{m-1}}{n} \sum_{k=1}^{n-1} \sin km\tau \arctan \frac{x \sin k\tau}{1 + x \cos k\tau}. \end{aligned}$$

The $\log(1-x)$ term will diverge as $x \rightarrow 1-$, but we write a similar formula changing m to $2n-m$ and subtract; the logarithmic terms cancel and we get an expression that is convergent when $x \rightarrow 1-$, namely

$$\int_0^x \frac{x^{m-1} - x^{2n-m-1}}{1-x^{2n}} dx = \frac{2(-1)^{m-1}}{n} \sum_{k=1}^{n-1} \sin \frac{km\pi}{n} \arctan \frac{x \sin \frac{k\pi}{n}}{1 + x \cos \frac{k\pi}{n}}.$$

We now let $x \rightarrow 1-$ and use the formula $\frac{\sin \varphi}{1+\cos \varphi} = \tan \frac{\varphi}{2}$ to get

$$\int_0^1 \frac{x^{m-1} - x^{2n-m-1}}{1-x^{2n}} dx = \frac{(-1)^{m-1}\pi}{n^2} \sum_{k=1}^{n-1} k \sin \frac{km\pi}{n}.$$

We now have

$$\sum_{k=1}^{n-1} k \sin \frac{km\pi}{n} = \frac{(-1)^{m-1}n}{2} \cot \frac{m\pi}{2n}$$

which is proved by exactly the same methods we used earlier. Indeed, with $u = e^{\frac{im\pi}{n}}$, the left side of the above formula is

$$\Im \left(z \left(\frac{z^n - 1}{z - 1} \right)' \Big|_{z=u} \right) = \Im \left(\frac{-u^n \cdot u}{(u-1)^2} + \frac{nu^{n-1} \cdot u}{u-1} + \frac{u}{(u-1)^2} \right).$$

But $u^n = (-1)^m$ and $\frac{u}{(u-1)^2}$ is unchanged under conjugation so that it has zero imaginary part. Hence the above reduces to

$$(-1)^m n \Im \left(\frac{1}{u-1} \right) = \frac{(-1)^{m-1}n}{2} \cot \frac{m\pi}{2n}.$$

Thus

$$\int_0^1 \frac{x^{m-1} - x^{2n-m-1}}{1-x^{2n}} dx = \frac{\pi}{2n} \cot \frac{m\pi}{2n}.$$

We write $q = 2n$, $p = m$ and remove the restriction that q be even by using the substitution $x = y^2$ when q is odd. We thus finally obtain

$$\int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \frac{\pi}{q} \cot \frac{p\pi}{q} \quad (q > p > 0).$$

*There is a misprint in §43 of [13] in this formula.

Eulerian integrals. Remarks on the gamma and beta functions. We shall now give a different method for evaluating the integrals that is based on the theory of the beta and gamma functions. These were first introduced by Euler (who else!) and later called *Eulerian integrals of the first and second kind* by Legendre. The discussion below is thus an opportunity to go into this part of analysis that Euler started. This derivation uses nothing that Euler did not know.

What we know now as the *gamma function* is the integral

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx \quad (0 < s < 1).$$

The condition on s is needed for convergence of the integral at 0. Euler would write this as

$$\Gamma(s) = \int_0^1 (-\log x)^{s-1} dx.$$

It was Legendre who wrote the integral in the infinite range and who introduced the notation $\Gamma(s)$ for it. In either representation it is trivial to check, using integration by parts, that

$$\Gamma(s+1) = s\Gamma(s), \quad \Gamma(1) = 1$$

and hence

$$\Gamma(n+1) = \int_0^1 (-\log x)^n dx = n!$$

Euler viewed the function $\Gamma(s)$ as an extension of the function $n!$, which is defined only over the natural numbers, to a function defined for all real values $s > 0$. Legendre called this the *Eulerian integral of the second kind*.

