

## Notation

$S_\omega$	the projective connection attached to a symmetric bidifferential $\omega$ (see (1.41)),
$T_X$	the holomorphic (or regular) tangent bundle of a complex manifold (or smooth projective variety) $X$ ,
$\Theta_X$	the sheaf of holomorphic vector fields of a complex manifold or a smooth projective variety $X$ ,
$\Omega_X^k$	the sheaf of holomorphic $k$ -forms of a complex manifold or a smooth projective variety $X$ ,
$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c$ ,	
$\widehat{\mathfrak{g}}_N = \mathfrak{g} \otimes_{\mathbb{C}} \left( \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) \right) \oplus \mathbb{C} \cdot c$ ,	
$\Delta$	the root system of a complex simple Lie algebra and its Cartan subalgebra $(\mathfrak{g}, \mathfrak{h})$ ,
$\Delta_+$	the set of positive roots,
$\theta$	the longest root of a complex simple Lie algebra $\mathfrak{g}$ ,
$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ ,	
$(\ , \ )$	Cartan-Killing form of a complex simple Lie algebra $\mathfrak{g}$ normalized as $(\theta, \theta) = 2$ ,
$P_+$	the set of integral dominant weights of a complex simple Lie algebra $\mathfrak{g}$ ,
$P_\ell = \{\lambda \in P_+ \mid (\lambda, \theta) \leq \ell\}$ ,	
$\mu^\dagger = -w(\mu)$	for $\mu \in P_\ell$ where $w$ is the longest element of the Weyl group of a complex simple Lie algebra $\mathfrak{g}$ ,
$\kappa = g^* + \ell$	where $g^*$ is the dual Coxeter number of a complex simple Lie algebra $\mathfrak{g}$ , ( $\kappa = n + 1 + \ell$ in case $\mathfrak{g} = \mathfrak{sl}(n + 1, \mathbb{C})$ ),
$c_v = \frac{\ell \dim \mathfrak{g}}{\kappa}$ ,	
$\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2\kappa}$ ,	
$\widehat{\Delta}_v = \Delta_\lambda + \Delta_{\mu_1} - \Delta_{\mu_2}$ ,	
$\widehat{\Delta}_v = \Delta_\lambda + \Delta_{\mu_1} - \Delta_{\mu_2}$	for a vertex $\mathbf{v} = \begin{pmatrix} \lambda \\ \mu_1 \ \mu_2 \end{pmatrix}$ ,
$X_n = \mathbb{C}^n \setminus \cup_{i < j} \Delta_{ij}$ ,	$\Delta_{ij} = \{(z_n, \dots, z_1) \in \mathbb{C}^n \mid z_i = z_j\}$ ,
$\mathcal{R}_n = \{(z_n, \dots, z_1) \in \mathbb{C}^n \mid  z_n  >  z_{n-1}  > \dots >  z_1 \}$ ,	

## Riemann Surfaces and Stable Curves

### 1.1 Compact Riemann surfaces

**1.1.1 Compact Riemann surfaces.** A connected compact one-dimensional complex manifold is called a compact Riemann surface or a closed Riemann surface. In these Lecture Notes we shall mainly consider compact Riemann surfaces; we often omit “compact”.

A compact Riemann surface  $R$  is an oriented closed surface, so it is diffeomorphic to the surface of a doughnut with  $g$  holes. The number  $g$  is called the *genus* of the Riemann surface  $R$  and often denoted by  $g(R)$ .

For a compact Riemann surface  $R$  of genus  $g$  its first homology group  $H_1(R, \mathbb{Z})$  is a free  $\mathbb{Z}$ -module of rank  $2g$ . Moreover, the intersection pairing  $\alpha \cdot \beta$ ,  $\alpha, \beta \in H_1(R, \mathbb{Z})$  is skew-symmetric and a bilinear mapping

$$H_1(R, \mathbb{Z}) \times H_1(R, \mathbb{Z}) \rightarrow \mathbb{Z}.$$

We can always find a symplectic basis  $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$  of  $H_1(R, \mathbb{Z})$ , that is,

$$\alpha_i \cdot \alpha_j = 0, \quad \beta_i \cdot \beta_j = 0, \quad \alpha_i \cdot \beta_j = \delta_{ij}.$$

Moreover, there are  $g$  linearly independent holomorphic one-forms on  $R$ . Let  $\Theta_R$  be the sheaf of germs of holomorphic vector fields on  $R$  and  $\omega_R$  be its dual. The sheaf  $\omega_R$  is called the *canonical sheaf*. It is the sheaf associated with a complex line bundle  $K_X$  called the *canonical line bundle* of  $R$ . It is well-known that  $H^0(R, \mathcal{O}_R(L))$  gives a holomorphic embedding of  $R$  into a complex projective space  $\mathbb{P}^{2g-3}$  if the degree of the line bundle  $L$  on  $R$  is at least  $2g + 1$ . Hence all compact Riemann surfaces have the structure of a projective variety. In the following sometimes we regard a compact Riemann surface as a projective variety.

**1.1.2 Complex analytic families of Riemann surfaces.** In the following, for a complex manifold  $M$  and a point  $Q$  on  $M$ , by  $T_Q(M)$  we mean the holomorphic

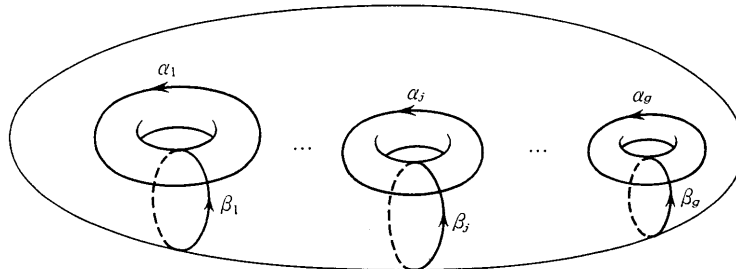


Figure 1.1 Riemann surface of genus  $g$ .

tangent vector space at the point  $Q$ . Also we assume that a complex manifold is connected, unless otherwise explicitly mentioned.

A holomorphic mapping  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  from an  $(m+1)$ -dimensional complex manifold  $\mathcal{C}$  to an  $m$ -dimensional complex manifold  $\mathcal{W}$  satisfying the following conditions is called a *smooth family of compact Riemann surfaces* or a *complex analytic family of compact Riemann surfaces* over the complex manifold  $\mathcal{W}$ .

- (1) The mapping  $\pi$  is proper. That is, for any compact set  $K$  of the complex manifold  $\mathcal{W}$ , the inverse image  $\pi^{-1}(K)$  is compact.
- (2) The mapping  $\pi$  is smooth and holomorphic. That is, for any point  $P \in \mathcal{C}$  the linear mapping  $(d\pi)_P: T_P(\mathcal{C}) \rightarrow T_{\pi(P)}(\mathcal{W})$  of the holomorphic tangent spaces is surjective.
- (3) For any point  $w \in \mathcal{W}$  the fibre  $\pi^{-1}(w)$  is connected.

By conditions (1) and (2) the fibre  $C_w = \pi^{-1}(w)$  over each point  $w \in \mathcal{W}$  is a compact Riemann surface. Also, for a point  $w_0 \in \mathcal{W}$  we call  $C_w$ ,  $w \in \mathcal{W}$ , a *deformation* of the compact Riemann surface  $C_{w_0}$ .

When a complex analytic family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  of compact Riemann surfaces and a holomorphic mapping  $h: \mathcal{S} \rightarrow \mathcal{W}$  are given, the fibre product  $\mathcal{C} \times_{\mathcal{W}} \mathcal{S} \rightarrow \mathcal{S}$  is a complex analytic family over  $\mathcal{S}$ . We call the family  $\mathcal{C} \times_{\mathcal{W}} \mathcal{S} \rightarrow \mathcal{S}$  the *pullback family* by the holomorphic mapping  $h$ .

**Example 1.1** On the product  $H \times \mathbb{C}$  of the upper half-plane  $H = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$  and the complex plane  $\mathbb{C}$ , the action of the group  $\mathbb{Z}^2$  is defined by

$$(m, n): \quad \begin{array}{ccc} H \times \mathbb{C} & \longrightarrow & H \times \mathbb{C}, \quad (m, n) \in \mathbb{Z}^2, \\ (\tau, \zeta) & \longmapsto & (\tau, \zeta + m + n\tau). \end{array}$$

Since the action is fixed point free and properly discontinuous, the quotient space  $\mathcal{E} = H \times \mathbb{C} / \mathbb{Z}^2$  has the structure of a two-dimensional complex manifold. The point  $(\tau, \zeta) \in H \times \mathbb{C}$  determines a point  $(\tau, [\zeta])$  of  $\mathcal{E}$ . The natural projection  $p: H \times \mathbb{C} \rightarrow H$  induces a holomorphic mapping

$$\pi: \quad \begin{array}{ccc} \mathcal{E} & \longrightarrow & H, \\ (\tau, [\zeta]) & \longmapsto & \tau. \end{array}$$

The holomorphic mapping  $\pi: \mathcal{E} \rightarrow H$  satisfies the above conditions (1), (2), (3), hence is a complex analytic family of compact Riemann surfaces over  $H$ . For a point  $\tau \in H$  the fibre  $E_\tau = \pi^{-1}(\tau)$  is an elliptic curve (a one-dimensional complex torus) with fundamental periods  $(1, \tau)$ .

**Example 1.2** The group  $SL(2, \mathbb{R})$  operates on the upper half-plane  $H$  by

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

For a positive integer  $n$  the normal subgroup  $\Gamma(n)$  of  $SL(2, \mathbb{Z})$  is defined by

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid \begin{array}{l} a \equiv d \equiv 1 \pmod{n} \\ b \equiv c \equiv 0 \pmod{n} \end{array} \right\}.$$

Then, it is known that for  $n \geq 3$  the action of the group  $\Gamma(n)$  has no fixed points on  $H$ . Moreover, the group  $\Gamma(n)$  acts on  $\mathcal{E}$  of Example 1.1 by

$$(\tau, [\zeta]) \longmapsto \left( \frac{a\tau + b}{c\tau + d}, \left[ \frac{\zeta}{c\tau + d} \right] \right).$$

The action is fixed point free. The holomorphic mapping  $\pi: \mathcal{E} \rightarrow H$  is compatible with the actions of  $\Gamma(n)$  on  $\mathcal{E}$  and  $H$ . Hence we have a holomorphic mapping of the quotient spaces  $\pi(n): \mathcal{E}(n) =: \mathcal{E}/\Gamma(n) \rightarrow C(n) := H/\Gamma(n)$ .

For  $n \geq 3$ ,  $\pi(n): \mathcal{E}(n) \rightarrow C(n)$  is also a complex analytic family of compact Riemann surfaces.

**1.1.3 Kodaira-Spencer mapping.** Let us study a complex analytic family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  of compact Riemann surfaces by using local coordinates. Choose a coordinate neighbourhood  $U$  of a point  $0 \in \mathcal{W}$  and local coordinates  $(w^1, \dots, w^m)$  of  $U$  with center 0. Let  $\pi^{-1}(U)$  be covered by open sets  $\{U_\lambda\}_{\lambda \in \Lambda} : \bigcup_{\lambda \in \Lambda} U_\lambda = \pi^{-1}(U)$ . Since  $\pi$  is a smooth holomorphic mapping, we can choose local coordinates of  $U_\lambda$  as  $(w^1, \dots, w^m, z_\lambda)$ . (That is,  $w^1, \dots, w^m$  is chosen as a part of the local coordinates of  $U_\lambda$  for all  $\lambda$ .) If  $U_\lambda \cap U_\mu \neq \emptyset$ , between the two local coordinates we have the following relation:

$$z_\lambda = f_{\lambda\mu}(z_\mu, w^1, \dots, w^m). \quad (1.1)$$

Here,  $f_{\lambda\mu}$  is holomorphic on  $U_\lambda \cap U_\mu$ . We may regard the complex manifold  $\pi^{-1}(U)$  as obtained by patching  $\{U_\lambda\}_{\lambda \in \Lambda}$  together by the relation (1.1). Then  $(w^1, \dots, w^m)$  are regarded as parameters of changing the patching. That is, for  $(w^1, \dots, w^m) = (0, \dots, 0)$  the manifold  $\pi^{-1}((0, \dots, 0))$  is a compact Riemann surface  $C_0$ , and it is obtained by patching by  $z_\lambda = f_{\lambda\mu}(z_\mu, 0, \dots, 0)$ . And for  $w = (w^1, \dots, w^m)$  we obtain  $C_w = \pi^{-1}(w)$  by changing the patching of  $C_0$  slightly by  $w$ . Hence, the degree one term of the Taylor expansion of (1.1) with respect to the variables  $w^1, \dots, w^m$  gives the first order approximation of deformations (usually called *infinitesimal deformation*) of the complex manifold  $C_0$ . This means that if  $U_\lambda \cap U_\mu \neq \emptyset$ , then

$$\left\{ \frac{\partial f_{\lambda\mu}}{\partial w^k}(z_\mu, 0, \dots, 0) \right\}, \quad 1 \leq k \leq m,$$

give information on the deformation of the compact Riemann surface  $C_0$ .

For  $U_\lambda \cap U_\mu \neq \emptyset$ , let us consider the holomorphic tangent vector field

$$\theta_{\lambda\mu}^{(k)} = \frac{\partial f_{\lambda\mu}}{\partial w^k}(z_\mu, 0, \dots, 0) \frac{\partial}{\partial z_\lambda} \quad (1.2)$$

on  $C_0 \cap U_\lambda \cap U_\mu$ . Note that in the above definition of the vector field  $\theta_{\lambda\mu}^{(k)}$  we use the vector field  $\frac{\partial}{\partial z_\lambda}$  on  $U_\lambda$ , but not  $\frac{\partial}{\partial z_\mu}$ . If we have  $U_\lambda \cap U_\mu \cap U_\nu \neq \emptyset$ , then by (1.1) we have

$$\begin{aligned} z_\lambda &= f_{\lambda\mu}(f_{\mu\nu}^1(z_\nu, w), w^1, \dots, w^m) \\ &= f_{\lambda\nu}(z_\nu, w^1, \dots, w^m). \end{aligned}$$

Hence, on  $U_\lambda \cap U_\mu \cap U_\nu$  we have

$$\begin{aligned}
\theta_{\lambda\nu}^{(k)} &= \frac{\partial f_{\lambda\nu}}{\partial w^k}(z_\nu, 0) \frac{\partial}{\partial z_\lambda} \\
&= \frac{\partial f_{\lambda\mu}}{\partial z_\mu}(f_{\mu\nu}(z_\nu, 0), 0) \cdot \frac{\partial f_{\mu\nu}}{\partial w^k}(z_\nu, 0) \frac{\partial}{\partial z_\lambda} \\
&\quad + \frac{\partial f_{\lambda\mu}}{\partial w^k}(f_{\mu\nu}(z_\nu, 0), 0) \frac{\partial}{\partial z_\lambda} \\
&= \frac{\partial f_{\mu\nu}}{\partial w^k}(z_\nu, 0) \left( \frac{\partial f_{\lambda\mu}}{\partial z_\mu} \frac{\partial}{\partial z_\lambda} \right) + \theta_{\lambda\mu}^{(k)} \\
&= \frac{\partial f_{\mu\nu}}{\partial w^k}(z_\nu, 0) \frac{\partial}{\partial z_\mu} + \theta_{\lambda\mu}^{(k)} \\
&= \theta_{\mu\nu}^{(k)} + \theta_{\lambda\mu}^{(k)}.
\end{aligned}$$

That is,  $\{\theta_{\lambda\mu}^{(k)}\}$  is a Čech one-cocycle with coefficients in holomorphic tangent vector fields. The cohomology class of  $H^1(C_0, \Theta)$  defined by the above one-cocycle is denoted by the same symbol  $\{\theta_{\lambda\mu}^{(k)}\}$ . Here, by  $\Theta$  we denote the sheaf of germs of holomorphic vector fields on  $C_0$ .

The cohomology class  $\{\theta_{\lambda\mu}^{(k)}\}$  is uniquely determined if we fix local coordinates  $(w^1, \dots, w^m)$ , and is independent of the choice of open covering  $\{U_\lambda\}_{\lambda \in \Lambda}$  of  $\pi^{-1}(U)$  by coordinate neighbourhoods and local coordinates  $(z_\lambda, w^1, \dots, w^m)$ . This is proved by direct calculations. Therefore, we can define a linear mapping from the holomorphic tangent vector space  $T_0\mathcal{W}$  at  $0 \in \mathcal{W}$  to  $H^1(C_0, \Theta)$  :

$$\begin{aligned}
\rho_0: \quad T_0\mathcal{W} &\longrightarrow H^1(C_0, \Theta), \\
\sum a_k \frac{\partial}{\partial w^k} &\longmapsto \sum a_k \{\theta_{\lambda\nu}^{(k)}\}.
\end{aligned} \tag{1.3}$$

This linear mapping is called the *Kodaira-Spencer mapping*.

By the way, in the definition of  $\{\theta_{\mu\nu}^{(k)}\}$  in (1.2), if, instead of putting  $w = 0$ , we put

$$\theta_{\lambda\mu}^{(k)}(w) = \frac{\partial f_{\lambda\mu}}{\partial w^k}(z_\mu, w) \frac{\partial}{\partial z_\lambda},$$

then by the same argument as above we can show that  $\{\theta_{\lambda\mu}^{(k)}(w)\} \in H^1(C_w, \Theta_{C_w})$ .

