Introduction

Consider a Galois theoretical embedding problem, i.e., the question of embedding a Galois extension M/K with Galois group $G = \mathrm{Gal}(M/K)$ into a larger Galois extension F/K, such that the Galois group $\mathrm{Gal}(F/K)$ is isomorphic to a specified group E and the restriction map from $\mathrm{Gal}(F/K)$ to G corresponds to a given homomorphism $\pi\colon E \to G$. How do we approach such a problem? How do we determine whether such an extension F/K exists? How do we find it if it does? Or (preferably) how do we find all of them if there are any?

The answers to these questions of course depend on the kind of embedding problem considered, both with respect to the nature of the groups and the nature of the field. For instance, if G and E are cyclic, it takes nothing more than elementary Galois theory to solve the problem over a finite field, whereas it takes class field theory (or at least some reasonably sophisticated algebraic number theory) to solve it over the field of rational numbers.

The methods of algebraic number theory and class field theory have in fact been put to good use in studying these kinds of problems: In the 1920's, Scholz [76] considered various solvable groups (mostly of small order) over algebraic number fields, and in the 1930's Scholz [77] and Reichardt [70] independently proved that all finite groups of odd prime power order could be realised as Galois groups over any algebraic number field, in both cases by building up the extensions through solving embedding problems 'along' a composition series. This approach culminated in the 1950's with Shafarevich's result that all solvable groups are Galois groups over all algebraic number fields; cf. [32] or [66].

Also in the 1930's, Witt [94] considered groups of prime power order p^n over fields of characteristic p, and essentially proved all involved embedding problems to be trivially solvable. We will touch briefly on this in Chapter 2. In the same paper, Witt solved the problem of embedding a biquadratic extension into an extension with the quaternion group as Galois group. We will get considerable mileage out of that result in Chapter 7.

It is clear that an embedding problem is in a sense a 'local' problem: We should be able to investigate it fully using only the extension M/K and the homomorphism $\pi\colon E \to G$, without having to consider 'global' structures like separable closures and absolute Galois groups.[1]

In 1932, Brauer [7] introduced a type of embedding problem for which the necessary 'local' information is readily available: In these so-called *Brauer type embedding problems*, where the kernel of π can be identified with a group of roots of

[1] This is not to say that separable closures and absolute Galois groups are not useful for theoretical considerations – section 6.2 of Chapter 6 is a good example of this – but simply that they tend to be unwieldy, if only because we do not generally know them.

unity inside M,[2] this information is contained in the cohomology group $H^2(G, M^*)$, or the relative Brauer group $\mathrm{Br}(M/K)$, whichever one prefers. This is not only decidedly 'local', but quite convenient, since both group cohomology and Brauer group theory are well-developed disciplines, providing ample tools for studying embedding problems.

We should mention here that Brauer was not in fact interested in the embedding problems. He was – naturally enough – interested in the Brauer group, and his concern was the classification of finite-dimensional central division algebras over the field K, and the connection to embedding problems was a reduction, as he writes, from 'non-commutative' to 'commutative' algebra. We will go the other way, using the fact that the structure of Brauer groups is reasonably well understood.

This monograph, then, is about Brauer type embedding problems, primarily the case where $\ker\pi$ has prime order, and some related embedding problem types. This topic brings together Galois theory, Brauer group theory, group cohomology and the theory of quadratic forms, and all of these subjects are covered in the text, in Chapters 1, 3, 4 and 5. (Chapter 1 differs from most introductions to Galois theory by not containing very many examples; on the other hand, there are plenty of explicitly given Galois extensions in later chapters.) In addition, Appendix A gives an introduction to pro-finite Galois theory.

Chapter 2 provides the set-up, i.e., the basic results, the definitions and the first examples. All done in as elementary a fashion as possible.[3] Most importantly, it introduces the *obstruction* to the embedding problem. This is an element in $H^2(G, M^*)$ expressing the solvability (or non-solvability) of the embedding problem. The existence of obstructions is what makes Brauer type embedding problems nice.

Chapter 6 deals with the problem of decomposing the obstruction into convenient factors. Again, section 6.1 is fairly elementary, while section 6.2 relies on pro-finite cohomology. This (and the subsequent section 6.3) is, however, the only place in the monograph we make real use of pro-finite cohomology, and it can be skipped without giving it a second thought, should one so desire.

Chapter 7 explores the connection between Brauer type embedding problems and quadratic forms. Specifically, it provides criteria for solvability for a number of embedding problems in terms of equivalences of quadratic forms, and describes how to find the solutions. This includes the famous result by Witt: A bi-quadratic extension $K(\sqrt{a}, \sqrt{b})/K$ in characteristic $\neq 2$ can be embedded in a Q_8-extension, if and only if the quadratic forms $\langle a, b, ab\rangle$ and $\langle 1, 1, 1\rangle$ are equivalent (over K).

The final chapter, Chapter 8, is concerned with reducing embedding problems. As it happens, in some cases embedding problems that are not of Brauer type can be reduced to Brauer type embedding problems and solved as such, thus extending the usefulness of Brauer type embedding problems. In particular, if the embedding problem is non-split with kernel of prime order p (and the characteristic of the involved fields is not p), it can be so reduced.

Another kind of embedding problem that can be reduced to Brauer type is the case where $\ker\pi$ is cyclic of order 4. Such an embedding problem reduces to two Brauer type embedding problems, and the results covering this reduction are given in section 8.3 of Chapter 8.

[2] See section 2.4 in Chapter 2 for details.

[3] Or at least reasonable. It is perfectly possible to describe Q_8-extensions without resorting to Brauer groups, but it is hard to motivate the arguments.

There are of course other ways of looking at embedding problems; cf. [55] and [32], which also contains extensive references. Particularly noteworthy is the study of embedding problems over *Hilbertian* fields (as defined in Chapter 1), where it can be shown that split-exact embedding problems with abelian kernel are always solvable. (The proof, while a little tedious, is not particularly deep, and consists mostly of producing so-called *regular* extensions with prescribed abelian Galois group. We refer to [19].) Most of the groups we will consider can be realised easily as Galois groups over $\mathbb{Q}$ (or any Hilbertian field) by invoking this result. This realisation is generally not, however, very explicit.

Another common approach is to realise the finite group G as a Galois group over $\mathbb{C}(t)$ (as mentioned in Chapter 1, this is always possible) and then attempt to *descend* to $\mathbb{Q}(t)$, followed by specialisation to $\mathbb{Q}$. Most realisations of finite simple groups (sporadic groups, projective special linear groups, etc.) over $\mathbb{Q}$ have been obtained in this manner. Here, a good reference is Malle & Matzat [55].

A powerful method, which we will make some use of, is the more general application of group cohomology. We will use it as it relates to the Brauer group, but it can easily be employed independently. Here, an important paper is Hoechsmann's *Zum Einbettungsproblem* [30], covering the basics of this approach. Also, Neukirch [65] made great contributions here, using the cohomological description of class field theory to consider problems similar to those of Shafarevich.

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Finally, two other things should probably be mentioned here as well: The manuscript was written in AMS-LaTeX, and computer calculations relating to examples were performed using Maple V.

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[4] Although, unfortunately, his questions are not actually *answered* there.