The *Eulerian integral of the first kind* is the integral

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx \quad (a > 0, b > 0).$$

The limits on a and b are again needed for convergence. This function is nowadays called the *beta function*. There is also an alternative form of it, as an infinite integral, namely

$$\int_0^{\infty} \frac{x^{a-1}}{(1+x)^{a+b}} dx = B(a, b);$$

the substitution $x \rightarrow \frac{1}{1+x}$ changes this to the standard form. We also have

$$\int_0^{\infty} \frac{x^{a-1}}{1+x^b} dx = \frac{1}{b} B\left(\frac{a}{b}, 1 - \frac{a}{b}\right)$$

as can be seen by the substitution $x \rightarrow x^b$. The relation between the beta and the gamma functions is given by the formula

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

In addition to the formula

$$\Gamma(n+1) = n! \quad (n = 0, 1, 2, \dots)$$

we have

$$\Gamma(1/2) = \sqrt{\pi}$$

and, more generally, the reflection formula

$$B(s, 1-s) = \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \quad (0 < s < 1),$$

both of which were known to Euler. We also need the formula obtained by logarithmic differentiation of the reflection formula. We have, with primes denoting derivatives with respect to s ,

$$-\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} = \pi \cot s\pi.$$

It is in 1729, in a letter to Goldbach* that the gamma function first makes its appearance. Euler's definition, in modern notation, is for positive real m :

$$(*) \quad \Gamma(1+m) = \lim_{n \rightarrow \infty} \frac{1.2.3 \dots n}{(m+1) \dots (m+n)} (n+1)^m,$$

or, in a form closer to what he preferred to write,

$$\Gamma(1+m) = \lim_{n \rightarrow \infty} \frac{1.2^m}{m+1} \frac{2^{1-m}.3^m}{m+2} \dots \frac{n^{1-m} \cdot (n+1)^m}{m+n}.$$

Subsequently he gave more details in his paper [14] of 1730/1 based on this definition. He knew all the basic properties of this function and used them in an essential manner in his work on the functional equation of the zeta function. We shall very briefly sketch the principal properties of Γ , all of which were known to Euler. For a very complete treatment see [15]; we follow this treatment in our brief discussion.

Let

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log(n+1)$$

so that

$$\lim_{n \rightarrow \infty} \gamma_n = \gamma$$

is Euler's constant. Then

$$\frac{z(z+1) \dots (z+n)}{n!} (n+1)^{-z} = ze^{z\gamma_n} \prod_{k=1}^{n-1} \left(1 + \frac{z}{k}\right) e^{-z/k}$$

so that we obtain Weierstrass's definition, now valid for all *complex* z ,

$$\frac{1}{\Gamma(z)} = ze^{z\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

This is a classic example of a canonical product of Weierstrass and defines an entire function with simple zeros precisely at $z = 0, -1, -2, \dots$. So the limit (*) exists for all $z \neq 0, -1, -2, \dots$ and defines Γ as a meromorphic function, vanishing nowhere and having simple poles precisely at $0, -1, -2, \dots$. It is immediate from the definition that

$$\Gamma(z+1) = z\Gamma(z).$$

If we start with the integral

$$\Pi(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt$$

and change t to ns and integrate by parts repeatedly, we get

$$\Pi(z, n) = \frac{1.2 \dots n}{z(z+1) \dots (z+n)} n^z$$

*See [1], Ch. 2, Vol. I, 3-6 (Euler to Goldbach).

so that

$$\Gamma(z) = \lim_{n \rightarrow \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt.$$

For $0 < x < 1$ we have $-\log(1-x) = x + x^2/2 + \dots > x$ so that $1-x < e^{-x}$. Hence

$$0 \leq \left(1 - \frac{t}{n}\right)^n |t^{z-1}| \leq e^{-t} t^{\Re(z)-1} \quad (\Re(z) > 0)$$

and so there is normal convergence and we can pass to the limit under the integral sign. Thus

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \int_0^1 (-\log u)^{z-1} du \quad (\Re(z) > 0).$$

Euler always used the second form of the integral. It is clear from this that $\Gamma(z) > 0$ if z is real and > 0 and that

$$\Gamma(1+z) = z! \quad (z = 1, 2, 3, \dots).$$

Thus $\Gamma(1+z)$ *interpolates* $z!$ when z is not a nonnegative integer, a point of view that was emphasized by Euler. From Euler's infinite product for $\sin z$ it follows at once that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}.$$