Moreover,  $\theta_{\lambda\mu}^{(k)}(w)$  is holomorphic in  $w$ . Hence, similarly to (1.3) we obtain the  $\mathcal{O}_{\mathcal{W}}$ -homomorphism

$$\begin{aligned}
\rho: \Theta_{\mathcal{W}} &\longrightarrow R^1\pi_*\Theta_{\mathcal{C}/\mathcal{W}}, \\
\sum a_k(w) \frac{\partial}{\partial w^k} &\longrightarrow \sum a_k(w) \{\theta_{\lambda\mu}^{(k)}(w)\};
\end{aligned} \tag{1.4}$$

here,  $\Theta_{\mathcal{C}/\mathcal{W}}$  is the sheaf of germs of relative holomorphic tangent vector fields of  $\pi: \mathcal{C} \rightarrow \mathcal{W}$ . The  $\mathcal{O}_{\mathcal{W}}$ -homomorphism  $\rho$  is also called the Kodaira-Spencer mapping.

Since the complex analytic family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  is a smooth holomorphic mapping, there exists an exact sequence of  $\mathcal{O}_{\mathcal{C}}$ -homomorphisms

$$0 \longrightarrow \Theta_{\mathcal{C}/\mathcal{W}} \longrightarrow \Theta_{\mathcal{C}} \longrightarrow \pi^*\Theta_{\mathcal{W}} \longrightarrow 0. \tag{1.5}$$

From the exact sequence we obtain a long exact sequence

$$\longrightarrow \pi_*\Theta_{\mathcal{C}} \longrightarrow \pi_*\pi^*\Theta_{\mathcal{W}} = \Theta_{\mathcal{W}} \xrightarrow{\tau} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{W}} \longrightarrow R^1\pi_*\Theta_{\mathcal{C}}.$$

It is easily seen that by the definition of  $\rho$  the Kodaira-Spencer mapping (1.4) is nothing but the  $\mathcal{O}_W$ -homomorphism  $\Theta_W \xrightarrow{\tau} R^1\pi_*\Theta_{C/W}$  of the long exact sequence.

**Remark 1.3** Similarly we can define a complex analytic family of compact complex manifolds and the Kodaira-Spencer mapping.

Let us calculate the Kodaira-Spencer mapping of the family of elliptic curves constructed in Example 1.2.

**Example 1.4** Let us first consider an elliptic curve  $E = \mathbb{C}/(1, \tau_0)$   $\tau_0 \in H$ . Let  $z$  be a global coordinate of the affine space  $\mathbb{C}$  and  $o$  the origin of the elliptic curve  $E$ . From the exact sequence

$$0 \rightarrow \Theta_E \rightarrow \Theta_E(*o) \xrightarrow{t} z^{-1}\mathbb{C}[z^{-1}] \frac{d}{dz} \rightarrow 0$$

we have the exact sequence

$$0 \rightarrow H^0(E, \Theta_E(*o)) \xrightarrow{t} z^{-1}\mathbb{C}[z^{-1}] \frac{d}{dz} \xrightarrow{\theta} H^1(E, \Theta_E) \rightarrow 0$$

where we put

$$\begin{aligned} \Theta_E(*o) &= \varinjlim_{m \rightarrow \infty} \Theta(mo) \\ H^0(E, \Theta_E(*o)) &= \varinjlim_{m \rightarrow \infty} H^0(E, \Theta(mo)). \end{aligned}$$

Note that for the Weierstrass  $\wp$ -function, we have

$$\frac{(-1)^n}{n!} \wp(z) \frac{d}{dz} \in H^0(E, \mathcal{O}_E((n+2)o)) \setminus H^0(E, \mathcal{O}_E((n+1)o))$$

for all non-negative integers  $n$ . Hence,  $z^{-1} \frac{d}{dz}$  has non-zero image  $\theta(z^{-1} \frac{d}{dz})$  in  $H^1(E, \Theta_E)$ . Let us describe the image. Let  $\{\mathcal{U}_\lambda\}_{\lambda=0}^N$  be a small open covering of  $E$  such that  $\mathcal{U}_0$  is a coordinate neighbourhood of  $o$  with local coordinate  $z$  and such that

$$o \notin \mathcal{U}_0 \cap \mathcal{U}_\mu, \quad \mu \geq 1.$$

Then the image of  $z^{-1} \frac{d}{dz}$  is given by the cocycle

$$\tau_{\lambda\mu} = \begin{cases} z^{-1} \frac{d}{dz}, & \lambda = 0, \quad \mu \geq 1, \quad \mathcal{U}_{\lambda\mu} \neq \emptyset \\ -z^{-1} \frac{d}{dz}, & \lambda \geq 1, \quad \mu = 0, \quad \mathcal{U}_{\lambda\mu} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Let  $p : \mathcal{E} \rightarrow H$  be the versal family constructed in Example 1.2. As we put  $E = \mathbb{C}/(1, \tau_0)$ , by using the above covering, we find an open covering  $\{\widehat{\mathcal{U}}_\lambda\}_{\lambda=0}^N$  of  $p^{-1}(V)$ , where  $V$  is an open neighbourhood of  $\tau_0$  such that  $p^{-1}(\tau_0) \cap \widehat{\mathcal{U}}_\lambda = \mathcal{U}_\lambda$ . Let  $(\tau_\lambda, z_\lambda)$  be local coordinates of  $\widehat{\mathcal{U}}_\lambda$  such that

$$\begin{aligned} \tau_\lambda &= \tau - \tau_0 \\ z_0 &= z \\ z_\lambda &= z_\mu + m_{\lambda\mu} + n_{\lambda\mu}\tau \end{aligned}$$

with  $m_{\lambda\mu}, n_{\lambda\mu} \in \mathbb{Q}$ . Then, the Kodaira-Spencer class  $\rho_{\tau_0}(\frac{\partial}{\partial \tau}) \in H^1(E, \Theta_E)$  is given by the cocycle

$$\theta_{\lambda\mu} = \{n_{\lambda\mu} \frac{d}{dz_\lambda}\}.$$

Let us consider the Dolbeault representations of the cocycles  $\{\tau_{\lambda\mu}\}$  and  $\{\theta_{\lambda\mu}\}$ .

First consider the cocycle  $\{\theta_{\lambda\mu}\}$ . Put

$$\psi_\lambda = \frac{(\bar{z}_\lambda - z_\lambda)}{2\sqrt{-1}\text{Im } \tau} \frac{d}{dz_\lambda}.$$

Then we have

$$\psi_\mu - \psi_\lambda = n_{\lambda\mu} \frac{d}{dz_\lambda} = \theta_{\lambda\mu}.$$

Therefore

$$\bar{\partial}\psi_\mu = \bar{\partial}\psi_\lambda = (2\sqrt{-1}\text{Im } \tau)^{-1} \frac{d}{dz} d\bar{z}$$

is the corresponding Dolbeault representative of  $\{\theta_{\lambda\mu}\}$ .

Next consider the cohomology class  $\{\tau_{\lambda\mu}\}$ . We may assume that

$$\{z \in \mathbb{C} \mid |z| \leq \epsilon\} \subset \mathcal{U}_0$$

and

$$\{z \in \mathbb{C} \mid |z| \leq \epsilon/2\} \subset \mathcal{U}_0 \setminus \bigcup_{\lambda=1}^N \mathcal{U}_\lambda.$$

Let  $f(z)$  be a real-valued  $C^\infty$  function in  $\mathcal{U}_0$  such that  $0 \leq f(z) \leq 1$  and

$$f(z) = \begin{cases} 0, & |z| < \epsilon/3 \\ 1, & |z| \geq \epsilon/2. \end{cases} \quad (1.6)$$

Put

$$\tilde{\tau}_\mu = \begin{cases} \frac{f(z)}{z} \frac{d}{dz} & \mu = 0 \\ 0 & \mu \geq 1. \end{cases}.$$

Then we have

$$\tau_{\lambda\mu} = \tilde{\tau}_\mu - \tilde{\tau}_\lambda$$

and

$$\omega = \bar{\partial}\tilde{\tau}_\lambda = \frac{1}{z} \left( \frac{\partial f(z)}{\partial \bar{z}} \right) \frac{d}{dz} d\bar{z}$$

represents the Dolbeault cohomology class of  $\{\tau_{\lambda\mu}\}$ . Since the Dolbeault cohomology class has a representative of the form

$$a \frac{d}{dz} d\bar{z}, \quad a \in \mathbb{C},$$

we will find the constant  $a$  for  $\omega$  defined above. Note that for any  $C^\infty$  function  $h$  on  $E$  we have

$$\int_E (\omega + \bar{\partial}h) \wedge dz = \int_E \omega \wedge dz.$$

We have

$$\begin{aligned} \int_{E-\{|z|<\epsilon\}} \frac{\partial f(z)}{\partial \bar{z}} \frac{1}{z} d\bar{z} \wedge dz &= \int_{E-\{|z|<\epsilon\}} \bar{\partial} \left( \frac{f(z)}{z} dz \right) \\ &= \int_{E-\{|z|<\epsilon\}} d \left( \frac{f(z)}{z} dz \right) \\ &= - \int_{|z|=\epsilon} \frac{1}{z} dz = -2\pi\sqrt{-1} \end{aligned}$$

by (1.6). On the other hand, we have

$$\int_E a d\bar{z} \wedge dz = 2\sqrt{-1} a \operatorname{Im} \tau.$$

Therefore we conclude that  $\{\tau_{\lambda\mu}\}$  and  $-\frac{\pi}{\operatorname{Im} \tau} \frac{d}{dz} d\bar{z}$  represent the same cohomology class. Thus we obtain the equality

$$\theta \left( z^{-1} \frac{d}{dz} \right) = -2\pi\sqrt{-1} \rho \left( \frac{\partial}{\partial \tau} \right).$$

Next let us consider another coordinate. Put

$$w = \exp(2\pi\sqrt{-1}z), \quad q_0 = \exp(2\pi\sqrt{-1}\tau_0).$$

Then the elliptic curve  $E$  can be regarded as the quotient manifold  $\mathbb{C}^*/\langle q_0 \rangle$ , where  $\langle q_0 \rangle$  means the infinite cyclic group acting on  $\mathbb{C}^*$  generated by the analytic automorphism

$$w \mapsto q_0 w.$$

In this coordinate, the origin of the elliptic curve corresponds to the point  $w = 1$ . Put

$$u = w - 1.$$

Then we have the exact sequence

$$0 \rightarrow \Theta_E \rightarrow \Theta_E(*o) \xrightarrow{\eta} u^{-1}\mathbb{C}[u^{-1}] \frac{d}{du} \rightarrow 0$$

and from this exact sequence we have the exact sequence

$$0 \rightarrow H^0(E, \Theta_E(*o)) \xrightarrow{\iota} u^{-1}\mathbb{C}[u^{-1}] \frac{d}{du} \xrightarrow{\theta} H^1(E, \Theta_E) \rightarrow 0.$$

The image  $\theta(u^{-1} \frac{d}{du})$  is calculated as follows. The image is defined by the cocycle

$$\omega_{\lambda\mu} = \begin{cases} u^{-1} \frac{d}{du} & \text{if } \lambda = 0, \mu \geq 1 \quad \mathcal{U}_{0\mu} \neq 0 \\ -u^{-1} \frac{d}{du} & \text{if } \lambda \geq 1, \mu = 0 \quad \mathcal{U}_{\lambda 0} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Since we have

$$\begin{aligned} u^{-1} \frac{d}{du} &= \frac{1}{(\exp(2\pi\sqrt{-1}z) - 1)2\pi\sqrt{-1} \exp(2\pi\sqrt{-1}z)} \frac{d}{dz} \\ &= \left( \frac{-1}{4\pi^2 z} + \text{holomorphic} \right) \frac{d}{dz} \end{aligned}$$

we conclude

$$\{\omega_{\lambda\mu}\} = -\frac{1}{4\pi^2} \{\tau_{\lambda\tau}\},$$



that is,

$$\theta(u^{-1} \frac{d}{dz}) = -\frac{1}{4\pi^2} \theta(z^{-1} \frac{d}{dz}) = -\frac{1}{2\pi\sqrt{-1}} \rho(\frac{d}{d\tau}).$$

In the following we need to consider degeneration of families of elliptic curves. In that case we shall use the parameter

$$q = \exp(2\pi\sqrt{-1}\tau).$$

Then we have

$$\begin{aligned} q \frac{d}{dq} &= \frac{1}{2\pi\sqrt{-1}} \frac{d}{d\tau} \\ \theta(u^{-1} \frac{d}{du}) &= -\theta(q \frac{d}{dq}). \end{aligned}$$

This formula will be used later.

**1.1.4 Kuranishi family.** Let us introduce the following definition, to help us understand another aspect of the Kodaira-Spencer mapping.

**Definition 1.5** If a complex analytic family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  of compact Riemann surfaces and the point  $0 \in \mathcal{W}$ ,  $C = \pi^{-1}(0)$ , satisfy the following condition (\*), the family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  is said to be *complete* at the point 0.

- (\*) If a complex analytic family  $\varpi: \mathcal{M} \rightarrow \mathcal{N}$  of compact Riemann surfaces, a point  $s \in \mathcal{N}$  and an analytic isomorphism  $f_0: M = \varpi^{-1}(s) \rightarrow C$  are given, then there exist a holomorphic mapping  $g$  from a neighbourhood  $U$  of the point  $s$  to  $\mathcal{W}$  and a holomorphic mapping  $f: \varpi^{-1}(U) \rightarrow \mathcal{C}$  such that they satisfy the following conditions:

$$\begin{aligned} g(s) &= 0, \\ f|_{\varpi^{-1}(s)} &= f_0, \end{aligned}$$

and such that, for any point  $u \in U$ ,  $f|_{\varpi^{-1}(u)}$  is an analytic isomorphism from  $\varpi^{-1}(u)$  to  $\pi^{-1}(g(u))$  with commutative diagram

$$\begin{array}{ccc} \varpi^{-1}(U) & \xrightarrow{f} & \mathcal{C} \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & \mathcal{W} \end{array} \quad (1.7)$$

In this case the family  $\varpi^{-1}(U) \rightarrow U$  is complex analytically isomorphic to the pullback of the family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  by the holomorphic mapping  $g$ . Moreover, for all  $g$  satisfying the above condition, if  $(dg)_s$  is uniquely determined, the family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  is said to be *versal* at the point 0. Moreover, if  $g$  is uniquely determined, the family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  is said to be *universal* at 0.

If an analytic family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  is semi-universal but not universal at the point 0, then the analytic automorphism group of the compact Riemann surface  $C = \pi^{-1}(0)$  gives an obstruction to being universal. The following theorem is important for applications.

**Theorem 1.6** *If a complex analytic family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  of compact Riemann surfaces of genus  $g \geq 2$  is complete at every point of a neighbourhood of a point  $0 \in \mathcal{W}$  and versal at 0, then the family  $\pi$  is universal at the point 0.*

The relationship between the Kodaira-Spencer mapping and a complete family or versal family is given in the following theorems.

