FIELDS INSTITUTE MONOGRAPHS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Large Deviations

Frank den Hollander

American Mathematical Society
Providence, Rhode Island

PREFACE

Large deviation theory is a part of probability theory that deals with the description of events where a sum of random variables deviates from its mean by more than a "normal" amount, i.e., beyond what is described by the central limit theorem. A precise calculation of the probabilities of such events turns out to be crucial for the study of integrals of exponential functionals of sums of random variables, which come up in a variety of different contexts. Large deviation theory finds application in probability theory, statistics, operations research, ergodic theory, information theory, statistical physics, financial mathematics, and the list goes on.

These lecture notes are an *introduction* to large deviations. Part A (Chapters I–V) describes *theory*, Part B (Chapters VI–X) describes *applications*. A glance at the table of contents shows what topics are covered. I have put much effort into conveying the main ideas without putting too much emphasis on technicalities. Most of the theory is driven by a few "key principles" and once these are understood the rest of the journey is safe sailing, give or take a storm or two. This is not to say that it is easy to grasp the full theoretical panorama. But the reader's patience will be rewarded when the ship enters the harbor of the applications.

Chapters I and II contain the basic large deviation theorems for i.i.d. random variables. Here the goal is to make the reader acquainted with the type of statements that are typical for the theory and to obtain results via explicit calculation of the rate function. Chapter III presents general definitions and theorems in a more abstract context. Here the goal is to expose the unified scheme that gives large deviation theory its overall structure and that can be made to work in many concrete cases. Chapter IV looks at large deviation theorems for Markov chains and explains how these can be obtained from the i.i.d. case via a change-of-measure argument. Chapter V considers random sequences with moderate dependence and shows how many of the results in Chapters I, II and IV can be put under a single heading, which in some sense closes the circle.

Chapters VI–X describe a selection of applications: statistical hypothesis testing; random walk in random environment; heat conduction with random sources and sinks; polymer chains; interacting diffusions. Here the theory comes to life and the reader gets to see the full impact of the results derived earlier. Except for the application in statistical hypothesis testing, which is put in mainly for didactical reasons, all applications are recent. Naturally their choice reflects my personal taste and involvement, since they circle around statistical physics and random media. But I think they offer a good sample of what large deviation theory is able to do in various different contexts. Each application is self-contained and tells a small story.

Many questions that come up during the exposition are posed as exercises to the reader. The solutions to these exercises are given in the Appendix. At the end I have included a list of frequently used words and symbols, with the number of the section where they appear first. This will help the reader to connect the different chapters.

Large deviation theory is a mixture of probability theory, convex analysis, variational calculus and set topology. As such it is mathematically both challenging and captivating. Even so, it is not obvious how to do justice to a vast area like large deviation theory in one hundred pages or so, especially when the goal is to cover both theory and applications. To focus ideas, I have restricted most of the exposition to random sequences, i.e., discrete-time random processes. This is a severe restriction indeed, but it makes the presentation much more user-friendly. The reader can expand his or her skills by turning to the monographs that are listed as references. Here a wealth of refinements and embellishments can be found, as well as beautiful and deep large deviation results for Brownian motion, random dynamical systems, Gibbs measures, interacting particle systems, Brownian motion among random obstacles, etc. These monographs also contain an extensive historical overview of the area.

The material presented here was taught as a graduate course at the Fields Institute for Research in Mathematical Sciences in Toronto, Canada, in the Fall of 1998, as part of the 1998-99 program on "Probability and its Applications". I am grateful to the staff of the Fields Institute for the hospitality I enjoyed as a visitor. I am grateful to the following colleagues for comments during the course: Rami Atar, Siva Athreya, Marek Biskup, Jürgen Gärtner, Takashi Hara, Remco van der Hofstad, Min Kang, Neal Madras, Anders Martin-Löf, George O'Brien, David Rolls, Tom Salisbury, Gordon Slade, Dean Slonowsky, Jan Swart and Stas Volkov.

I am indebted to Siva Athreya, Nina Gantert, Jürgen Gärtner, Andreas Greven, Remco van der Hofstad, Wolfgang König and George O'Brien for reading a draft of the manuscript and for suggesting improvements.

Special thanks go to Marek Biskup, who helped me to prepare the manuscript, both in terms of content and exposition, and who worked hard on the layout and on the figures. Marek was a constant companion in the enterprise of bringing the job to a good end.

Frank den Hollander
Nijmegen, October 1999

Contents