In particular, taking $z = 1/2$ we have

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Reverting back to Euler's definition with $m = \frac{1}{2}$, we can write this as

$$\frac{1}{2}\sqrt{\pi} = \sqrt{\frac{2.4}{3.3} \frac{4.6}{5.5} \dots}$$

or

$$\frac{\pi}{4} = \frac{2.4}{3.3} \frac{4.6}{5.5} \dots,$$

which is a famous result of Wallis [16]. It would seem that this result of Wallis acted as a guide to Euler when he set about his theory of the gamma function. From this point of view, the Eulerian infinite products for $\sin x$ are far-reaching generalizations of the Wallis product for $\frac{\pi}{4}$.

The beta function is defined as

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx \quad (\Re(a), \Re(b) > 0).$$

The fundamental formula

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

may be proved, in post-Eulerian fashion, as follows. We have

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty e^{-t} e^{-u} t^{a-1} u^{b-1} dt du.$$

Change to new variables $t = x^2$, $u = y^2$ and use polar coordinates in the xy -plane to get

$$\Gamma(a)\Gamma(b) = 2 \int_0^\infty e^{-r^2} (r^2)^{a+b-1} d(r^2) \int_0^{\pi/2} (\cos \theta)^{2a-1} (\sin \theta)^{2b-1} d\theta.$$

Changing to $w = \cos^2 \theta$ we get in the end

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b).$$

Finally

$$\Gamma'(1) = -\gamma.$$

To prove this we logarithmically differentiate Weierstrass's infinite product to get

$$-\Gamma'(1) = 1 + \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)} - \frac{1}{n} \right) = \gamma.$$

It is very interesting to look at Euler's derivation of the formula

$$n! = \int_0^1 (-\log x)^{n-1} dx.$$

Indeed, once he obtained this, he would use this as the definition of the gamma function. To derive this Euler first starts out from the formula, established by expanding $(1-x)^n$ by the binomial theorem,

$$\int_0^1 x^e (1-x)^n dx = \frac{1.2 \dots n}{(e+1)(e+2) \dots (e+n+1)}.$$

He then writes $e = \frac{f}{g}$ and rewrites this as

$$\frac{1.2.3 \dots n}{(f+g)(f+2g) \dots (f+ng)} = \frac{f+(n+1)g}{g^{n+1}} \int_0^1 x^{\frac{f}{g}} (1-x)^n dx.$$

He now wants to take $f \rightarrow 1$ and let $g \rightarrow 0$; the left side will go to $n!$, but the behavior of the right side is not clear. Euler writes the limiting form of the right side as

$$\int_0^1 \frac{x^{\frac{1}{g}} (1-x)^n}{g^{n+1}} dx$$

and proceeds to determine it! He makes the substitution

$$x = y^{\frac{g}{1+g}}$$

and obtains the expression

$$\frac{f+(n+1)g}{f+g} \int_0^1 \frac{(1-x^{\frac{g}{1+g}})^n}{g^n} dx.$$

He now evaluates the limit of the expression inside the integral for $g \rightarrow 0$, $f \rightarrow 1$ by a "known" method, namely l'Hopital's rule, to get

$$\lim_{g \rightarrow 0} \left(\frac{1-x^{\frac{g}{1+g}}}{g} \right)^n = (-\log x)^n$$

and hence finally

$$n! = \int_0^1 (-\log x)^n dx$$

(see [17] for a fuller account).

After this brief introduction let us return to our integrals. We wish to prove that

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \frac{\pi}{q \sin(\frac{p}{q}\pi)} \quad (0 < p < q, \quad p, q \text{ integers}).$$

Now

$$\int_0^1 \frac{x^{q-p-1}}{1+x^q} dx = \int_1^\infty \frac{x^{p-1}}{1+x^q} dx$$

as can be seen by the substitution $x \rightarrow \frac{1}{x}$. Hence

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \int_0^\infty \frac{x^{p-1}}{1+x^q} dx = \frac{1}{q} B\left(\frac{p}{q}, 1 - \frac{p}{q}\right).$$

Thus

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \frac{\pi}{q \sin\left(\frac{p}{q}\right)\pi} \quad (0 < p < q, \quad p, q \text{ integers})$$

as we wanted to prove.