**Theorem 1.7** *If the Kodaira-Spencer mapping of a complex analytic family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  is surjective at a point  $w \in \mathcal{W}$ , then the family is complete at the point  $w$ .*

**Theorem 1.8** *A complex analytic family  $\pi: \mathcal{C} \rightarrow \mathcal{W}$  is versal at a point  $0 \in \mathcal{W}$  if and only if the family  $\pi$  is complete at every point of a neighbourhood of the point  $0$ , and the Kodaira-Spencer mapping  $\rho_0: T_0\mathcal{W} \rightarrow H^1(\pi^{-1}(0), \Theta)$  at the point  $0$  is an isomorphism.*

**1.1.5 Period mappings.** Let  $R$  be a compact Riemann surface of genus  $g$  and  $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$  be a symplectic basis of  $H_1(R, \mathbb{Z})$ . A period matrix  $\tau$  of the Riemann surface  $R$  is given by

$$\tau := \left( \int_{\beta_i} \omega_j \right) \cdot \left( \int_{\alpha_i} \omega_j \right)^{-1} \quad (1.8)$$

where  $\{\omega_1, \dots, \omega_g\}$  is a basis of holomorphic one-forms on  $R$ . Note that the period matrix  $\tau$  is independent of the choice of the basis of holomorphic one-forms but depends on the choice of symplectic basis. Let  $\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_g, \tilde{\beta}_1, \dots, \tilde{\beta}_g\}$  be another symplectic basis of  $H_1(R, \mathbb{Z})$ . Then we have

$$\begin{pmatrix} \tilde{\beta}_1 \\ \vdots \\ \tilde{\beta}_g \\ \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_g \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \\ \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix}$$

where the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is in the group  $Sp(g, \mathbb{Z})$  of symplectic integral matrices defined by

$$Sp(g, \mathbb{Z}) := \left\{ X \in M_{2g}(\mathbb{Z}) \mid {}^t X \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} X = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \right\}.$$

Let  $\tilde{\tau}$  be the period matrix of the Riemann surface  $R$  defined by the symplectic basis  $\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_g, \tilde{\beta}_1, \dots, \tilde{\beta}_g\}$ . Then we have

$$\tilde{\tau} = (A\tau + B)(C\tau + D)^{-1}.$$

A basis  $\{\omega_1, \dots, \omega_g\}$  of holomorphic one-forms of  $R$  is called a *normalized basis* with respect to a symplectic basis  $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$  of  $H_1(R, \mathbb{Z})$ , if we have

$$\int_{\alpha_i} \omega_j = \delta_{ij}, \quad \int_{\beta_i} \omega_j = \tau_{ij}, \quad \tau = (\tau_{ij}). \quad (1.9)$$

Let  $\pi: \mathcal{R} \rightarrow \mathbf{T}$  be the universal family of compact Riemann surfaces of genus  $g$  over the Teichmüller space  $\mathbf{T}$ . For each point  $t \in \mathbf{T}$ , put  $R_t = \pi^{-1}(t)$ . There is a canonical isomorphism

$$H^1(R_t, \Theta_{R_t}) \simeq T_t \mathbf{T} \quad (1.10)$$

which is nothing but the one induced by the Kodaira-Spencer mapping. Taking the dual there is a canonical isomorphism

$$T_t \mathbf{T}^* \simeq H^0(R_t, \omega_{R_t}^{\otimes 2}). \quad (1.11)$$

In the following we often identify the two sides of (1.11). Then, since  $\mathbf{T}$  is simply connected, the sheaf  $R_1 \pi_* \mathbb{Z}$  is trivial on  $\mathbf{T}$  and we find global sections

$$\{\widehat{\alpha}_1, \dots, \widehat{\alpha}_g, \widehat{\beta}_1, \dots, \widehat{\beta}_g\}$$

of the sheaf  $R_1 \pi_* \mathbb{Z}$  such that for each point  $t \in \mathbf{T}$ ,

$$\{\widehat{\alpha}_1(t), \dots, \widehat{\alpha}_g(t), \widehat{\beta}_1(t), \dots, \widehat{\beta}_g(t)\}$$

is a symplectic basis of  $H_1(R_t, \mathbb{Z})$ . The sheaf  $\pi_* \omega_{\mathcal{R}/\mathbf{T}}$  is a free  $\mathcal{O}_{\mathbf{T}}$ -module of rank  $g$ . Let  $\{\widehat{\omega}_1, \dots, \widehat{\omega}_g\}$  be an  $\mathcal{O}_{\mathbf{T}}$ -free basis of  $\pi_* \omega_{\mathcal{R}/\mathbf{T}}$ . Then the matrix

$$\Pi(t) := \left( \int_{\widehat{\beta}_i(t)} \widehat{\omega}_j(t) \right) \cdot \left( \int_{\widehat{\alpha}_i(t)} \widehat{\omega}_j(t) \right)^{-1} \quad (1.12)$$

defines a holomorphic mapping

$$\Pi : \mathbf{T} \rightarrow \mathfrak{S}_g \quad (1.13)$$

from the Teichmüller space to the Siegel upper half-plane of degree  $g$ . Moreover, there exists a group homomorphism

$$\Phi : \text{Mod}_g \rightarrow Sp(g, \mathbb{Z}) \quad (1.14)$$

of the mapping class group  $\text{Mod}_g$  of genus  $g$  to the symplectic group such that

$$\Pi(\gamma t) = (A_\gamma \Pi(t) + B_\gamma)(C_\gamma \Pi(t) + D_\gamma)^{-1}. \quad (1.15)$$

where

$$\Phi(\gamma) = \begin{pmatrix} A_\gamma & B_\gamma \\ B_\gamma & D_\gamma \end{pmatrix}. \quad (1.16)$$

**1.1.6 Jacobian variety of a Riemann surface.** For a compact Riemann surface  $R$  let  $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$  be a symplectic basis and  $\{\omega_1, \dots, \omega_g\}$  be a normalized basis of holomorphic one-forms. In the following

$$(R, \{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\})$$

is called a *weakly marked Riemann surface*. Let  $\tau \in \mathfrak{S}_g$  be the corresponding period matrix. We let  $J(R)$  be a  $g$ -dimensional complex torus with period matrix  $(I_g, \tau)$ . Then, we can define a holomorphic mapping  $j : R \rightarrow J(R)$  by

$$j(Q) := \left( \int_{P_0}^Q \omega_1, \dots, \int_{P_0}^Q \omega_g \right) \quad (1.17)$$

where the base point  $P_0 \in R$  is fixed once and for all. If we change the base point, the mapping  $j$  differs only by a translation of the complex torus  $J(R)$ . The mapping  $j$  is called the *Albanese* mapping and the complex torus  $J(R)$  is usually called the Jacobian variety of the Riemann surface  $R$ .

The Jacobian variety  $J(R)$  may be defined intrinsically by

$$J(R) = H^0(R, \omega_R)^* / H_1(R, \mathbb{Z}). \quad (1.18)$$

Precisely speaking, this is the definition of the Albanese variety of the Riemann surface  $R$ . Since we have

$$\int_{\alpha_i} \omega_j = \delta_{ij}$$

for a normalized basis  $\{\omega_1, \dots, \omega_g\}$  of  $H^0(R, \omega_R)$  we may regard  $\{\alpha_1, \dots, \alpha_g\}$  as the corresponding dual basis of  $H^0(R, \omega_R)^*$ . Then the equality

$$\int_{\beta_i} \omega_j = \tau_{ij}$$

implies that

$$\beta_i = \sum_{j=1}^g \tau_{ij} \alpha_j.$$

Therefore, we can express  $J(R)$  as a complex torus  $\mathbb{C}^g / (I_g, \tau)$  defined above.

The Picard variety  $Pic^0(R)$  of the Riemann surface  $R$  is defined by

$$Pic^0(R) := H^1(R, \mathcal{O}_R) / H^1(R, \mathbb{Z}). \quad (1.19)$$

Choose again a normalized basis  $\{\omega_1, \dots, \omega_g\}$  of  $H^0(R, \omega_R)$ . Then  $\{\bar{\omega}_1, \dots, \bar{\omega}_g\}$  is a basis of  $H^1(R, \mathcal{O}_R)$ . By the Hodge decomposition theorem, we have an isomorphism

$$H^1(R, \mathbb{C}) \simeq H^1(R, \mathcal{O}_R) \oplus H^0(R, \omega_R).$$

Let  $\{\alpha_1^*, \dots, \alpha_g^*, \beta_1^*, \dots, \beta_g^*\}$  be the basis of the cohomology group  $H^1(R, \mathbb{Z})$  dual to the basis  $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$  of  $H_1(R, \mathbb{Z})$ . Since we may regard  $H^1(R, \mathbb{Z})$  as a lattice of  $H^1(R, \mathbb{C})$ , by the above isomorphism we may express

$$\begin{aligned} \alpha_i^* &= \sum_{j=1}^g a_{ij} \bar{\omega}_j + \sum_{j=1}^g \bar{a}_{ij} \omega_j \\ \beta_i^* &= \sum_{j=1}^g b_{ij} \bar{\omega}_j + \sum_{j=1}^g \bar{b}_{ij} \omega_j. \end{aligned}$$

Put

$$A = (a_{ij}) \quad B = (b_{ij}).$$

By the equalities

$$\begin{aligned} \delta_{ik} = \langle \alpha_k, \alpha_i^* \rangle &= \sum_{j=1}^g a_{ij} \int_{\alpha_k} \bar{\omega}_j + \sum_{j=1}^g \bar{a}_{ij} \int_{\alpha_k} \omega_j \\ &= a_{ik} + \bar{a}_{ik} \\ 0 = \langle \beta_k, \alpha_i^* \rangle &= \sum_{j=1}^g a_{ij} \int_{\beta_k} \bar{\omega}_j + \sum_{j=1}^g \bar{a}_{ij} \int_{\beta_k} \omega_j \\ &= \sum_{j=1}^g a_{ij} \bar{\tau}_{jk} + \sum_{j=1}^g \bar{a}_{ij} \tau_{jk}. \end{aligned}$$

and the similar equalities for  $\beta_i^*$ , we have

$$\begin{pmatrix} A & \bar{A} \\ B & \bar{B} \end{pmatrix} \begin{pmatrix} I_g & \bar{\tau} \\ I_g & \tau \end{pmatrix} = I_{2g}$$

Put

$$Y = (-2\pi\sqrt{-1}\text{Im } \tau)^{-1}. \quad (1.20)$$

Note that  $Y$  is a symmetric matrix. We have

$$\begin{pmatrix} I_g & \bar{\tau} \\ I_g & \tau \end{pmatrix}^{-1} = \begin{pmatrix} -\tau Y & -\bar{\tau} Y \\ Y & Y \end{pmatrix}.$$

Hence the lattice induced by  $H^1(R, \mathbb{Z})$  in  $H^1(R, \mathcal{O}_R)$  is given by the rows of the matrix

$$\begin{pmatrix} -\tau Y \\ Y \end{pmatrix}$$

where we use  $\{\bar{\omega}_1, \dots, \bar{\omega}_g\}$  as a basis of  $H^1(R, \mathcal{O}_R)$ . Hence, we may express  $Pic^0(R)$  as a complex torus  $\mathbb{C}^g / (Y\tau, Y)$ . Thus, there is an isomorphism

$$\lambda : J(R) \rightarrow Pic^0(R)$$

given by

$$\lambda \begin{bmatrix} z_1 \\ \vdots \\ z_g \end{bmatrix} = \begin{bmatrix} Y \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} \end{bmatrix}. \quad (1.21)$$

Note that the isomorphism  $\lambda$  is induced from a  $\mathbb{C}$ -linear isomorphism

$$\begin{aligned} H^0(R, \omega_R)^* &\rightarrow H^1(R, \mathcal{O}_R) \\ \sum a_i \alpha_i &\mapsto \sum_{ij} (y_{ij} a_j) \bar{\omega}_i \end{aligned} \quad (1.22)$$

where  $(y_{ij}) = Y$ .

The Albanese and Picard varieties of a compact Riemann surface are dual to each other and they are principally polarized abelian varieties. The above isomorphism  $\lambda : J(R) \rightarrow Pic^0(R)$  is a canonical one. The sheaf  $\Theta_{J(R)}$  of holomorphic vector fields on the Jacobian variety is trivial and we have a canonical isomorphism

$$\Theta_{J(R)} \simeq T_0 J(R) \otimes \mathcal{O}_{J(R)}$$

Moreover, the Albanese mapping  $j : R \rightarrow J(R)$  induces an isomorphism

$$H^1(J(R), \mathcal{O}) \simeq H^1(R, \mathcal{O})$$

by  $d\bar{z}_j \mapsto \bar{\omega}_j$ . Therefore, we have a canonical isomorphism

$$\iota : H^1(J(R), \Theta_{J(R)}) \simeq T_0 J(R) \otimes H^1(R, \mathcal{O}_R).$$

where  $T_0 J(R)$  is the tangent space of  $J(R)$  at the origin. By the above isomorphism  $\lambda$ , we may identify  $T_0 J(R)$  with  $H^1(R, \mathcal{O}_R)$ . Hence, we may rewrite

$$\iota : H^1(J(R), \Theta_{J(R)}) \simeq H^1(R, \mathcal{O}_R) \otimes H^1(R, \mathcal{O}_R). \quad (1.23)$$

Since  $T_0 J(R)$  is identified with  $H^0(R, \omega_R)^*$  by (1.19), by (1.8)) the mapping  $\iota$  is written explicitly as

$$\iota \left( \frac{\partial}{\partial z_i} d\bar{z}_j \right) = \sum_{k=1}^g y_{ik} \bar{\omega}_k \otimes \bar{\omega}_j \quad (1.24)$$

where the matrix  $(y_{ij}) = Y$  is defined in (1.20). Put

$$\mathfrak{T} := \{T = (t_{ij}) \mid T \text{ is } g \times g \text{ complex matrix with } \det \begin{pmatrix} I_g & T \\ I_g & \bar{T} \end{pmatrix} \neq 0\}$$

and we let  $\mathbb{Z}^{2g}$  act on  $\mathfrak{T} \times \mathbb{C}^g$  by

$$\underline{n} : (T, z) \rightarrow (T, z + \underline{n}_1 + \underline{n}_2 T), \quad \underline{n} = (\underline{n}_1, \underline{n}_2), \quad \underline{n}_i \in \mathbb{Z}^g.$$

The quotient space  $\pi : \mathcal{T} = \mathfrak{T} \times \mathbb{C}^g / \mathbb{Z}^{2g} \rightarrow \mathfrak{T}$  is a versal family of complex tori. Let  $\mu : \mathcal{A} \rightarrow \mathfrak{S}_g$  be the family of abelian varieties over the Siegel upper half-plane such that the fibre  $A_\tau = \mu^{-1}(\tau)$  is the complex torus with period matrix  $(I_g, \tau)$ . There

is a natural embedding  $i : \mathfrak{S}_g \hookrightarrow \mathfrak{T}$  and the induced family  $i^*\mathcal{T} \rightarrow \mathfrak{S}_g$  is isomorphic to the family  $\mu : \mathcal{A} \rightarrow \mathfrak{S}_g$ . We call the mapping

$$i \circ \Pi : \mathbf{T} \rightarrow \mathfrak{T}$$

also the period mapping. At a point  $\tau \in \mathfrak{S}_g \subset \mathfrak{T}$  corresponding to the Riemann surface  $R$  with symplectic basis  $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ , by the Kodaira-Spencer mapping we may identify the tangent space  $T_\tau \mathfrak{T}$  of  $\mathfrak{T}$  at  $\tau$  with  $H^1(J(R), \Theta_{J(R)})$ . Hence, the period mapping  $i \circ \Pi$  induces a natural mapping

$$H^1(R, \Theta_R) \rightarrow H^1(J(R), \Theta_{J(R)}) = H^1(R, \mathcal{O}_R) \otimes H^1(R, \mathcal{O}_R)$$

where we use the isomorphism  $\iota$  (1.22). The dual of this mapping is given by the natural multiplication map

$$\begin{aligned} H^0(R, \omega_R) \otimes H^0(R, \omega_R) &\rightarrow H^0(R, \omega_R^{\otimes 2}) \\ \nu \otimes \mu &\mapsto \nu\mu. \end{aligned} \quad (1.25)$$

Let us calculate the Kodaira-Spencer class of a tangent vector  $\frac{\partial}{\partial t_{ij}}$  at a point  $\tau \in \mathfrak{S}_g \subset \mathfrak{T}$ . Let  $\mathcal{U}$  be an open neighbourhood of the point  $\tau_0$  and let  $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$  be a small open covering of  $\mathcal{U}$ . We choose local coordinates  $(t_{ij}, z_\lambda^1, \dots, z_\lambda^g)$ ,  $1 \leq i \leq j \leq g$  in such a way that  $(z_\lambda^1, \dots, z_\lambda^g)$  are deduced from the global affine coordinates  $(z_1, \dots, z_g)$  of the vector space  $\mathbb{C}^g$ . Thus, if  $\mathcal{U}_\lambda \cap \mathcal{U}_\mu \neq \emptyset$ , then we have

$$z_\lambda^i - z_\mu^i = m_{\lambda\mu}^i + \sum_{k=1}^g n_{\lambda\mu}^k t_{ik} \quad (1.26)$$

where  $m_{\lambda\mu}^i, n_{\lambda\mu}^i \in \mathbb{R}^g$ ,  $i = 1, 2, \dots, g$ . Therefore, the Kodaira-Spencer class  $\rho\left(\frac{\partial}{\partial t_{ij}}\right)$  is given by the cocycle  $\theta_{\lambda\mu}$  defined by

$$\theta_{\lambda\mu} = n_{\lambda\mu}^j \frac{\partial}{\partial z_\lambda^i}. \quad (1.27)$$

Now put

$$\begin{pmatrix} \psi_\lambda^1 \\ \vdots \\ \psi_\lambda^g \end{pmatrix} = Y \begin{pmatrix} \bar{z}_\lambda^1 - z_\lambda^1 \\ \vdots \\ \bar{z}_\lambda^g - z_\lambda^g \end{pmatrix}.$$

where  $Y = (-2\sqrt{-1}\text{Im}(t_{ij}))^{-1}$ . Then we have

$$\psi_\mu^j \frac{\partial}{\partial z_\mu^i} - \psi_\lambda^j \frac{\partial}{\partial z_\lambda^i} = \theta_{\lambda\mu}.$$

Thus

$$\bar{\partial}\left(\psi_\mu^j \frac{\partial}{\partial z_\mu^i}\right) = \bar{\partial}\left(\psi_\lambda^j \frac{\partial}{\partial z_\lambda^i}\right)$$

is a global form and is expressed as

$$\sum_{k=1}^g y_{jk} \frac{\partial}{\partial z_i} d\bar{z}_k. \quad (1.28)$$

The Dolbeault cohomology class (1.0-20) is the Kodaira-Spencer class  $\rho(\frac{\partial}{\partial t_{ij}})$  and by (1.22) we have

$$\iota(\rho(\frac{\partial}{\partial t_{ij}})) = \sum_{\ell=1}^g \sum_{k=1}^g \{(y_{i\ell}\bar{\omega}_\ell) \otimes (y_{jk}\bar{\omega}_k)\}$$

Hence, by (1.22) the dual of this class in  $H^0(R, \omega_R) \otimes H^0(R, \omega_R)$  is  $\omega_i \otimes \omega_j$ . Hence, by (1.26) we conclude

$$\Pi^*(d\tau_{ij}) = \Pi^* \circ i^*(dt_{ij}) = \omega_i \omega_j \quad (1.29)$$

where we use the identification (1.11). Thus we have proved the following Proposition.

