Preface

This research monograph deals with the interaction between the theory of Coxeter groups on one hand and the relationships among several Hopf algebras of recent interest on the other hand. It is aimed at upper-level graduate students and researchers in these areas. The viewpoint is new and leads to a lot of simplification.

0.1 The first part: Chapters 1-3

The first part, aside from Chapter 2, consists of standard material. The first two chapters are related to Coxeter theory, while the third chapter is related to Hopf algebras. We hope that they will make the second part more accessible.

Chapter 1 provides an introduction to some standard Coxeter theory written in a language suitable for our purposes. The emphasis is on the gate property and the projection maps of Tits, which are crucial in almost everything that we do. The reader may be required to accept many facts on faith, since most proofs are omitted. This chapter is a prerequisite for Chapter 5.

Chapter 2 is completely self-contained. It begins with some standard material on left regular bands (LRBs). We then develop some new material on pointed faces, lunes and bilinear forms on LRBs, largely inspired by the descent theory of Coxeter groups (Chapter 5). We also introduce the concept of a projection poset which generalizes the concept of an LRB to take into account some nonassociative examples.

Chapter 3 provides a brief discussion on cofree coalgebras, the coradical filtration and the antipode, which are standard notions in the theory of Hopf algebras. We then briefly discuss three examples of Hopf algebras which have now become standard: namely, the Hopf algebras of symmetric functions Λ, noncommutative symmetric functions $\mathrm{N}\Lambda$ and quasi-symmetric functions $\mathrm{Q}\Lambda$.

0.2 The second part: Chapters 4-8

The second part consists of mostly original work. The well-prepared reader may start directly with this part and refer back to the first part as necessary. Chapter 4 provides a brief overview of this work, which is spread over the next four chapters. Chapter 5 is related to Coxeter theory, while Chapters 6, 7 and 8 are related to Hopf algebras. Each of them is kept as self-contained as possible; the reader may even read them as different papers. A more detailed overview is given in the introductory section of each of these four chapters. The results in the second part, which are stated without credit, are new to our knowledge.

0.3 Future work

At many points in this monograph we say, "This will be explained in a future work." We plan to write a follow-up to this monograph, where these issues will be taken up. Our main motivation is not merely to prove new results or reprove existing results but rather to show that these ideas have a promising future.

0.4 Acknowledgements

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0.5 Notation

$\mathbb{K}$ stands for a field of characteristic 0. For P a set, we write $\mathbb{K}P$ for the vector space over $\mathbb{K}$ with basis the elements of P and $\mathbb{K}P^*$ for its dual space. A word is written in italics if it is being defined at that place. While looking for a particular concept, the reader is advised to search both the notation and the subject index. The notation $[n]$ stands for the set $\{1, 2, \ldots, n\}$. The table below indicates the main letter conventions that we use.

subsets	S, T, U, V
compositions	α, β, γ
partitions	λ, μ, ρ
faces or set compositions	F, G, H, K, N, P, Q
chambers	C, D, E
pointed faces or fully nested set compositions	(F, D), (P, C)
flats or set partitions	X, Y
lunes or nested set partitions	L, M

We write Σ for the set of faces and $\mathcal{C}$ for the set of chambers. Otherwise we use roman script for the above sets. For example, Q is the set of pointed faces, and L is the set of flats. For the coalgebras and algebras constructed from such sets, we use the calligraphic script $\mathcal{M}$, $\mathcal{N}$ and so on. There are some inevitable conflicts of notation; however, the context should keep things clear. For example, we also use the above letters F, M, K, H and S to denote various bases, V for a vector space, H for a Hopf algebra and S for an antipode.