The treatment of the second integral is a little more delicate since we cannot separate the two terms. We make the substitution $y = 1 - x^q$ to get, with $s = p/q$ as before,

$$\int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \frac{1}{q} \int_0^1 \frac{(1-y)^{s-1} - (1-y)^{-s}}{y} dy.$$

To separate the two terms we introduce a factor y^h . We have

$$\begin{aligned} \int_0^1 \frac{y^{s-1} + y^{-s}}{y^{1-h}} dy &= B(s, h) - B(1-s, h) \\ &= \frac{\Gamma(s)\Gamma(h)}{\Gamma(s+h)} - \frac{\Gamma(1-s)\Gamma(h)}{\Gamma(1-s+h)} \\ &= \frac{1}{h} \left(\frac{\Gamma(s)\Gamma(1+h)}{\Gamma(s+h)} - \frac{\Gamma(1-s)\Gamma(1+h)}{\Gamma(1-s+h)} \right) \\ &= \frac{1}{h} (a(s, h) - a(1-s, h)) \end{aligned}$$

where

$$a(s, h) = \frac{\Gamma(s)\Gamma(1+h)}{\Gamma(s+h)}.$$

But

$$a(s, 0) = a(1-s, 0) = 1$$

so that

$$\int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \lim_{h \rightarrow 0} \int_0^1 \frac{y^{s-1} + y^{-s}}{y^{1-h}} dy = a'(s, 0) - a'(1-s, 0)$$

where the primes denote derivatives with respect to h . As

$$a'(s, 0) = \Gamma'(1) - \frac{\Gamma'(s)}{\Gamma(s)}$$

we have, for $a'(s, 0) - a'(1-s, 0)$, the expression

$$-\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} = -\frac{d}{ds} \log(\Gamma(s)\Gamma(1-s)) = \pi \cot s\pi$$

by the reflection formula. Hence we have finally shown that

$$\int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \frac{\pi \cot(p/q)\pi}{q}.$$

As mentioned earlier we may replace p/q by a real parameter s with $0 < s < 1$ in the final formulae.

3.7. Multizeta values

More than 30 years after his great discoveries on the zeta values, Euler introduced a new theme, one whose richness is far from being completely exhausted even today. In a remarkable paper [9] published in 1775 Euler introduced the *double zeta series*

$$\zeta(s_1, s_2) = \sum_{n_1 > n_2 > 0} \frac{1}{n_1^{s_1} n_2^{s_2}} \quad (s_1, s_2 \in \mathbf{Z}, s_1 \geq 2, s_2 \geq 1).$$

The condition $s_1 \geq 2$ is needed for convergence, although Euler operated boldly with the case when $s_1 = 1$ also. The above definition, which is the one currently used, is a slight modification of the one Euler used. Euler allowed n_1 to be equal to n_2 so that

$$\zeta_{\text{Euler}}(s_1, s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2).$$

In the paper [9] Euler obtained remarkable identities involving these, such as

$$\zeta(2, 1) = \zeta(3)$$

as well as the more general

$$\zeta(p, 1) + \zeta(p-1, 2) + \cdots + \zeta(2, p-1) = \zeta(p+1).$$

From this he derived

$$2\zeta(p-1, 1) = (p-1)\zeta(p) - \sum_{2 \leq q \leq p-2} \zeta(q)\zeta(p-q).$$

In his paper cited above, Euler's aim was to express any $\zeta(a, b)$ with $a + b = r$ as a linear combination with rational coefficients of $\zeta(r)$ and the $\zeta(p)\zeta(r-p)$, generalizing the above formula. He succeeded in doing this for r odd but not for r even. We shall comment on this later.