**Proposition 1.9** *Under the above assumption and notation, we have*

$$\begin{aligned} \Pi^* \left( \sum_{i < j} \frac{\partial}{\partial \tau_{ij}} (\log \det(C_\gamma \tau + D_\gamma)) d\tau_{ij} \right) \\ = \sum_{i < j} \omega_i \omega_j \Pi^* \left( \frac{\partial}{\partial \tau_{ij}} \log \det(C_\gamma \tau + D_\gamma) \right). \end{aligned} \quad (1.30)$$

**1.1.7 Symmetric bidifferential.** Let us consider a symmetric bidifferential form on a compact Riemann surface. First we recall basic facts on theta functions. For a point  $\tau \in \mathfrak{S}_g$  the theta function  $\theta(\tau, z)$  is defined as

$$\theta(\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp\{2\pi\sqrt{-1}(\frac{1}{2}{}^t m \tau m + {}^t m z)\}.$$

The theta function  $\theta(\tau, z)$  can be considered as a holomorphic section of the line bundle which defines a principal polarization of the abelian variety  $A_\tau = \mathbb{C}^g / (I_g, \tau)$ . For  $e \in \mathbb{C}^g$  we denote the corresponding point of  $A_\tau$  by  $[e]$ . The zeroes of the theta function  $\theta(\tau, z)$  define a divisor on  $A_\tau$  called the *theta divisor* and is denoted by  $\Theta$ .

The theta function  $\theta[e](\tau, z)$  with characteristic  $e = \begin{bmatrix} \delta \\ \epsilon \end{bmatrix}$ ,  $\delta, \epsilon \in \mathbb{R}^g$  is defined as

$$\theta[e](\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp 2\pi\sqrt{-1} \left\{ \frac{1}{2}{}^t (m + \delta) \tau (m + \delta) + {}^t (m + \delta) (z + \epsilon) \right\}.$$

Let  $R$  be a compact Riemann surface of genus  $g$ . Let us fix a symplectic basis of  $H_1(R, \mathbb{Z})$  and let  $\tau \in \mathfrak{S}_g$  be the corresponding period matrix. Then the abelian variety  $A_\tau$  is the Jacobian variety  $J(R)$  of  $R$ . Let  $Div^d(R)$  be the group of divisors of degree  $d$  on a compact Riemann surface  $R$ . We can extend the mapping  $j : R \rightarrow J(R)$  to the mapping  $j : Div^d(R) \rightarrow J(R)$  by putting

$$j \left( \sum_{i=1}^{m+d} P_i - \sum_{k=1}^m Q_k \right) := \sum_{i=1}^{m+d} j(P_i) - \sum_{k=1}^m j(Q_k).$$

Note that for two divisors  $D, D' \in Div^d(R)$ ,  $j(D) = j(D')$  if and only if  $D$  and  $D'$  are linearly equivalent, that is, there is a meromorphic function  $f$  on  $R$  such that  $D = D' + (f)$ . Let  $Pic^d(R)$  be the set of isomorphism classes of line bundles of degree  $d$  on the compact Riemann surface  $R$ . Then, by Abel's theorem, the mapping  $j : Div^d(R) \rightarrow J(R)$  factors through  $j : Pic^d(R) \rightarrow J(R)$ , where we

identify  $Pic^d(R)$  with the quotient  $Div^d(R)/\sim$  by the linear equivalence relation. Moreover, for any divisor  $D$  on  $R$  we denote the corresponding line bundle on  $R$  by  $[D]$ . Also in the following we always identify  $Pic^0(R)$  with  $J(R)$  via the canonical isomorphism  $\lambda$ . The following Theorem due to Riemann is the most fundamental in the theory of theta functions associated with a Riemann surface.

**Theorem 1.10** *There is a divisor  $\Delta \in Div^{g-1}(R)$  with*

$$[2\Delta] = \omega_R \quad (1.31)$$

such that for any point  $P \in R$  and  $e \in \mathbb{C}^g$  we have:

- (1) If  $\theta(\tau, e) \neq 0$ , then the divisor  $D$  of  $R$  defined by the equation in the variable  $Q$

$$\theta(\tau, j(Q) - j(P) - e) = 0$$

is an effective divisor of degree  $g$  with

$$H^1(R, \mathcal{O}_R(D)) = 0$$

and

$$[e] = j(D - P - \Delta).$$

- (2) If  $\theta(\tau, e) = 0$ , then for some effective divisor  $E$  of degree  $g - 1$  we have

$$[e] = j(E - \Delta).$$

Moreover,  $\dim_{\mathbb{C}} H^1(R, \mathcal{O}_R(E))$  is the multiplicity of the theta divisor  $\Theta$  at the point  $[e]$  and is the smallest integer  $d$  such that

$$\theta(B - A - e) \equiv 0$$

for all effective divisors  $A, B$  of degree less than or equal to  $d - 1$ .

Put

$$L_0 = [\Delta]. \quad (1.32)$$

Hence, we have

$$L_0^{\otimes 2} = \omega_R. \quad (1.33)$$

For a half-period (i.e. a two-torsion point)  $\alpha \in J(C)$ , that is,  $2\alpha = 0$  in  $J(R)$ , we let  $L_\alpha$  be the line bundle on the Riemann surface  $R$  such that  $L_\alpha \otimes L_0^{-1}$  is the half period  $\alpha$ . Hence, in particular

$$L_\alpha^{\otimes 2} = \omega_R. \quad (1.34)$$

**Corollary 1.11** *We have*

$$h^0(R, L_\alpha) = h^1(R, L_\alpha) = \text{mult}_\alpha \Theta \quad (1.35)$$

and  $h^0(R, L_\alpha)$  is even for  $\alpha$  even and odd for  $\alpha$  odd, where a half-period  $\alpha$  is called odd (resp. even), if we have

$$\alpha = \frac{1}{2} \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} \begin{pmatrix} \tau \\ I_g \end{pmatrix}, \quad \delta, \epsilon \in \mathbb{Z}^g$$

with  $\delta \cdot {}^t\epsilon$  even (resp. odd).

For the proofs of Theorem 1.10 and Corollary 1.11 we refer the reader to [Fay]. Let  $\alpha$  be a non-singular odd half-period, that is,  $\text{mult}_\alpha \Theta = 1$ . Then, by the above corollary, the line bundle  $L_\alpha$  has a non-zero global section  $h_\alpha$  satisfying

$$h_\alpha^2 = \sum_{i=1}^g \frac{\partial \theta[\alpha]}{\partial z_i}(\tau, 0) \omega_i \quad (1.36)$$



where  $\{\omega_1, \dots, \omega_g\}$  is the normalized basis of holomorphic one-forms and we use (1.34). The *prime form*  $E(P, Q)$  of the Riemann surface  $R$  is defined as

$$E(P, Q) := \frac{\theta[\alpha](\tau, j(P - Q))}{h_\alpha(P)h_\alpha(Q)} \quad (1.37)$$

for points  $P, Q \in R$ . Note that the prime form is independent of the choice of  $\alpha$  and is a holomorphic section of the line bundle  $\pi_1 L_0^{-1} \otimes \pi_2 L_0^{-1} \otimes \delta^*(\Theta)$  on  $R \times R$ , where  $\pi_i$  is the projection of  $R \times R$  to the  $i$ -th factor and  $\delta : R \times R \rightarrow J(R)$  is the mapping sending  $(P, Q) \in R \times R$  to  $j(Q - P)$ . Moreover, we have

$$E(P, Q) = -E(Q, P) \quad (1.38)$$

and  $E(P, Q)$  vanishes to first order along the diagonal in  $R \times R$  and does not vanish outside the diagonal. Let  $x$  (resp.  $y$ ) be a local coordinate of  $R$  with center  $P$  (resp.  $Q$ ) and by abuse of notation let us write

$$\begin{aligned} h_\alpha(P) &= h_\alpha(x)\sqrt{dx} \\ h_\alpha(Q) &= h_\alpha(y)\sqrt{dy}. \end{aligned}$$

Then the prime form  $E(P, Q)$  is expressed as

$$E(P, Q) = E(x, y)(\sqrt{dy})^{-1}(\sqrt{dx})^{-1}.$$

Now let us define the bidifferential  $\omega(x, y)dxdy$  by

$$\omega(x, y)dxdy := \frac{\partial^2}{\partial x \partial y} \log E(x, y)dxdy. \quad (1.39)$$

Then  $\omega(x, y)dxdy$  is a meromorphic bidifferential on  $R \times R$  holomorphic outside the diagonal and has a pole of order two along the diagonal. That is,  $\omega(x, y)dxdy \in H^0(R \times R, p_1^* \omega_R \otimes p_2^* \omega_R(2\Delta))$  Moreover, by (1.38) we have

$$\omega(x, y) = \omega(y, x). \quad (1.40)$$

**Proposition 1.12** *For  $f \in \mathbb{C}^g$ , if  $[f] \in J(R)$  is a non-singular point of the theta divisor  $\Theta$ , then we have*

$$\omega(x, y)dxdy = \frac{\partial^2 \log \theta}{\partial x \partial y}(\tau, j(x) - j(y) - f)dxdy.$$

For a proof, see [Fay].

For a symmetric bidifferential  $\hat{\omega}(x, y)dxdy$  on  $R \times R$  holomorphic outside the diagonal, we always have an expansion along the diagonal

$$\hat{\omega}(x, y)dxdy = \left( \sum_{k=k_0}^2 \frac{c_{-k}}{(x-y)^k} + \text{holomorphic} \right).$$

The coefficient  $c_{-2}$  is independent of the choice of local coordinates and called the *bi-residue* and written as  $\text{Res}^2(\hat{\omega})$ .

**Corollary 1.13** *There exists a symmetric bidifferential  $\omega$  which is holomorphic outside the diagonal and has a pole of order 2 on the diagonal with  $\text{Res}^2 \omega = 1$ .*

In the following, by a meromorphic symmetric bidifferential  $\omega(x, y)dxdy$  of a Riemann surface  $R$  we mean that  $\omega(y, x) = \omega(x, y)$ ,  $\omega$  is holomorphic on  $R \times R$  outside the diagonal and has a pole of order 2 along the diagonal.

**Lemma 1.14** *Let  $\omega(x, y)dxdy$  and  $\tilde{\omega}(x, y)dxdy$  be symmetric bilinear differentials on  $R \times R$  holomorphic outside the diagonal such that they have poles of order 2 on the diagonal with*

$$\text{Res}^2(\omega) = \text{Res}^2(\tilde{\omega})$$

*Then the difference*

$$\Omega(x, y)dxdy = \omega(x, y)dxdy - \tilde{\omega}(x, y)dxdy$$

*is a holomorphic symmetric bidifferential on  $R \times R$ .*

The space of symmetric bilinear differentials on  $R \times R$  having a pole of order two on the diagonal and holomorphic outside the diagonal is written as

$$H^0(R \times R, p_1^*\omega_R \otimes p_2^*\omega_R(2\Delta))^{(\iota)}$$

where  $\iota$  is the automorphism of  $R \times R$  exchanging the components. Note that

$$\omega_{R \times R} = p_1^*\omega_R \otimes p_2^*\omega_R$$

Let us define the *projective connection*  $S_\omega(z)dz^2$  attached to a symmetric bidifferential  $\omega$  with biresidue 1 of a compact Riemann surface  $R$  by

$$S_\omega(z)dz^2 = 6 \lim_{w \rightarrow z} \left( \omega(w, z)dwdz - \frac{dwdw}{(w-z)^2} \right). \quad (1.41)$$

The projective connection  $S_\omega(z)dz^2$  does depend on a choice of a local coordinate  $z$ . The following theorem is well-known and is obtained by direct calculations.

**Theorem 1.15**

$$S_\omega(z)dz^2 = S_\omega(w)dw^2 - \{z; w\}dw^2$$

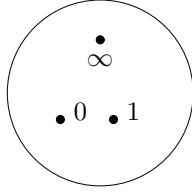
where  $\{z; w\}$  is the Schwarzian derivative defined by

$$\{z; w\} = \frac{d^3z/dw^3}{dw^3/dw} - \frac{3}{2} \left( \frac{d^2z/dw^2}{dw^2/dw} \right)^2.$$

**1.2 Stable  $N$ -pointed curves**

**1.2.1 Nodal curve.** By a nodal curve  $C$  we mean a closed analytic subvariety in a complex projective space  $\mathbb{P}^n$  whose singular points are nodes, that is, local analytically isomorphic to a neighbourhood  $U_\epsilon = \{(z, w) \mid zw = 0, |z| < \epsilon, |w| < \epsilon\}$  of  $zw = 0$  at the origin. If we use the language of algebraic geometry, a nodal curve is a semi-stable curve, that is, a reduced connected complete algebraic curve defined over the complex numbers  $\mathbb{C}$ . Let  $P$  be a node of a nodal curve  $C$ . There is a neighbourhood of  $P$  isomorphic to the above  $U_\epsilon$  for sufficiently small  $\epsilon$ . Then we can desingularize the node  $P$  by separating  $U_+ = U_\epsilon \cap \{w = 0\} = \{z \mid |z| < \epsilon\}$  and  $U_- = U_\epsilon \cap \{z = 0\} = \{w \mid |w| < \epsilon\}$ . Hence, instead of the point  $P$  we have two points  $P_+ = 0 \in U_+$  and  $P_- = 0 \in U_-$ . The new variety  $\tilde{C}$  thus obtained is called the curve obtained by normalization at the point  $P$ . There is a natural holomorphic mapping  $\varpi : \tilde{C} \rightarrow C$  such that  $\varpi$  is the identity mapping from  $\tilde{C} \setminus \{P_+, P_-\}$  to  $C \setminus \{P\}$  and  $\varpi^{-1}(P) = \{P_+, P_-\}$ .

Conversely, from a compact Riemann surface we can construct a nodal curve. Let  $P_+$  and  $P_-$  be two distinct points on a compact Riemann surface. Choose a local coordinate  $z$  (resp.  $w$ ) of  $R$  with center  $P_+$  (resp.  $P_-$ ). Let us identify  $P_+$  with  $P_-$  and obtain a new space  $C$ . By  $P$  we denote the point of  $C$  corresponding to  $P_\pm$ . We identify a neighbourhood of  $P$  in  $C$  with a neighbourhood of  $zw = 0$  at



**Figure 1.2** Stable 3-pointed curves of genus 0.

the origin. Identify  $C \setminus \{P\}$  with  $R \setminus \{P_+, P_-\}$ . Then  $C$  is a nodal curve. If we choose two other points on  $C \setminus \{P\}$  and repeat the same process we obtain a nodal curve with two nodes. In this way any nodal curve is obtained from a compact Riemann surface.

### 1.2.2 Stable $N$ -pointed curves.

**Definition 1.16** The data  $\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N)$  consisting of a curve  $C$  and points  $Q_1, \dots, Q_N$  on  $C$  is called an *stable  $N$ -pointed curve* if the following conditions are satisfied.

- (1) The curve  $C$  is a compact Riemann surface or a nodal curve.
- (2)  $Q_1, Q_2, \dots, Q_N$  are non-singular points of the curve  $C$ .
- (3) If an irreducible component  $C_i$  is a projective line (i.e. Riemann sphere)  $\mathbb{P}^1$  (resp. a rational curve with one double point, resp. an elliptic curve), the sum of the number of intersection points of  $C_i$  and other components and the number of  $Q_j$ 's on  $C_i$  is at least three (resp. one).
- (4)  $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = g$ .

Note that the above condition (3) is equivalent to saying that  $\text{Aut}(\mathfrak{X})$  is a finite group, so that  $\mathfrak{X}$  has no infinitesimal automorphisms. In the following we often add the following condition (Q) for an  $N$ -pointed stable curve  $\mathfrak{X}$ .

(Q) Each component  $C_i$  contains at least one  $Q_j$ .

The meaning of the condition (Q) will be clarified in the following Lemma 1.21 and Lemma 1.22.

**Example 1.17** For  $g = 0$ , by the property (3),  $N \geq 3$ . Any 3-pointed stable curve of genus 0 is isomorphic to the curve in Figure 1.2.

Any 4-pointed stable curve of genus 0 is isomorphic to one of the following two stable curves in Figure 1.3.

Any 1-pointed stable curve of genus 1 is isomorphic to one of the following two stable curves in Figure 1.4.

**Definition 1.18** Let  $C$  be a curve and  $Q$  a non-singular point on  $C$ . The  $n$ -th infinitesimal neighbourhood  $s^{(n)}$  of  $C$  at the point  $Q$  is the  $\mathbb{C}$ -algebra isomorphism

$$s^{(n)} : \mathcal{O}_{C,Q} / \mathfrak{m}_Q^{n+1} \simeq \mathbb{C}[[\xi]] / (\xi^{n+1}) \quad (1.42)$$

where  $\mathfrak{m}_Q$  is the maximal ideal of  $\mathcal{O}_{C,Q}$  consisting of germs of holomorphic functions vanishing at  $Q$ .

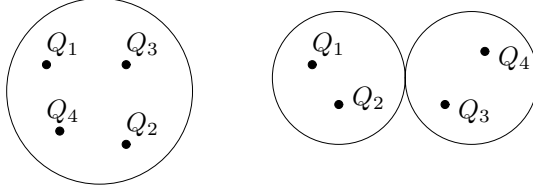


Figure 1.3 Stable 4-pointed curves of genus 0.

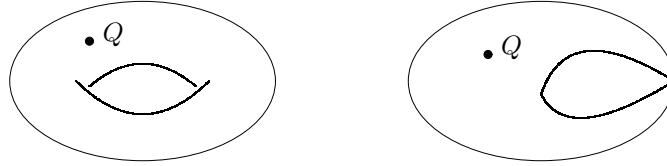


Figure 1.4 Stable 1-pointed curves of genus 1.