Euler's identities for the double zeta values. The starting point to prove the identities described above is the obvious relation

$$\zeta(p)\zeta(q) = \zeta(p, q) + \zeta(q, p) + \zeta(p+q) \quad (p, q > 1).$$

This is immediate from the formula

$$\zeta(p)\zeta(q) = \sum_{m, n} \frac{1}{m^p} \frac{1}{n^q} = \sum_{m > n} + \sum_{n > m} + \sum_{m=n}.$$

The idea now is to look for what happens when p or q is 1 in view of the general principle that at the boundary of the regions where identities hold, interesting things happen. Now both $\zeta(1, q)$ and $\zeta(1)\zeta(q)$ are given by logarithmically diverging series and so one might expect that by examining more closely this divergence we may obtain identities at the points when one of the two values equals 1. The method used below is Euler's.

We start with the partial fraction expansion

$$\begin{aligned} (x+a)^{-p} x^{-q} &= \sum_{0 \leq i \leq q-1} \binom{-p}{i} a^{-(p+i)} x^{-(q-i)} \\ &\quad + \sum_{0 \leq j \leq p-1} \binom{-q}{j} (-a)^{-(q+j)} (x+a)^{-(p-j)}. \end{aligned}$$

To see this, write

$$(x+a)^{-p}x^{-q} = \sum_{0 \leq i \leq q-1} A_i x^{-(q-i)} + \sum_{0 \leq j \leq p-1} B_j (x+a)^{-(p-j)}.$$

Then

$$(x+a)^{-p} = \sum_i A_i x^i + ((x+a)^{-p} \sum_j B_j (x+a)^j) x^q.$$

Then $x = 0$ gives $A_0 = a^{-p}$, and differentiating i times at $x = 0$ gives $A_i = \binom{-p}{i} a^{-(p+i)} = A_i$. We do the same thing at $x = -a$ to get the B_j . This expansion is valid when $p, q \geq 1$. For $p = 2, q = 1$ this is

$$\frac{1}{xa^2} - \frac{1}{(x+a)a^2} = \frac{1}{(x+a)^2x} + \frac{1}{(x+a)^2a}.$$

Let us sum over $a, x = 1, 2, \dots, N$. Write

$$\zeta_N(r) = \sum_{1 \leq n \leq N} \frac{1}{n^r}, \quad \zeta_N(r, s) = \sum_{N \geq m > n \geq 1} \frac{1}{m^r n^s}.$$

Then we get

$$\zeta_N(2)\zeta_N(1) - \sum_{1 \leq a, x \leq N} \frac{1}{x+a} \frac{1}{a^2} = 2 \sum_{1 \leq a, x \leq N} \frac{1}{(x+a)^2} \frac{1}{a}.$$

To estimate the right side we use

$$\zeta_N(2, 1) < \sum_{1 \leq a, x \leq N} \frac{1}{(x+a)^2} \frac{1}{a} < \zeta(2, 1)$$

so that the right side $\rightarrow 2\zeta(2, 1)$. We shall now look at the left side. It is

$$\begin{aligned} \zeta_N(2)\zeta_N(1) - \sum_{1 \leq a \leq N} \frac{1}{a^2} \sum_{a+1 \leq b \leq a+N} \frac{1}{b} \\ = \zeta_N(2)\zeta_N(1) - \zeta_N(1, 2) - \sum_{1 \leq a \leq N} \frac{1}{a^2} \sum_{N+1 \leq j \leq N+a} \frac{1}{j}. \end{aligned}$$

We claim that the last term on the right side of this relation goes to 0. Indeed,

$$\sum_{N+1 \leq j \leq N+a} \frac{1}{j} < \int_N^{N+a} \frac{dx}{x} = \log \frac{N+a}{N} = \log \left(1 + \frac{a}{N}\right) < \frac{a}{N}$$

so that the last term in question is majorized by

$$\sum_{1 \leq a \leq N} \frac{1}{a^2} \frac{a}{N} = \frac{1}{N} \sum_{1 \leq a \leq N} \frac{1}{a} = O\left(\frac{\log N}{N}\right) \rightarrow 0.$$