Taking the limit  $n \rightarrow \infty$  in the isomorphism (1.42), we have the isomorphism

$$s^{(\infty)} : \widehat{\mathcal{O}}_{C,Q} \simeq \mathbb{C}[[\xi]]. \quad (1.43)$$

The isomorphism  $s^{(\infty)}$  is called the *formal neighbourhood* of  $C$  at  $Q$ .

**Definition 1.19** The data  $\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N; s_1, s_2, \dots, s_N)$  is called an *stable  $N$ -pointed curve of genus  $g$  with formal neighbourhoods* if

- (1)  $(C; Q_1, Q_2, \dots, Q_N)$  is an  $N$ -pointed stable curve of genus  $g$ .
- (2)  $s_j$  is a formal neighbourhood of  $C$  at  $Q_j$ .

Similarly an stable  $N$ -pointed curve with  $n$ -th formal neighbourhoods

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \dots, Q_N; s_1^{(n)}, s_2^{(n)}, \dots, s_N^{(n)})$$

is defined .

**1.2.3 Residue pairing.** Let  $C$  be a nodal curve, that is,  $C$  is a reduced curve with at most ordinary double points and proper over  $\mathbb{C}$ . Let  $\Omega_C^1$  be the sheaf of Kähler differentials of the curve  $C$  and  $\omega_C$  be the dualizing sheaf of the curve  $C$ . Near a singular point  $P$ , the curve  $C$  is analytically isomorphic to the variety defined by

$$xy = 0.$$

In these coordinates the sheaf  $\Omega_C^1$  is expressed as

$$\Omega_C^1 = (\mathcal{O}_C dx + \mathcal{O}_C dy) / (x dy + y dx) \mathcal{O}_C. \quad (1.44)$$

On the other hand, near the singular point  $P$  the dualizing sheaf  $\omega_C$  is the invertible sheaf generated by the differential  $\zeta$  given by  $dx/x$  outside  $x = 0$  and  $-dy/y$  outside  $y = 0$ . Moreover, outside the singular points of the curve  $C$ , the sheaves  $\Omega_C^1$  and  $\omega_C$  coincide. Thus, we have

$$\Omega_C^1 = \mathfrak{m} \omega_C \quad (1.45)$$

where  $\mathfrak{m}$  is the defining ideal sheaf of the singular points of  $C$ . Hence, we have the following exact sequence.

$$0 \rightarrow \Omega_C^1 \rightarrow \omega_C \rightarrow \omega_C \otimes \mathcal{O}_{C_{Sing}} \rightarrow 0.$$

Let  $\nu : \tilde{C} \rightarrow C$  be the normalization of the curve  $C$ . We let  $\{P_1, \dots, P_q\}$  be the set of double points of the curve  $C$  and for each double point  $P_i$ , put  $\nu^{-1}(P_i) = \{P_{i,+}, P_{i,-}\}$ . Then, we have the following exact sequence.

$$0 \rightarrow \omega_C \rightarrow \nu_* \omega_{\tilde{C}} \left( \sum_i^k (P_{i,+} + P_{i,-}) \right) \xrightarrow{r} \bigoplus_i^q \mathbb{C} \rightarrow 0 \quad (1.46)$$

where at each double point  $P_i$ , the mapping  $r$  is given by

$$\text{res}_{P_{i,+}}(\tau) - \text{res}_{P_{i,-}}(\tau).$$

This means that a local holomorphic section of the dualizing sheaf  $\omega_C$  is regarded as a local meromorphic section of a one-form on  $\tilde{C}$  which has a pole of order one at  $P_{i,+}$  and  $P_{i,-}$  such that the sum of the residues is zero and the one-form is holomorphic outside  $P_{i,\pm}$ 's. In the following we shall often use this interpretation. The following Lemma is an easy consequence of this interpretation.

**Lemma 1.20** *Let  $\tau$  be a meromorphic section of the dualizing sheaf  $\omega_C$  holomorphic at the double points. Then the sum of the residues of  $\tau$  is zero.*

**Lemma 1.21** *Assume that the  $N$ -pointed stable curve*

$$\mathfrak{X} = (C; Q_1, Q_2, \dots, Q_N; s_1, s_2, \dots, s_N)$$

*with formal neighbourhoods satisfies the condition (Q). By  $t_j$  we denote the Laurent expansions at  $Q_j$  with respect to a formal parameter  $\xi_j = s^{-1}(\xi)$ . Then the following homomorphisms are injective:*

$$t = \oplus t_j \quad : \quad H^0(C, \mathcal{O}(* \sum_{j=1}^N Q_j)) \longrightarrow \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) \quad (1.47)$$

$$t = \oplus t_j \quad : \quad H^0(C, \omega_C(* \sum_{j=1}^N Q_j)) \longrightarrow \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) d\xi_j \quad (1.48)$$

where  $\omega_C$  is the dualizing sheaf of the curve  $C$  and

$$H^0(C, \mathcal{O}(* \sum_{j=1}^N Q_j)) = \text{inj} \lim_{n \rightarrow \infty} H^0(C, \mathcal{O}(n \sum_{j=1}^N Q_j)).$$

**Proof** By condition (Q) if an element  $f \in H^0(C, \mathcal{O}(* \sum_{j=1}^N Q_j))$  is zero at an irreducible component of the curve  $C$ , then  $f = 0$ . Hence the mapping (1.47) is injective. Similarly, if an element  $\omega \in H^0(C, \omega_C(* \sum_{j=1}^N Q_j))$  is zero at an irreducible component of the curve  $C$ , then  $\omega = 0$ , and hence the injectivity of (1.48) follows.  $\square$

By this Lemma  $H^0(C, \mathcal{O}(* \sum_{j=1}^N Q_j))$  (resp.  $H^0(C, \omega_C(* \sum_{j=1}^N Q_j))$ ) can be regarded as a subspace of  $\bigoplus_{j=1}^N \mathbb{C}((\xi_j))$  (resp.  $\bigoplus_{j=1}^N \mathbb{C}((\xi_j)) d\xi_j$ ). There is the residue pairing

$$\begin{aligned} \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) \times \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) d\xi_j &\rightarrow \mathbb{C} \\ ((f(\xi_1), \dots, f(\xi_N), g(\xi_1)d\xi_1, \dots, g(\xi_N)d\xi_N)) &\mapsto \sum_{j=1}^N \operatorname{Res}_{\xi_j=0} (f(\xi_j)g(\xi_j)d\xi_j). \end{aligned} \quad (1.49)$$

The following Theorem plays an important role in our theory.

**Theorem 1.22** *Under the residue pairing the vector space  $H^0(C, \mathcal{O}(*\sum_{j=1}^N Q_j))$  and the vector space  $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$  are annihilators of each other.*

**Proof** Since the residue is independent of the choice of local coordinates, we may assume that the formal neighbourhood gives holomorphic coordinates in a neighbourhood of the point  $Q_j$ . For any positive integers  $m, n$  we have the exact sequence

$$0 \rightarrow \mathcal{O}_C(-m \sum_{j=1}^N Q_j) \rightarrow \mathcal{O}_C(n \sum_{j=1}^N Q_j) \xrightarrow{P} \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}\xi_j^k \rightarrow 0. \quad (1.50)$$

From this exact sequence we have the long exact sequence of cohomology groups

$$H^0(C, \mathcal{O}_C(n \sum_{j=1}^N Q_j)) \rightarrow \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}\xi_j^k \xrightarrow{c} H^1(C, \mathcal{O}_C(-m \sum_{j=1}^N Q_j)) \rightarrow 0.$$

Note that by the Serre duality  $H^1(C, \mathcal{O}_C(-m \sum_{j=1}^N Q_j))$  is dual to

$$H^0(C, \omega_C(m \sum_{j=1}^N Q_j)).$$

On the image of  $c$  the dual pairing

$$\langle \cdot, \cdot \rangle : H^1(C, \mathcal{O}_C(-m \sum_{j=1}^N Q_j)) \times H^0(C, \mathcal{O}_C(m \sum_{j=1}^N Q_j)) \rightarrow \mathbb{C}$$

is given by

$$\langle c((g_j(\xi_j))), \tau \rangle = \sum_{j=1}^N \operatorname{Res}_{\xi_j=0} (g_j(\xi_j)\tau_j) \quad (1.51)$$

where  $(g_j(\xi_j)) \in \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}\xi_j^k$ ,  $\tau \in H^0(C, \mathcal{O}_C(m \sum_{j=1}^N Q_j))$  and  $\tau_j$  is the Laurent expansion of  $\tau$  at the point  $Q_j$  with respect to  $\xi_j$ .  $(g_j(\xi_j)) \in \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}\xi_j^k$  is the image of a global meromorphic function  $g \in H^0(C, \mathcal{O}_C(-m \sum_{j=1}^N Q_j))$  if and only if

$$\sum_{j=1}^N \operatorname{Res}_{\xi_j=0} (g_j(\xi_j)\tau_j) = 0$$

for any  $\tau \in H^0(C, \omega_C(m \sum_{j=1}^N Q_j))$ . The proof of (1.51) will be given later.

Assume that  $(f_j(\xi_j)) \in \bigoplus_{j=1}^N \mathbb{C}((\xi_j))$  is annihilated by  $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$  by the residue pairing. Fix a positive integer  $m$ . For

$$f_j(\xi_j) = \sum_{k=-n_j}^{\infty} a_k^{(j)} \xi_j^k, \quad j = 1, 2, \dots, N$$

put

$$f_{j,m}(\xi_j) = \sum_{k=-n_j}^{m-1} a_k^{(j)} \xi_j^k, \quad j = 1, 2, \dots, N.$$

Put  $n = \max_j n_j$ . Since  $(f_j(\xi_j))$  is annihilated by  $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$ ,  $(f_{j,m}(\xi_j))$  is annihilated by  $H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$ .  $(f_{j,m}(\xi_j))$  is the image of a meromorphic function  $f^{(m)} \in H^0(C, \mathcal{O}_C(n\sum_{j=1}^N Q_j))$ . Let  $f_j^{(m)}(\xi_j)$  be the Laurent expansion of  $f^{(m)}$  at  $Q_j$  with respect to  $\xi_j$ . Then

$$g_j(\xi_j) := f_j(\xi_j) - f_j^{(m)}(\xi_j) \equiv 0 \pmod{\xi_j^m}.$$

Since  $f^{(m)}$  is a global meromorphic function  $(f_j^{(m)}(\xi_j))$  is annihilated by

$$H^0(C, \omega_C(*\sum_{j=1}^N Q_j)).$$

$(g_j(\xi_j))$  is also annihilated by  $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$ . Assume that  $g_k(\xi_k) \neq 0$  so that

$$g_k(\xi_k) = b_s \xi_k^s + b_{s+1} \xi_k^{s+1} + \dots, \quad b_s \neq 0.$$

Then  $s \geq m > 0$ . Hence there exists a meromorphic one-form

$$\omega \in H^0(C, \omega_C((s+1)Q_k))$$

which has a pole of order  $s+1$  at  $Q_k$  and is holomorphic on  $C \setminus Q_k$ . Hence we have

$$\sum_{j=1}^N \operatorname{Res}_{\xi_j=0} g_j \omega_j = \operatorname{Res}_{\xi_k=0} g_k \omega_k = b_s \neq 0,$$

where  $\omega_j$  is the Laurent expansion of  $\omega$  at  $Q_j$  with respect to  $\xi_j$ . This contradicts the fact that  $\{g_j(\xi_j)\}$  is annihilated by  $H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$ . Thus we conclude that  $g_j(\xi_j) = 0$ ,  $j = 1, 2, \dots, N$ . Thus  $f_j(\xi_j)$  is the Laurent expansion of  $f^{(m)}$  at  $Q_j$  with respect to  $\xi_j$ .

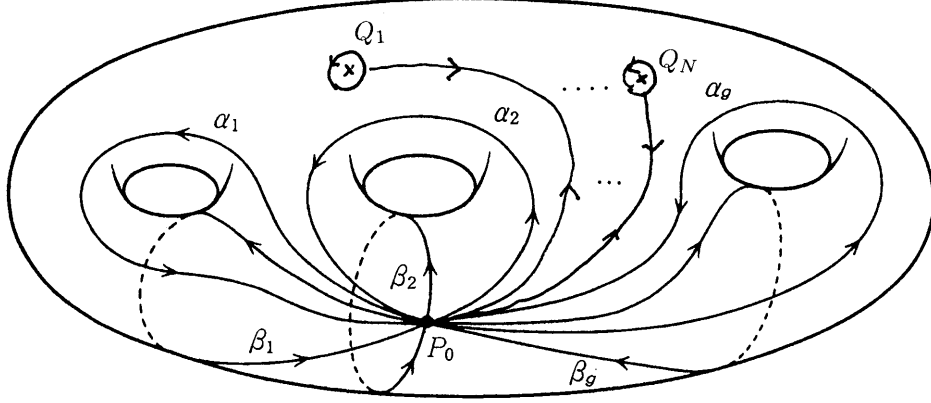
Let us show (1.51). First assume that  $C$  is a compact Riemann surface. For a small positive number  $\varepsilon$  put  $D_\varepsilon^{(j)} = \{\xi_j \mid |\xi_j| < \varepsilon\}$  and  $C_\varepsilon = C \setminus \sum_{j=1}^N D_\varepsilon^{(j)}$ . Cut  $C_\varepsilon$  along a closed curve so that the cut surface  $R'_\varepsilon$  is simply connected (see Figure 1.5).

Choose an open covering  $\{U_i\}_{i=1}^t$  of  $C$  such that

$$Q_j \in U_j, \quad j = 1, 2, \dots, N, \quad Q_j \notin U_i, \quad i \neq j.$$

Choose  $U_i$  small enough so that for  $j = 1, \dots, N$  there exist

$$h_j \in H^0(U_j, \mathcal{O}_C(m\sum_{j=1}^N Q_j))$$



**Figure 1.5** Cut the surface  $C_\varepsilon$  along a closed curve.

with  $p(h_j) = g_j(\xi_j)$ . Set  $h_i = 0$  for  $i \geq N + 1$ . Then,

$$h_{ab} = h_b - h_a \in H^0(U_a \cap U_b, \mathcal{O}_C(-n \sum_{j=1}^N Q_j))$$

and  $\{h_{ab}\}$  is a one-cocycle which represents

$$c((g_j(\xi_j))) \in H^1(C, \mathcal{O}_C(-n \sum_{j=1}^N Q_j)).$$

The duality pairing  $H^1(C, \mathcal{O}_C(-m \sum_{j=1}^N Q_j)) \times H^0(C, \mathcal{O}_C(n \sum_{j=1}^N Q_j)) \rightarrow \mathbb{C}$  is given by the integration

$$\frac{1}{2\pi i} \int_C \omega \wedge \tau$$

where  $\omega \in H_{\bar{\partial}}^{(0,1)}(C, \mathcal{O}_C(-m \sum_{j=1}^N Q_j))$  is a Dolbeault cohomology class. Let  $\mathcal{D}_C^{(0,0)}$  (resp.  $\mathcal{D}_C^{(0,k)}$ ) be the sheaf of germs of complex-valued  $C^\infty$  functions (resp. type  $(0, k)$ -forms) on  $C$  and define

$$\mathcal{D}_C^{(0,i)}(-n \sum_{j=1}^N Q_j) = \mathcal{O}_C(-n \sum_{j=1}^N Q_j) \otimes_{\mathcal{O}_C} \mathcal{D}_C^{(0,i)}.$$

Then we have

$$\begin{aligned} & H_{\bar{\partial}}^{(0,1)}(C, \mathcal{O}_C(-m \sum_{j=1}^N Q_j)) \\ & \simeq \frac{\text{Ker} \left\{ \bar{\partial} : H^0(C, \mathcal{D}_C^{(0,1)}(-m \sum_{j=1}^N Q_j)) \rightarrow H^0(C, \mathcal{D}_C^{(0,2)}(-m \sum_{j=1}^N Q_j)) \right\}}{\bar{\partial} \left( H^0(C, \mathcal{D}_C^{(0,0)}(-m \sum_{j=1}^N Q_j)) \right)}. \end{aligned}$$



The Dolbeault cohomology class associated with  $\{h_{ab}\} = c(g_j(\xi_j))$  is given as follows. There exist  $s_a \in H^0(U_a, \mathcal{D}_C^{(0,0)}(-n \sum_{j=1}^N Q_j))$  such that

$$h_{ab} = s_b - s_a.$$

Then we have

$$h_a - s_a = h_b - s_a.$$

for any  $a, b$  so that they define a global function  $h$  on  $C$ . Note that  $h$  has a pole at  $Q_j$  so that  $\bar{\partial}h$  is not the zero cohomology class. The class  $\bar{\partial}h$  represents the Dolbeault cohomology class associated with the Čech one-cocycle  $\{h_{ab}\}$ .

$$\bar{\partial}h = -\bar{\partial}s_a$$

on  $U_a$ . Then the pairing  $\langle c((g_j(\xi_j)), \tau) \rangle$  is calculated as follows.

$$\begin{aligned} \langle c((g_j(\xi_j)), \tau) \rangle &= \frac{1}{2\pi i} \int_C \bar{\partial}h \wedge \tau \\ &= \frac{1}{2\pi i} \int_C d(h\tau) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{C_\varepsilon} d(h\tau) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\partial C} h\tau \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^N \frac{1}{2\pi i} \int_{\partial D_\varepsilon^{(j)}} (h_j - s_j)\tau \\ &\quad (\text{integrations along } \alpha_i, \beta_i \text{ and } l_j \text{ cancel out}) \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^N \frac{1}{2\pi i} \left\{ \int_{\partial D_\varepsilon^{(j)}} h_j\tau - \int_{\partial D_\varepsilon^{(j)}} s_j\tau \right\} \\ &= \sum_{j=1}^N \text{Res}_{Q_j}(h_j\tau) - \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{D_\varepsilon^{(j)}} d(s_j\tau). \end{aligned}$$

Since  $s_j\tau$  is  $C^\infty$  in a neighbourhood of  $Q_j$ , we have

$$\lim_{\varepsilon \rightarrow 0} \int_{D_\varepsilon^{(j)}} d(s_j\tau) = 0.$$

Also, we have  $\text{Res}_{Q_j}(h_j\tau) = \text{Res}_{\xi_j=0}(g_j(\xi_j)\tau_j)$  where  $\tau_j$  is the Laurent expansion of  $\tau$  at  $Q_j$  with respect to  $\xi_j$ ; hence we conclude

$$\langle c((g_j(\xi_j)), \tau) \rangle = \sum_{j=1}^N \text{Res}_{\xi_j=0} g_j(\xi_j)\tau_j.$$

This is the desired result.

Next assume that  $C$  has nodes. For simplicity assume that  $C$  has only one node  $P$ . Let  $\pi : \tilde{C} \rightarrow C$  be the normalization map and  $\pi^{-1}(P) = \{P_+, P_-\}$ . Then  $\pi^*\omega_C = \omega_{\tilde{C}}(P_+ + P_-)$ . Hence

$$\pi^*H^0(C, \omega_C(m \sum_{j=1}^N Q_j)) \subset H^0(\tilde{C}, \omega_{\tilde{C}}(m \sum_{j=1}^N Q_j)).$$

This implies that on  $\tilde{C}$ ,  $(g_j(\xi_j))$  is annihilated by  $H^0(\tilde{C}, \omega_{\tilde{C}}(m \sum_{j=1}^N Q_j))$ . Hence, by the above arguments there exists  $\tilde{g} \in H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(m \sum_{j=1}^N Q_j))$  such that  $p(\tilde{g}) = (g_j(\xi_j))$ . On the other hand  $(g_j(\xi_j))$  is annihilated by  $H^0(C, \omega_C(m \sum_{j=1}^N Q_j))$ . Hence for  $\tau \in \pi^* H^0(C, \omega_C(m \sum_{j=1}^N Q_j)) \setminus H^0(\tilde{C}, \omega_{\tilde{C}}(m \sum_{j=1}^N Q_j))$  we have

$$\sum_{j=1}^N \operatorname{Res}_{Q_j}(\tilde{g}\tau) = 0.$$

On the other hand  $\tilde{g}\tau$  has poles of order one at  $P_+$  and  $P_-$ . Therefore we conclude

$$\operatorname{Res}_{P_+}(\tilde{g}\tau) + \operatorname{Res}_{P_-}(\tilde{g}\tau) = 0. \quad (1.52)$$

Since we have

$$\operatorname{Res}_{P_+}(\tilde{g}\tau) = \tilde{g}(P_+) \operatorname{Res}_{P_+} \tau, \quad \operatorname{Res}_{P_-}(\tilde{g}\tau) = \tilde{g}(P_-) \operatorname{Res}_{P_-} \tau$$

and

$$\operatorname{Res}_{P_+} \tau + \operatorname{Res}_{P_-} \tau = 0,$$

we have

$$\tilde{g}(P_+) = \tilde{g}(P_-).$$

This means that  $\tilde{g}$  is the pullback of a meromorphic function

$$g \in H^0(C, \mathcal{O}_C(m \sum_{j=1}^N Q_j)).$$

This shows that (1.51) is valid for a nodal curve.

The characterization of  $H^0(C, \omega_C(* \sum_{j=1}^N Q_j))$  is proved similarly.  $\square$

### 1.3 Deformation of pointed curves

Let  $C$  be a compact Riemann surface of genus  $g$ . A deformation of  $C$  is, by definition, a proper *smooth* holomorphic mapping  $\pi : X \rightarrow Y$  of complex spaces with a prescribed point  $y \in Y$  such that  $\pi^{-1}(y)$  is isomorphic to the Riemann surface  $C$  (see 1.1.2). It is more convenient to generalize the notion of complex analytic family  $f : X \rightarrow Y$  in such a way that the underlying space  $Y$  is a complex space, not necessarily a complex manifold.

For example, if  $Y = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ , the deformation of  $C$  is called an infinitesimal (or first order) deformation of  $C$ . The infinitesimal deformations of the Riemann surface  $C$  are parameterized by the cohomology group  $H^1(C, \Theta_C)$ , where  $\Theta_C$  is the sheaf of germs of holomorphic vector fields on  $C$ . (See 1.1.2.)

More generally, we can define a deformation of the data

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \dots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \dots, \eta_N^{(n)})$$

of an  $N$ -pointed Riemann surface of genus  $g$  with  $n$ -th infinitesimal neighbourhoods. The infinitesimal deformations of an  $N$ -pointed Riemann surface of genus  $g$  with  $n$ -th infinitesimal neighbourhoods are parameterized by the cohomology group  $H^1(C, \Theta_C(-(n+1) \sum_{j=1}^N Q_j))$ . If  $C$  is a singular semi-stable curve, a deformation of  $C$  is defined as a proper *flat* holomorphic mapping  $\pi : X \rightarrow Y$  of complex spaces with a prescribed point  $y \in Y$  such that  $\pi^{-1}(y)$  is isomorphic to the curve  $C$ . In this case, the infinitesimal deformations of the curve  $C$  are parameterized by the

cohomology group  $Ext_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C)$  and the infinitesimal deformations of a stable  $N$ -pointed curve

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \dots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \dots, \eta_N^{(n)})$$

with  $n$ -th infinitesimal neighbourhoods are parameterized by the cohomology group

$$Ext_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C(-(n+1) \sum_{j=1}^N Q_j)).$$

(See, for example, [Ar], [DM, Section 1], [SGA7, Exposé VI, 6].) Here,  $\Omega_C^1$  is the sheaf of Kähler differentials on the curve  $C$ . In our situation, we may regard the exact sequence

$$0 \rightarrow \mathcal{I}_C/\mathcal{I}_C^2 \rightarrow \Omega_Y^1 \otimes \mathcal{O}_C \rightarrow \Omega_C^1 \rightarrow 0.$$

as a definition of the sheaf  $\Omega_C^1$  where  $\mathcal{I}_C$  is the sheaf of defining ideals of  $C = f^{-1}(y_0)$  of a deformation  $\pi : X \rightarrow Y$  of the curve  $C$ .) Put  $\Theta_C = \underline{Hom}_{\mathcal{O}_C}(\Omega_C^1, \mathcal{O}_C)$ . There is an exact sequence

$$\begin{aligned} 0 \rightarrow H^1(C, \Theta_C(-(n+1) \sum_{j=1}^N Q_j)) &\rightarrow Ext_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C(-(n+1) \sum_{j=1}^N Q_j)) \\ &\rightarrow H^0(C, \underline{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C)) \rightarrow 0. \end{aligned} \quad (1.53)$$

If the stable curve  $C$  has  $q$  double points  $P_1, P_2, \dots, P_q$ , then we have

$$\underline{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C)_Q = \begin{cases} \mathbb{C}, & \text{if } Q = P_j, \quad i = 1, 2, \dots, q \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have

$$H^0(C, \underline{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C)) \simeq \mathbb{C}^q.$$

Each element of  $H^1(C, \Theta_C(-(n+1) \sum_{j=1}^N Q_j))$  corresponds to an infinitesimal deformation of the data

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \dots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \dots, \eta_N^{(n)})$$

preserving the singularities.

**Definition 1.23** The data  $(\pi : Y \rightarrow B; s_1, s_2, \dots, s_N; \eta_1, \eta_2, \dots, \eta_N)$  is called a (holomorphic) family of  $N$ -pointed stable curves of genus  $g$  with formal neighbourhoods if the following conditions are satisfied.

- (1)  $Y$  and  $B$  are connected complex manifolds,  $\pi : Y \rightarrow B$  is a proper flat holomorphic map and  $s_1, s_2, \dots, s_N$  are holomorphic sections of  $\pi$ .
- (2) For each point  $b \in B$  the data  $(Y_b := \pi^{-1}(b); s_1(b), s_2(b), \dots, s_N(b))$  is an  $N$ -pointed stable curve of genus  $g$ .
- (3)  $\eta_j$  is an  $\mathcal{O}_B$ -algebra isomorphism

$$\eta_j : \widehat{\mathcal{O}}/s_j(B) := \varprojlim_{n \rightarrow \infty} \mathcal{O}_Y/I_j^n \simeq \mathcal{O}_B[[\xi]],$$

where  $I_j$  is the defining ideal of  $s_j(B)$  in  $Y$ .

Similarly we define a family of stable  $N$ -pointed curves of genus  $g$

$$(\pi : Y \rightarrow B; s_1, s_2, \dots, s_N; \tilde{\eta}_1^{(n)}, \tilde{\eta}_2^{(n)}, \dots, \tilde{\eta}_N^{(n)})$$

with  $n$ -th formal neighbourhoods, by changing the third condition by

(3')  $\tilde{\eta}_j^{(n)}$  is an  $\mathcal{O}_B$ -algebra isomorphism

$$\tilde{\eta}_j^{(n)} : \mathcal{O}_Y/I_j^{n+1} \simeq \mathcal{O}_B[[\xi]]/I_j^{(n+1)}.$$

For a holomorphic mapping  $f : D \rightarrow B$ , we can define the pullback  $f^*\mathfrak{X}^{(n)}$  (resp.  $f^*\mathfrak{X}$ ) of the family of  $N$ -pointed stable curves with  $n$ -th infinitesimal neighbourhoods (resp. formal neighbourhoods) over the base space  $D$  by the mapping  $f$ .

**Definition 1.24** A family

$$\mathfrak{X} = (\pi^{(n)} : \mathcal{C}^{(n)} \rightarrow \mathcal{B}^{(n)}; s_1^{(n)}, s_2^{(n)}, \dots, s_N^{(n)}; \tilde{\eta}_1^{(n)}, \tilde{\eta}_2^{(n)}, \dots, \tilde{\eta}_N^{(n)})$$

of stable  $N$ -pointed curves of genus  $g$  with  $n$ -th formal neighbourhoods is said to be *versal* (resp. *universal*) at a point  $b \in \mathcal{B}^{(n)}$  if for any deformation

$$\mathfrak{Y} = (\pi : X \rightarrow Y; s_1, \dots, s_N; \hat{\eta}_1, \dots, \hat{\eta}_N)$$

of  $\pi^{(n)-1}(b) = (C; Q_1, \dots, Q_N; \eta_1, \dots, \eta_N)$  with prescribed point  $y \in Y$  there exists a holomorphic mapping (resp. unique holomorphic mapping)  $f$  from a neighbourhood of  $y$  in  $Y$  to  $\mathcal{B}^{(n)}$  such that the pullback  $f^*\mathfrak{X}$  is isomorphic to  $\mathfrak{Y}$  in a neighbourhood of  $y$  in  $Y$  and such that  $df$  is uniquely determined at the base point. If the family is versal (resp. universal) at each point of  $\mathcal{B}^{(n)}$ , the family  $\mathfrak{X}$  is called a versal (resp. universal) family.

In the following we mainly consider a family of stable pointed curves with formal coordinates, which is versal as a family of pointed curves. Hence, in this section we only consider a family of stable pointed curves. The Kodaira-Spencer mapping of a family of stable pointed curves is given by the following theorem. For a proof see [TUY, p. 499–503].

**Theorem 1.25** For a family  $(\pi : Y \rightarrow B; s_1, s_2, \dots, s_N)$  of  $N$ -pointed stable curve of genus  $g$  and for each point  $b \in B$ , there exists a  $\mathbb{C}$ -linear mapping

$$\rho_b : T_b B \rightarrow \text{Ext}_{\mathcal{O}_{Y_b}}^1(\Omega_{Y_b}^1, \mathcal{O}_{Y_b}(-\sum_{j=1}^N s_j(b))), \quad (1.54)$$

where  $Y_b = \pi^{-1}(b)$ .

The  $\mathbb{C}$ -linear mapping  $\rho_b$  is called the *Kodaira-Spencer mapping* of the family  $(\pi : Y \rightarrow B; s_1, s_2, \dots, s_N)$  at the point  $b$ .

A sheaf version of Theorem 1.25 is the following:

**Corollary 1.26** If  $(\pi : Y \rightarrow B; s_1, s_2, \dots, s_N)$  is a family of  $N$ -pointed smooth curves of genus  $g$ , the Kodaira-Spencer mapping  $\rho_s$  induces an  $\mathcal{O}_B$ -module homomorphism

$$\rho : \Theta_B \rightarrow R^1 \pi_* \underline{\text{Hom}}(\Omega_{Y/B}^1, \Theta_Y(-\sum_{j=1}^N s_j(B))).$$

The criterion of versality of a family is given by the following:

**Proposition 1.27** A family

$$(\pi : \mathcal{C} \rightarrow \mathcal{B}; s_1, s_2, \dots, s_N)$$

of stable  $N$ -pointed curves of genus  $g$  is versal at a point  $b \in \mathcal{B}$  if and only if the Kodaira-Spencer mapping

$$\rho_b : T_b \mathcal{B} \rightarrow \text{Ext}_{\mathcal{O}_{C_b}}^1(\Omega_{\mathcal{O}_{C_b}}^1, \mathcal{O}_{C_b}(-\sum_{j=1}^N s_j(b)))$$

is an isomorphism at the point  $b$ .

Let us prove the existence of the versal family of stable  $N$ -pointed curves.

**Theorem 1.28** *For each  $N$ -pointed stable curve  $(C; Q_1, \dots, Q_N)$  of genus  $g$ , there exists a family*

$$\mathfrak{F} = (\pi : \mathcal{C} \rightarrow \mathcal{B}; s_1, s_2, \dots, s_N)$$

with prescribed point  $b \in \mathcal{B}$  such that  $\pi^{-1}(b)$  is isomorphic to  $(C; Q_1, \dots, Q_N)$  and such that the family  $\mathfrak{F}$  is versal at  $b$ . Moreover,  $\mathcal{C}$  and  $\mathcal{B}$  are complex manifolds and the family  $\mathfrak{F}$  is versal at each point of a small neighbourhood of  $b$  in  $\mathcal{B}$ . If the  $N$ -pointed stable curve has trivial automorphism group, then the family  $\mathfrak{F}$  is universal at  $b$ .

**Proof** The theorem is a consequence of deformation theory ([Ar], [Sc], [SGA 7]). Since we need an explicit description of a versal family, we give here a method to construct a versal family of the first infinitesimal neighbourhoods.

By our assumption, the curve  $C$  has only ordinary double points. Hence, by deformation theory, there exists a versal family  $\pi^{(\phi)} : \mathcal{C}^{(\phi)} \rightarrow \mathcal{B}^{(\phi)}$  with specified point  $x \in \mathcal{B}^{(\phi)}$  such that  $C_x = \pi^{(\phi)-1}(x) \simeq C$  and the Kodaira-Spencer mapping

$$\rho_x : T_x \mathcal{B}^{(\phi)} \rightarrow \text{Ext}_{\mathcal{O}_{C_x}}^1(\Omega_{\mathcal{O}_{C_x}}^1, \mathcal{O}_{C_x})$$

is an isomorphism. (Since the automorphism group of  $C$  may not be trivial, the family  $\pi : \mathcal{C}^{(\phi)} \rightarrow \mathcal{B}^{(\phi)}$  may not be universal at the point  $x$  but only versal.) Put

$$\mathcal{B} = \mathcal{C}^{(\phi)N} \setminus \left( \bigcup_{i < j} \Delta_{ij} \cup \{ \text{singular points of } \mathcal{C}^{(\phi)N} \} \right)$$

where

$$\Delta_{ij} = \{ (x_1, \dots, x_N) \in \mathcal{C}^{(\phi)N} \mid x_i = x_j \}$$

is the  $(i, j)$ -th diagonal. There is a natural holomorphic mapping  $p : \mathcal{B} \rightarrow \mathcal{B}^{(\phi)}$ . Put also

$$\mathcal{C} = \mathcal{C}^{(\phi)} \times_{\mathcal{B}^{(\phi)}} \mathcal{B}$$

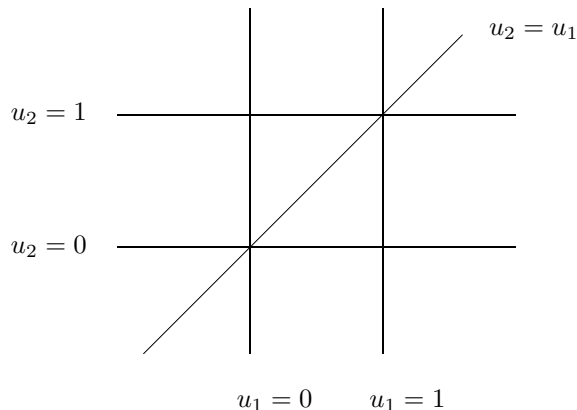
and let  $\pi : \mathcal{C} \rightarrow \mathcal{B}$  be the projection to the second factor. By our definition,  $(Q_1, \dots, Q_N) \in p^{-1}(x)$ . Put  $x_0 = (Q_1, \dots, Q_N) \in \mathcal{B}$ . Then we have  $\pi^{-1}(x_0) = C_x \times x_0 \simeq C$ . Moreover, we can define holomorphic sections

$$s_j : \mathcal{B} \rightarrow \mathcal{C}$$

by

$$s_j((P_1, \dots, P_N)) = (P_j, P_1, \dots, P_N) \in \mathcal{C}^{(\phi)} \times_{\mathcal{B}^{(\phi)}} \mathcal{B}.$$

Then we have  $s_j(x_0) = (Q_j, x_0)$ . It is easy to show that  $\mathfrak{F} = (\pi : \mathcal{C} \rightarrow \mathcal{B}; s_1, \dots, s_N)$  is versal at each point of  $\mathcal{B}$ .  $\square$



**Figure 1.6** The moduli space of ordered four-pointed projective lines :  $B_2 = \mathbb{C}^2 \setminus \{4 \text{ lines}\}$ .