We thus have

$$\zeta_N(2)\zeta_N(1) - \zeta_N(1, 2) \rightarrow 2\zeta(2, 1).$$

On the other hand,

$$\zeta_N(2)\zeta_N(1) = \zeta_N(1, 2) + \zeta_N(2, 1) + \zeta_N(3)$$

so that

$$\zeta_N(2)\zeta_N(1) - \zeta_N(1, 2) \rightarrow \zeta(2, 1) + \zeta(3)$$

also. So

$$\zeta(2, 1) + \zeta(3) = 2\zeta(2, 1)$$

which is the required identity. The same method works for $p \geq 2, q = 1$. The partial fraction expansion becomes

$$\frac{1}{a^p} \frac{1}{x} - \frac{1}{(x+a)} \frac{1}{a^p} = \frac{1}{(x+a)^p} \frac{1}{x} + \sum_{0 \leq j \leq p-2} \frac{1}{a^{j+1}} \frac{1}{(x+a)^{p-j}}.$$

Once again we sum over $a, x = 1, 2, \dots, N$. For $r \geq 2$ we have

$$\zeta_N(r, j+1) < \sum_{1 \leq a, x \leq N} \frac{1}{a^{j+1}} \frac{1}{(x+a)^r} < \zeta(r, j+1)$$

so that the right side tends to

$$2\zeta(p, 1) + \zeta(p-1, 2) + \dots + \zeta(2, p-1).$$

The left side is

$$\zeta_N(p)\zeta_N(1) - \zeta_N(1, p) - \sum_{1 \leq a \leq N} \frac{1}{a^p} \sum_{N+1 \leq j \leq N+a} \frac{1}{j}$$

and the last term of this relation, as before, is $O\left(\frac{\zeta_N(p-1)}{N}\right) = o(1)$. Hence

$$\zeta_N(p)\zeta_N(1) - \zeta_N(1, p) \rightarrow 2\zeta(p, 1) + \zeta(p-1, 2) + \dots + \zeta(2, p-1).$$

But

$$\zeta_N(p)\zeta_N(1) = \zeta_N(p, 1) + \zeta_N(1, p) + \zeta_N(p+1)$$

so that

$$\zeta_N(p)\zeta_N(1) - \zeta_N(1, p) \rightarrow \zeta(p, 1) + \zeta(p+1)$$

also. Hence

$$\zeta(p, 1) + \zeta(p-1, 2) + \dots + \zeta(2, p-1) = \zeta(p+1).$$

This leads to one of Euler's many formulae. We have

$$2\zeta(p+1) = 2\zeta(p, 1) + \sum_{2 \leq q \leq p-1} [\zeta(p-q+1, q) + \zeta(q, p-q+1)]$$

so that, writing $p = \lambda - 1$ as Euler does, we get

$$2\zeta(\lambda-1, 1) = (\lambda-1)\zeta(\lambda) - \sum_{2 \leq q \leq \lambda-2} \zeta(\lambda-q)\zeta(q).$$

For $\lambda = 3$ we get the formula $\zeta(2, 1) = \zeta(3)$. This formula suggests that for each λ , perhaps each $\zeta(m, n)$ with $m+n = \lambda$ is a linear combination of $\zeta(\lambda)$ and the $\zeta(p)\zeta(q)$ for $p+q = \lambda$. This is true in many cases as Euler himself found out, but not always [18a].

Multizeta values. The double zeta series can obviously be generalized further. We then obtain the *multizeta values* defined by

$$\zeta(s_1, s_2, \dots, s_r) = \sum_{n_1 > n_2 > \dots > n_r} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_r^{s_r}} \quad (s_i \in \mathbf{Z}, s_i \geq 1, s_1 \geq 1).$$

The identities of Euler discussed above can be generalized to these multizeta values as we shall see below. In recent years the MZV have generated tremendous interest because of their relationship to some fundamental objects in number theory, topology, and algebraic geometry. Some basic conjectures have been formulated, and this area remains an extremely hot one today.