Let  $\mathfrak{F}(\pi := \mathcal{C} \rightarrow \mathcal{B}; s_1, s_2, \dots, s_N)$  be a versal family of  $N$ -pointed stable curves of genus  $g$ . Put

$$\Sigma = \{P \in \mathcal{C} \mid d\pi_P : T_P\mathcal{C} \rightarrow T_{\pi(P)}\mathcal{B} \text{ is not surjective}\} \quad (1.55)$$

$$D = \pi(\Sigma). \quad (1.56)$$

The set  $\Sigma$  is called the *critical locus* of the family and  $D$  is called the *discriminant locus* of the family. The following lemma is a consequence of the deformation theory of singular curves with ordinary double points. (See for example [Ar], [DM, Section 1] or [SGA 7, Exposé VI, 6].)

**Lemma 1.29** *For a versal family of stable  $N$ -pointed curves of genus  $g$*

$$(\pi : \mathcal{C} \rightarrow \mathcal{B}; s_1, s_2, \dots, s_N),$$

assume  $2g - 2 + N \geq 1$ .

(1) *We have*

$$\dim \mathcal{B} = 3g - 3 + N$$

$$\dim \mathcal{C} = 3g - 2 + N.$$

(2) *The critical locus  $\Sigma$  is a smooth subvariety of codimension 2 in  $\mathcal{C}$ .*

(3) *The discriminant locus  $D$  is a divisor with normal crossings in  $\mathcal{B}$ .*

**Example 1.30** Let  $(\mathbb{P}^1; Q_1, \dots, Q_N)$  be an  $N$ -pointed projective line (smooth curve of genus 0). By an automorphism of  $\mathbb{P}^1$ , the first three points  $Q_1, Q_2, Q_3$  can be mapped to the points  $0, 1, \infty$ . We may regard  $\mathbb{P}^1$  as  $\mathbb{C} \cup \{\infty\}$ . Let  $u$  be a global coordinate of  $\mathbb{C}$ . Thus,  $(\mathbb{P}^1; Q_1, \dots, Q_N)$  is isomorphic to  $(\mathbb{P}^1; 0, 1, \infty, u_1, \dots, u_{N-3})$ . Put

$$B_m = \{(u_1, \dots, u_m) \in \mathbb{C} \mid u_j \neq 0, 1, \infty, u_i \neq u_j, i \neq j\}.$$

For example,  $B_1 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ , and  $B_2 = \mathbb{C}^2 \setminus$  the lines in the figure 1.6.

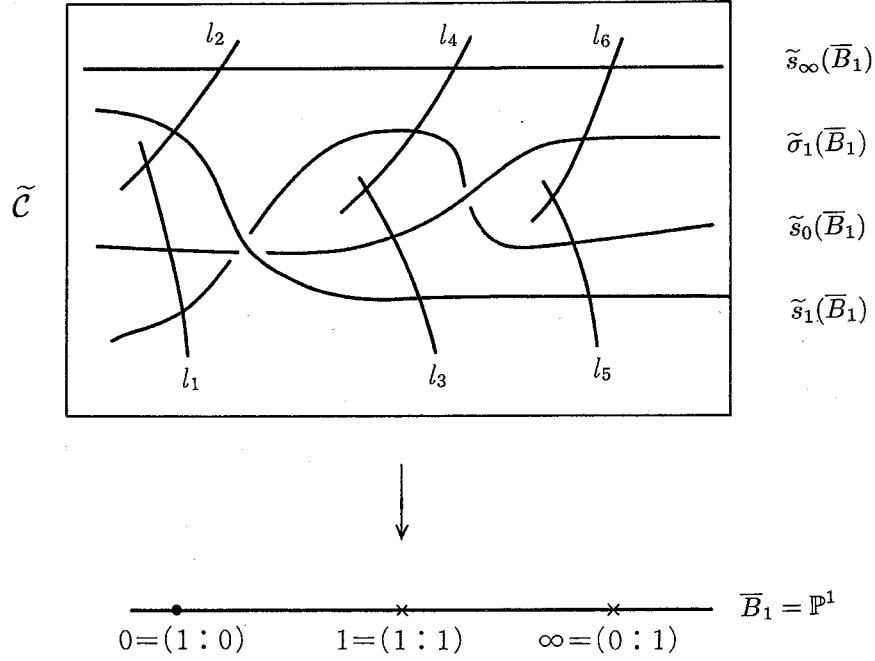


Figure 1.7 The universal family of stable four-pointed curves of genus 0.

Let  $\pi : \mathcal{C} = \mathbb{P}^1 \times B_m \rightarrow B_m$  be the projection to the second factor. For any point  $(u_1, \dots, u_m) \in B_m$  put

$$\begin{aligned} s_k((u_1, \dots, u_m)) &= k, \quad k = 0, 1, \infty \\ \sigma_i((u_1, \dots, u_m)) &= u_i, \quad i = 1, 2, \dots, m. \end{aligned}$$

Note that  $s_k$  and  $\sigma_i$  are holomorphic sections of  $\pi$ . Then  $B_m$  is the moduli space of ordered  $(m+3)$ -pointed projective lines and  $(\pi : \mathcal{C} \rightarrow B_m; s_0, s_1, s_\infty; \sigma_1, \dots, \sigma_m)$  is the universal family of ordered  $(m+3)$ -pointed projective lines. The extension of the universal family to that of stable  $(m+3)$ -pointed curves of genus 0 was first studied by Terada [T] in a different context. The detailed study of the moduli space of stable  $N$ -pointed curves of genus 0 was done by Gerritzen, Herrlich and van der Put [GHP]. In the case  $m=1$  the extension can be easily described. The compactification  $\overline{B}_1$  of  $B_1$  is  $\mathbb{P}^1$ . Let  $\tilde{\mathcal{C}}$  be the blowing up of  $\mathbb{P}^1 \times \overline{B}_1$  at the points  $(0,0)$ ,  $(1,1)$  and  $(\infty, \infty)$  and  $\tilde{\pi} : \tilde{\mathcal{C}} \rightarrow \overline{B}_1$  be the natural holomorphic mapping. Then there are natural extensions  $\tilde{s}_k$  and  $\tilde{\sigma}_i$  of  $s_k$  and  $\sigma_1$ , respectively, which are holomorphic sections of  $\tilde{\pi}$ . Then,  $(\tilde{\pi} : \tilde{\mathcal{C}} \rightarrow \overline{B}_1; \tilde{s}_0, \tilde{s}_1, \tilde{s}_\infty, \tilde{\sigma}_1)$  is the universal family of ordered stable four-pointed curves of genus 0 (1.30). The fibres of  $\tilde{\pi}$  over the compactified points  $0, 1, \infty$  are stable four-pointed curves of genus 0 with ordinary double point. This example will be considered in §5.4 below.

By Knudsen [Kn], the compactification  $\overline{B}_2$  of  $B_2$  by means of ordered five-pointed stable curves of genus 0 is nothing but our  $\tilde{\mathcal{C}}$ . For the explicit description by means of coordinates we refer the reader to [T] and [GHP].

### 1.4 Versal family of stable pointed curves

Let us consider a versal family  $\mathfrak{F} = (\pi : \mathcal{C} \rightarrow \mathcal{B}; s_1, s_2, \dots, s_N)$  of stable  $N$ -pointed curves of genus  $g$ . We assume that we have a local (formal) coordinate  $\eta$  with center  $s_N(\mathcal{B})$ . In the following we need to consider locally a family  $\mathfrak{F}$ . For that purpose we introduce the following local coordinates of  $\mathcal{C}$ .

For a point  $P \in \Sigma$  of the critical locus of  $\pi$ , we can choose local coordinates  $(u_1, u_2, \dots, u_{M-1}, z, w)$  of  $\mathcal{C}$  with center  $P$  and local coordinates  $(\tau_1, \tau_2, \dots, \tau_M)$  of  $\mathcal{C}$  with center  $\pi(P)$  such that the holomorphic mapping  $\pi$  is given by

$$(u_1, u_2, \dots, u_{M-1}, z, w) \longmapsto (u_1, u_2, \dots, u_{M-1}, zw) = (\tau_1, \tau_2, \dots, \tau_M).$$

In other words, we have

$$\pi^* \tau_k = \begin{cases} u_k, & k = 1, 2, \dots, M-1 \\ zw, & k = M. \end{cases}$$

For a point  $P \in \mathcal{C} \setminus \Sigma$  we can choose local coordinates

$$(u_1, u_2, \dots, u_M, z)$$

of  $\mathcal{C}$  with center  $P$  and local coordinates  $(\tau_1, \tau_2, \dots, \tau_M)$  of  $\mathcal{C}$  with center  $\pi(P)$  such that the holomorphic mapping is given by

$$(u_1, u_2, \dots, u_{M-1}, z) \longmapsto (u_1, u_2, \dots, u_M) = (\tau_1, \tau_2, \dots, \tau_M).$$

An  $\mathcal{O}_{\mathcal{C}}$ -module  $\Omega_{\mathcal{C}/\mathcal{B}}^1$  is defined by the exact sequence

$$\pi^{-1}\Omega_{\mathcal{B}}^1 \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{C}} \rightarrow \Omega_{\mathcal{C}}^1 \rightarrow \Omega_{\mathcal{C}/\mathcal{B}}^1 \rightarrow 0.$$

The sheaf  $\Omega_{\mathcal{C}/\mathcal{B}}^1$  is called the sheaf of germs of relative 1-forms of the family  $\pi : \mathcal{C} \rightarrow \mathcal{B}$ . Let us describe the sheaf  $\Omega_{\mathcal{C}/\mathcal{B}}^1$  by using the above local coordinates. In a neighbourhood of a point  $P \in \mathcal{C} \setminus \Sigma$ , the sheaf  $\Omega_{\mathcal{C}/\mathcal{B}}^1$  is locally isomorphic to  $\mathcal{O}_{\mathcal{C}} dz$ . In a small neighbourhood of  $P \in \Sigma$ , we have an  $\mathcal{O}_{\mathcal{C}}$ -module isomorphism

$$\Omega_{\mathcal{C}/\mathcal{B}}^1 \simeq (\mathcal{O}_{\mathcal{C}} dz + \mathcal{O}_{\mathcal{C}} dw) / \mathcal{O}_{\mathcal{C}}(wdz + zdw). \quad (1.57)$$

Moreover, we have the following lemma.

**Lemma 1.31** *The following sequence*

$$0 \rightarrow \pi^{-1}\Omega_{\mathcal{B}}^1 \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{C}} \rightarrow \Omega_{\mathcal{C}}^1 \rightarrow \Omega_{\mathcal{C}/\mathcal{B}}^1 \rightarrow 0 \quad (1.58)$$

*is exact and gives a locally free resolution of the sheaf  $\Omega_{\mathcal{C}/\mathcal{B}}^1$ .*

Let  $\omega_{\mathcal{C}/\mathcal{B}}$  be the relative dualizing sheaf of  $\pi : \mathcal{C} \rightarrow \mathcal{B}$ . Since  $\mathcal{C}$  and  $\mathcal{B}$  are non-singular and  $\pi$  is flat, we have an  $\mathcal{O}_{\mathcal{C}}$ -module isomorphism

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \omega_{\mathcal{C}} \otimes (\pi^* \omega_{\mathcal{B}}^{-1})$$

where  $\omega_Y$  is the dualizing sheaf (canonical sheaf) of a complex manifold  $Y$ . The relative dualizing sheaf  $\omega_{\mathcal{C}/\mathcal{B}}$  is described locally as follows. (See also (1.44).)

In a small neighbourhood of a point  $P \in \mathcal{C} \setminus \Sigma$ , we have

$$\omega_{\mathcal{C}/\mathcal{B}} = \Omega_{\mathcal{C}/\mathcal{B}}^1 \simeq \mathcal{O}_{\mathcal{C}} dz.$$

In a small neighbourhood of a point  $P \in \Sigma$ , we have

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}}(dz \wedge dw) \otimes (d\tau_M)^{-1}.$$



In particular, we have

$$\omega_{\mathcal{C}/\mathcal{B}} \simeq \begin{cases} \mathcal{O}_{\mathcal{C}} \frac{dz}{z} & \text{if } z \neq 0 \\ \mathcal{O}_{\mathcal{C}} \frac{dw}{w} & \text{if } w \neq 0 \end{cases}$$

with relation

$$\frac{dz}{z} + \frac{dw}{w} = 0$$

if  $zw \neq 0$ .

**Lemma 1.32** *There exists an exact sequence*

$$0 \rightarrow \Omega_{\mathcal{C}/\mathcal{B}}^1 \rightarrow \omega_{\mathcal{C}/\mathcal{B}} \rightarrow \omega_{\mathcal{C}/\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_{\Sigma} \rightarrow 0.$$

**Proof** The mapping  $\Omega_{\mathcal{C}/\mathcal{B}}^1 \rightarrow \omega_{\mathcal{C}/\mathcal{B}}$  is given locally in a neighbourhood of a point  $P \in \mathcal{C} \setminus \Sigma$  by

$$dz \mapsto dz$$

and in a neighbourhood of a point  $P \in \Sigma$  by

$$\begin{aligned} dz &\mapsto z(dz \wedge dw) \otimes (d\tau_M)^{-1} \\ dw &\mapsto w(dz \wedge dw) \otimes (d\tau_M)^{-1}. \end{aligned}$$

In particular, we have

$$\begin{aligned} dz &\mapsto z \frac{dz}{z} \quad \text{if } z \neq 0 \\ dw &\mapsto w \frac{dw}{w} \quad \text{if } w \neq 0. \end{aligned}$$

This proves Lemma 1.32. □

**Lemma 1.33** *Put*

$$\Theta_{\mathcal{C}/\mathcal{B}} = \underline{\text{Hom}}_{\mathcal{O}_{\mathcal{C}}}(\Omega_{\mathcal{C}/\mathcal{B}}^1, \mathcal{O}_{\mathcal{C}}). \quad (1.59)$$

*Then  $\Theta_{\mathcal{C}/\mathcal{B}}$  is an invertible  $\mathcal{O}_{\mathcal{C}}$ -module and there is an isomorphism*

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \underline{\text{Hom}}_{\mathcal{O}_{\mathcal{C}}}(\omega_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}). \quad (1.60)$$

*Hence,  $\Theta_{\mathcal{C}/\mathcal{B}}$  is an invertible sheaf.*

**Proof** By (1.58) it is easy to show that in a neighbourhood of a point  $P \in \mathcal{C} \setminus \Sigma$  we have an isomorphism

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}} \frac{\partial}{\partial z}$$

and in a neighbourhood of a point  $P \in \Sigma$  we have an isomorphism

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \mathcal{O}_{\mathcal{C}} \left( z \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} \right). \quad (1.61)$$

By this fact and (1.57), we have the desired result. □

From the exact sequence (1.58) we obtain the following Corollary.

**Corollary 1.34** *The following sequence*

$$0 \rightarrow \Theta_{\mathcal{C}/\mathcal{B}} \rightarrow \Theta_{\mathcal{C}} \rightarrow \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \rightarrow \underline{\text{Ext}}_{\mathcal{O}_{\mathcal{C}}}^1(\Omega_{\mathcal{C}/\mathcal{B}}^1, \mathcal{O}_{\mathcal{C}}) \rightarrow 0$$

*of  $\mathcal{O}_{\mathcal{C}}$ -modules is exact.*

**Lemma 1.35** *There exists an exact sequence*

$$0 \rightarrow \Theta_{\mathcal{B}}(-\log D) \rightarrow \Theta_{\mathcal{B}} \xrightarrow{t} \pi_* \underline{\text{Ext}}_{\mathcal{O}_{\mathcal{C}}}^1(\Omega_{\mathcal{C}/\mathcal{B}}^1, \mathcal{O}_{\mathcal{C}}) \rightarrow 0 \quad (1.62)$$

where

$$\Theta_{\mathcal{B}}(-\log D) = \{ v \in \Theta_{\mathcal{B}} \mid v(\mathcal{I}_D) \subset \mathcal{I}_D \}$$

and  $\mathcal{I}_D$  is the sheaf of defining ideals of  $D$  in  $\mathcal{B}$ .