The shuffle identities between the MZV. I follow Deligne [19]. Let $[1, r]$ denote the set $\{1, 2, \dots, r\}$. A *shuffle* of $[1, r]$ and $[1, s]$ is a bijection σ from the disjoint sum $I(r, s)$ of $[1, r]$ and $[1, s]$, identified with $[1, r] \times \{1\} \cup [1, s] \times \{2\}$, onto $[1, r + s]$, such that σ is strictly increasing on each of $[1, r] \times \{1\}$ and $[1, s] \times \{2\}$. A shuffle is thus a way of splicing together two ordered sequences into one sequence such that the order within each sequence is preserved. Thus if $r = 3, s = 2$, then there are 10 shuffles; a shuffle in this case is determined as soon as we decide the images of $(1, 2)$ and $(2, 2)$. If $\sigma(1, 2) = a$ where $a = 1, 2, 3, 4$, then $\sigma(2, 2)$ has to be in $\{a + 1, \dots, 5\}$, so that the total number is $\sum_{1 \leq a \leq 4} (5 - a) = 10$. A *shuffle with equalities* is heuristically just like a shuffle, but the two sequences are allowed to overlap when we splice them together. Thus a shuffle with equalities of $[1, r]$ and $[1, s]$ is a map σ of $I(r, s)$ onto $[1, t]$ where $t \leq r + s$ which is strictly increasing on each of $[1, r] \times \{1\}$ and $[1, s] \times \{2\}$. Let $\text{Sh}(r, s)$ and $\text{Sh}^+(r, s)$ be the set of shuffles and shuffles with equalities of $[1, r]$ and $[1, s]$.

If f, g are two functions on positive integers, then

$$\sum_{0 < n_1 < N} f(n_1) \sum_{0 < n_2 < N} g(n_2) = \sum_{n_1 > n_2} + \sum_{n_2 > n_1} + \sum_{n_1 = n_2}.$$

The terms on the right correspond to the 3 shuffles with equalities $\sigma \in \text{Sh}^+(1, 1)$, according to whether $\sigma(1, 1) >, <, = \sigma(1, 2)$. This can be generalized immediately and leads us to the following identities:

$$\zeta(a_1, \dots, a_r) \zeta(b_1, \dots, b_s) = \sum_{\sigma \in \text{Sh}^+(r, s)} \zeta(c_1, \dots, c_t)$$

where $[1, t]$ is the range of σ and

$$c_k = \begin{cases} a_u & \text{if } \sigma(u, 1) = k \\ b_v & \text{if } \sigma(v, 2) = k \\ a_u + b_v & \text{if } \sigma(u, 1) = \sigma(v, 2) = k. \end{cases}$$

For instance,

$$\begin{aligned} \zeta(a_1, a_2) \zeta(b) &= \zeta(b, a_1, a_2) + \zeta(a_1, b, a_2) \\ &\quad + \zeta(a_1, a_a, b) + \zeta(a_1 + b, a_2) + \zeta(a_1, a_2 + b). \end{aligned}$$

MZV as iterated integrals. We have already seen that Euler had an integral representation of $\zeta(2)$, namely,

$$\zeta(2) = \int_0^1 \frac{-\log(1-t)}{t} dt.$$

We can write this as

$$\zeta(2) = \int_{1 > t > s > 0} \frac{dt}{t} \frac{ds}{1-s}.$$

More generally, let us define the *dilogarithm* by

$$\text{Li}_2(z) = \int_{z > t > s > 0} \frac{dt}{t} \frac{ds}{1-s}.$$

Then

$$\text{Li}_2(z) = \sum_{n > 0} \frac{z^n}{n^2}, \quad \zeta(2) = \text{Li}_2(1).$$

There is an obvious generalization of this to multizeta values. We define

$$\omega_0 = \frac{dt}{t}, \quad \omega_1 = \frac{dt}{1-t},$$

and for any sequence $\mathbf{i} = (i(1), i(2), \dots, i(r))$,

$$\omega(\mathbf{i}) = \omega_{i(1)}\omega_{i(2)} \dots \omega_{i(r)}.$$

We define the *iterated integral*

$$\text{It} \int_0^z \omega(\mathbf{i}) := \int_{z > t_1 > t_2 > \dots > t_r > 0} \omega_{i(1)}\omega_{i(2)} \dots \omega_{i(r)} dt_1 dt_2 \dots dt_r.$$

It is then an easy calculation, by expanding

$$\frac{1}{1-t} = \sum_{n \geq 0} t^n,$$

that

$$\text{It} \int_0^z \omega(0, 1) = \int_{z > t_1 > t_2 > 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2}.$$

More generally, let s_1, s_2, \dots, s_r be integers ≥ 1 , and let $\mathbf{i} = (i(1), \dots, i(r))$ be any sequence of 0's and 1's such that $i(m) = 1$ precisely for $m = s_1, s_1 + s_2, \dots, s_1 + s_2 + \dots + s_r$. Write

$$\omega_{s_1, s_2, \dots, s_r} = \omega(\mathbf{i}).$$

Then

$$\text{It} \int_0^z \omega(\mathbf{i}) = \text{Li}_r(z)$$

where $\text{Li}_r(z)$ is the *polylogarithm* defined by

$$\text{Li}_r(z) = \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \dots n_r^{s_r}}.$$

If $s_1 \geq 2$ we can take $z = 1$ and get

$$\zeta(s_1, s_2, \dots, s_r) = \text{Li}_r(1) = \text{It} \int_0^z \omega_{s_1, s_2, \dots, s_r}.$$

The representation of the MZV as iterated integrals gives rise to a second family of shuffle identities among the MZV and, more generally, among the polylogarithms. To see this, note that

$$\{t > 0\} \times \{s > 0\} \approx \{t > s > 0\} \sqcup \{s > t > 0\}$$

(\sqcup means that the union is disjoint) where \approx means that the two sides are equal up to a set of measure 0, namely the set where $t = s$. This can be generalized immediately to the following:

$$\begin{aligned} \{t_1 > t_2 > \dots > t_a > 0\} \times \{s_1 > s_2 > \dots > s_b > 0\} \\ \approx \bigsqcup_{\sigma \in \text{Sh}(a,b)} \{u_{\sigma 1} > u_{\sigma 2} > \dots > u_{\sigma a+b} > 0\} \end{aligned}$$

where

$$u_{\sigma k} = \begin{cases} t_i & \text{if } \sigma(i) = k \\ s_j & \text{if } \sigma(j) = k. \end{cases}$$

Integration and Fubini's theorem now lead at once to the shuffle identities

$$\begin{aligned} \text{It} \int_0^z \omega_{s_1, s_2, \dots, s_a} \cdot \text{It} \int_0^z \omega_{t_1, t_2, \dots, t_b} \\ = \sum_{\sigma \in \text{Sh}(a, b)} \text{It} \int_0^z \sigma[\omega_{s_1, s_2, \dots, s_a}, \omega_{t_1, t_2, \dots, t_b}]. \end{aligned}$$

Here

$$\sigma[\omega_{s_1, s_2, \dots, s_a}, \omega_{t_1, t_2, \dots, t_b}]$$

means the form obtained by splicing the two forms according to σ . If $s_1, t_1 \geq 2$, we can take $z = 1$ and get the shuffle identity between the MZV's:

$$\zeta(s_1, s_2, \dots, s_a) \zeta(t_1, t_2, \dots, t_b) = \sum_{\sigma \in \text{Sh}(a, b)} \sigma[(s_1, \dots, s_a), (t_1, \dots, t_b)].$$

Here

$$\sigma[(s_1, \dots, s_a), (t_1, \dots, t_b)]$$

means the sequence obtained by splicing the two sequences $(s_i), (t_j)$ according to σ .

The reason why the shuffles with equalities do not enter in this identity is that the regions where two of the variables are equal have measure zero and so do not contribute to the integrals. These identities are very different from the earlier ones. For instance, the first set of identities gives

$$\zeta(2)\zeta(3) = \zeta(2, 3) + \zeta(3, 2) + \zeta(5)$$

while the second set gives

$$\zeta(2)\zeta(3) = \zeta(2, 3) + 4\zeta(3, 2) + 5\zeta(4, 1).$$

There is a third set of identities due to Ecalle [20], and it is an open question whether all the identities between the MZV's are consequences of these three sets. For more detailed results and discussions of open questions as well as the connections of the MZV to other branches of mathematics, the reader should consult the literature. There is an interpretation of the various integrals as periods; see [18b].

Notes and references

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- (1) This article is a review of the entire history of Euler's discovery of the zeta values.
- (2) The forgotten article of Euler was published in a Swiss journal and includes a list of the exact values of $\zeta(n)$ for even $n \leq 26$; see I-14, 177-186.
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