**Proof** First note that the sheaf  $\Theta_{\mathcal{B}}(-\log D)$  is a sheaf of germs of vector fields on  $\mathcal{B}$  tangent to  $D$ . Since  $\pi : \mathcal{C} \rightarrow \mathcal{B}$  is a versal family, using the Kodaira-Spencer mapping and (1.53), for each point  $s \in \mathcal{B}$  we have an exact sequence

$$0 \rightarrow H^1(C_s, \Theta_{C_s}(-\sum_{j=1}^N s_j(s))) \rightarrow T_s \mathcal{B} \rightarrow H^0(C_s, \underline{\text{Ext}}_{\mathcal{O}_{C_s}}^1(\Omega_{C_s}^1, \mathcal{O}_{C_s})) \rightarrow 0.$$

Each element of  $H^1(C_s, \Theta_{C_s}(-\sum_{j=1}^N s_j(s)))$  corresponds to a tangent vector of  $\mathcal{B}$  at  $s$  preserving the singularities of  $C_s$ . Hence the sheaf version of the above sequence is also exact.  $\square$

**Theorem 1.36** *Let  $(\pi : \mathcal{C} \rightarrow \mathcal{B}; s_1, s_2, \dots, s_N)$  be a versal family of stable  $N$ -pointed curves of genus  $g$ . Then there exists an  $\mathcal{O}_{\mathcal{B}}$ -module isomorphism*

$$\rho : \Theta_{\mathcal{B}}(-\log D) \xrightarrow{\sim} R^1 \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \quad (1.63)$$

where we put  $S_j = s_j(\mathcal{B})$  and  $S = \sum_{j=1}^N S_j$ .

**Proof** Applying  $\underline{\text{Hom}}_{\mathcal{O}_{\mathcal{C}}}(\cdot, \mathcal{O}_{\mathcal{C}})$  to the exact sequence (1.58), we obtain the exact sequence

$$0 \rightarrow \Theta_{\mathcal{C}/\mathcal{B}} \rightarrow \Theta_{\mathcal{C}} \rightarrow \pi^{-1} \Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \rightarrow \underline{\text{Ext}}_{\mathcal{O}_{\mathcal{C}}}^1(\Omega_{\mathcal{C}/\mathcal{B}}^1, \mathcal{O}_{\mathcal{C}}) \rightarrow 0.$$

This exact sequence splits into the following short exact sequences:

$$0 \rightarrow \Theta_{\mathcal{C}/\mathcal{B}} \rightarrow \Theta_{\mathcal{C}} \xrightarrow{\kappa} \mathcal{M} \rightarrow 0. \quad (1.64)$$

$$0 \rightarrow \mathcal{M} \rightarrow \pi^{-1} \Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \rightarrow \underline{\text{Ext}}_{\mathcal{O}_{\mathcal{C}}}^1(\Omega_{\mathcal{C}/\mathcal{B}}^1, \mathcal{O}_{\mathcal{C}}) \rightarrow 0. \quad (1.65)$$

Let  $\mathcal{T}$  be the sheaf of germs of holomorphic vector fields on  $\mathcal{C}$  preserving  $n$ -th infinitesimal neighbourhoods. The sheaf  $\mathcal{T}$  is given by

$$\mathcal{T} = \{ v \in \Theta_{\mathcal{C}} \mid v(\mathcal{I}_S) \subset \mathcal{I}_S \}. \quad (1.66)$$

The sheaf  $\mathcal{T}$  is an  $\mathcal{O}_{\mathcal{C}}$ -submodule of  $\Theta_{\mathcal{C}}$  and coincides with  $\Theta_{\mathcal{C}}$  outside  $\bigcup_{j=1}^N S_j$ . For a point  $P \in S_j$  we let  $(u_1, u_2, \dots, u_M, z)$  be local coordinates of  $\mathcal{C}$  with center  $P$  such that  $(u_1, u_2, \dots, u_M)$  are the coordinates of  $\mathcal{B}$  with center  $\pi(P)$  and such that  $S_j$  is defined by the equation  $z = 0$  in a neighbourhood of  $P$ . Then, in a neighbourhood of  $P$  the sheaf  $\mathcal{T}$  is generated by

$$z \frac{\partial}{\partial z}, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_M}$$

as an  $\mathcal{O}_{\mathcal{C}}$ -module. Hence  $\mathcal{T}$  is locally free on  $\mathcal{C}$ .

Let us examine the exact sequences (1.64) and (1.65). Since the support of the sheaf  $\underline{\text{Ext}}_{\mathcal{O}_{\mathcal{C}}}^1(\Omega_{\mathcal{C}/\mathcal{B}}^1, \mathcal{O}_{\mathcal{C}})$  is in  $\Sigma$ , the sheaf  $\mathcal{M}$  is equal to  $\pi^{-1} \Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}}$  on  $\mathcal{O}_{\mathcal{C}} \setminus \Sigma$ . By using the above local coordinates of  $\mathcal{C}$  with center  $P \in S_j$ , the restriction of the mapping  $\kappa$  in (1.64) to  $\mathcal{T}$  in a neighbourhood of  $P$  is given by

$$a(u, z) z \frac{\partial}{\partial z} + \sum B_j(u, z) \frac{\partial}{\partial u_j} \mapsto \sum B_j(u, z) \frac{\partial}{\partial u_j}.$$

Hence  $\kappa : \mathcal{T} \rightarrow \mathcal{M}$  is surjective and its kernel is  $\Theta_{\mathcal{C}/\mathcal{B}}(-(n+1)S)$  in a neighbourhood of  $P$ . On the other hand, on  $\mathcal{B} \setminus \bigcup_{j=1}^N S_j$  the sheaf  $\mathcal{T}$  is equal to  $\Theta_{\mathcal{C}}$ . Thus we have an exact sequence

$$0 \rightarrow \Theta_{\mathcal{C}/\mathcal{B}}(-S) \rightarrow \mathcal{T} \rightarrow \mathcal{M} \rightarrow 0. \quad (1.67)$$

From the exact sequence (1.67) we obtain the long exact sequence

$$\begin{aligned} 0 \rightarrow \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \xrightarrow{\tau} \pi_*\mathcal{T} \rightarrow \pi_*\mathcal{M} \xrightarrow{\rho} R^1\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) \\ \rightarrow R^1\pi_*\mathcal{T} \rightarrow R^1\pi_*\mathcal{M} \rightarrow 0. \end{aligned} \quad (1.68)$$

Put  $\mathcal{B}_0 = \mathcal{B} \setminus D$ ,  $\mathcal{C}_0 = \pi^{-1}(\mathcal{B}_0)$ ,  $\pi_0 = \pi|_{\mathcal{C}_0}$ . Then on  $\mathcal{B}_0$ ,  $\pi_{0*}\mathcal{M} = \Theta_{\mathcal{B}}$  and the homomorphism  $\rho$  is the Kodaira-Spencer mapping by Corollary 1.26. Since our family is versal,  $\rho$  is an isomorphism on  $\mathcal{B}_0$ . Therefore, the sheaf homomorphism  $\tau$  in (1.68) is an isomorphism on  $\mathcal{B}_0$ . But on  $\mathcal{B}_0$  we have  $\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) = 0$ . Therefore,  $\pi_*\mathcal{T} = 0$  on  $\mathcal{B}_0$ . As  $\mathcal{T}$  is locally free,  $\pi_*\mathcal{T}$  is torsion-free, hence  $\pi_*\mathcal{T} = 0$  on  $\mathcal{B}$ . This also implies

$$\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S)) = 0 \quad (1.69)$$

on  $\mathcal{B}$ .

Next we show that  $\rho$  in (1.68) is an isomorphism. For that purpose it is enough to show that  $R^1\pi_*\mathcal{T}$  is locally free. Because, if  $R^1\pi_*\mathcal{T}$  is locally free, as  $\rho$  is an isomorphism on  $\mathcal{B}_0$ , Coker  $\rho$  is a torsion subsheaf of  $R^1\pi_*\mathcal{T}$ , hence zero. By the cohomology theory of coherent sheaves,

$$\chi(\mathcal{T} \otimes \mathcal{O}_{C_s}) = \dim_{\mathbb{C}} H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) - \dim_{\mathbb{C}} H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s})$$

is independent of  $s \in \mathcal{B}$ , where  $C_s = \pi^{(n)-1}(s)$ . (See, for example, [BS].) Moreover, if  $\dim_{\mathbb{C}} H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s})$  is independent of  $s$ , say  $k$ , since we have  $H^2(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0$ ,  $R^1\pi_*\mathcal{T}$  is a locally free  $\mathcal{O}_{\mathcal{B}}$ -module of rank  $k$  on  $\mathcal{B}$ . Therefore, it is enough to show that  $H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0$  for all  $s \in \mathcal{B}$ .

Since  $C_s$  is a locally complete intersection in  $\mathcal{C}$ , we have an exact sequence

$$0 \rightarrow \Theta_{C_s} \rightarrow \Theta_{\mathcal{C}} \otimes \mathcal{O}_{C_s} \rightarrow \mathcal{O}_{C_s}(N) \rightarrow \underline{Ext}_{\mathcal{O}_{C_s}}^1(\Omega_{\mathcal{O}_{C_s}}^1, \mathcal{O}_{C_s}) \rightarrow 0$$

where  $N$  is the normal bundle of  $C_s$  in  $\mathcal{C}$  which is a trivial bundle of rank  $3g-3+N$ . (See, for example, [Ar].) From this exact sequence we obtain two short exact sequences

$$\begin{aligned} 0 \rightarrow \Theta_{C_s} \rightarrow \Theta_{\mathcal{C}} \otimes \mathcal{O}_{C_s} \rightarrow M_s \rightarrow 0, \\ 0 \rightarrow M_s \rightarrow \mathcal{O}_{C_s}(N) \rightarrow \underline{Ext}_{\mathcal{O}_{C_s}}^1(\Omega_{\mathcal{O}_{C_s}}, \mathcal{O}_{C_s}) \rightarrow 0. \end{aligned}$$

Similarly as above we have an exact sequence

$$0 \rightarrow \Theta_{C_s}(-\sum_{j=1}^N Q_j) \rightarrow \mathcal{T} \otimes \mathcal{O}_{C_s} \rightarrow M_s \rightarrow 0,$$

where  $Q_j = s_j(s)$ . This gives a long exact sequence

$$\begin{aligned} 0 &= H^0(C_s, \Theta_{C_s}(-\sum Q_j)) \rightarrow H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) \rightarrow H^0(C_s, M_s) \\ &\xrightarrow{\rho} H^1(C_s, \Theta_{C_s}(-\sum Q_j)) \rightarrow H^1(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}). \end{aligned} \quad (1.70)$$

The cohomology group  $H^0(C_s, M_s)$  parameterizes infinitesimal displacements of  $C_s$  in  $\mathcal{C}$ . (For the details see Tsuboi [Ts].) Since  $\pi : \mathcal{C} \rightarrow \mathcal{B}$  is a versal family,

infinitesimal displacements of  $C_s$  in  $\mathcal{C}$  and infinitesimal deformations of  $C_s$  coincide. Hence the homomorphism  $\rho$  in (1.70) is an isomorphism. Hence we have

$$H^0(C_s, \mathcal{T} \otimes \mathcal{O}_{C_s}) = 0.$$

Finally we shall show that  $\pi_*\mathcal{M}$  is isomorphic to  $\Theta_{\mathcal{B}}(-\log D)$ . From (1.65) we obtain an exact sequence

$$0 \rightarrow \pi_*\mathcal{M} \rightarrow \Theta_{\mathcal{B}} \xrightarrow{t} \pi_*(\underline{\text{Ext}}_{\mathcal{O}_{\mathcal{C}}}^1(\Omega_{\mathcal{C}/\mathcal{B}}^1, \mathcal{O}_{\mathcal{C}})).$$

The homomorphism  $t$  is the same as the one appearing in the exact sequence (1.62). Hence  $t$  is surjective. Therefore, by Lemma 1.35  $\pi_*\mathcal{M}$  is isomorphic to  $\Theta_{\mathcal{B}}(-\log D)$ .  $\square$

**Remark 1.37** The homomorphism  $\rho$  in the above Theorem 1.36 is also called the *Kodaira-Spencer mapping*. The above proof shows that there exists an exact sequence

$$0 \rightarrow \Theta_{\mathcal{C}/\mathcal{B}}(-\sum S_j) \rightarrow \mathcal{T} \rightarrow \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}} \rightarrow \underline{\text{Ext}}_{\mathcal{O}_{C_s}}^1(\Omega_{C_s}^1, \mathcal{O}_{C_s}) \rightarrow 0$$

where  $\mathcal{T}$  is the subsheaf of  $\Theta_{\mathcal{C}}$  defined in (1.66). Choose a small Stein open set  $\mathcal{U} \subset \mathcal{B}$  and a vector field  $v \in H^0(\mathcal{U}, \Theta_{\mathcal{B}}(-\log D))$ . Choose also a Stein open covering  $\{\mathcal{U}_j\}_{j \in J}$  of  $\pi^{-1}(\mathcal{U})$ . Then  $v$  also defines an element

$$\pi^*v \in H^0(\mathcal{U}_j, \pi^{-1}\Theta_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{C}}),$$

whose image in  $\underline{\text{Ext}}_{\mathcal{O}_{C_s}}^1(\Omega_{C_s}^1, \mathcal{O}_{C_s})$  is zero, since the tangent vector  $v$  is a direction of an infinitesimal deformation preserving singularities. Therefore, if  $\mathcal{U}_j$  is small enough, we can find an element  $v_j \in H^0(\mathcal{U}_j, \mathcal{T})$  which is mapped to  $\pi^*v$ . Then we have

$$v_{ij} = v_j - v_i \in H^0(\mathcal{U}_i \cap \mathcal{U}_j, \Theta_{\mathcal{C}/\mathcal{B}}(-S))$$

and  $\{v_{ij}\}$  defines an element

$$[\{v_{ij}\}] \in H^1(\pi^{-1}(\mathcal{U}), \Theta_{\mathcal{C}/\mathcal{B}}(-S)).$$

The mapping

$$v \longmapsto [\{v_{ij}\}]$$

is nothing but the Kodaira-Spencer mapping  $\rho$  in Theorem 1.36.

**Lemma 1.38** *Let  $\mathfrak{F} = (\pi : \mathcal{C} \rightarrow \mathcal{B}; s_1, s_2, \dots, s_N)$  be a versal family of  $N$ -pointed smooth curves of genus  $g$ . Let  $X, Y$  be holomorphic vector fields of  $\mathcal{B}$  and*

$$\rho : \Theta_{\mathcal{B}} \rightarrow R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S)$$

*be the Kodaira-Spencer mapping. Then we have*

$$\rho([X, Y]) = [\rho(X), \rho(Y)] + X(\rho(Y)) - Y(\rho(X))$$

*where by putting*

$$\rho(X) = \{\theta_{\lambda\mu}\}, \quad \rho(Y) = \{\tau_{\lambda\mu}\}$$

*$\rho([X, Y])$  is a cocycle  $\{\vartheta_{\lambda\mu}\}$  defined as*

$$\vartheta_{\lambda\mu} = [\theta_{\lambda\mu}, \tau_{\lambda\mu}]$$

*and  $X(\rho(Y))$  is a cocycle  $\{\tau'_{\lambda\mu}\}$  defined as*

$$\tau'_{\lambda\mu} = X(\tau_{\lambda\mu}).$$

**Proof** We may assume that  $\mathcal{B}$  is small enough. Let  $\{U_{\lambda\mu}\}$  be a sufficiently fine open covering of  $\mathcal{C}$ . Let  $(z_1, \dots, z_m)$  be coordinates of  $\mathcal{B}$  and  $(z_1, \dots, z_m, \xi_\lambda)$  be local coordinates of  $U_\lambda$ . Moreover, we may assume that for  $j \in \{1, 2, \dots, N\} \subset \Lambda$ ,  $U_j$  is a coordinate neighbourhood of  $s_j(\mathcal{B})$  such that  $\xi_j$  induces an  $n$ -th infinitesimal neighbourhood  $\eta_j$  and that  $s_j(\mathcal{B})$  is defined by  $\xi_j = 0$ . For  $U_\lambda \cap U_\mu \neq \emptyset$ , we have

$$\xi_\lambda = h_{\lambda\mu}(\xi_\mu, z_1, \dots, z_m).$$

The vector field  $X$  induces an infinitesimal transformation of the coordinates  $(z_1, \dots, z_m)$ :

$$(z_1, \dots, z_m) \mapsto (z_1 + \epsilon_1 a_1(z), \dots, z_m + \epsilon_1 a_m(z))$$

where  $\epsilon_1$  is a dual number, that is,  $\epsilon_1^2 = 0$  and

$$X = \sum_{j=1}^m a_j(z) \frac{\partial}{\partial z_j}.$$

For simplicity we express (1.4) as  $z \mapsto z + \epsilon_1 X$ . Since our family is versal, we have

$$g_{\lambda\mu}(\xi_\mu, z + \epsilon_1 X) = g_{\lambda\mu}(\xi_\mu, z) + \epsilon_1 \ell_{\lambda\mu}(\xi_\mu, z)$$

where we put

$$\begin{aligned} \theta_{\lambda\mu} &= \ell_{\lambda\mu} \frac{d}{d\xi_\mu} \\ \tau_{\lambda\mu} &= m_{\lambda\mu} \frac{d}{d\xi_\mu}. \end{aligned}$$

Similarly we have

$$g_{\lambda\mu}(\xi_\mu, z + \epsilon_2 Y) = g_{\lambda\mu}(\xi_\mu, z) + \epsilon_2 m_{\lambda\mu}(\xi_\mu, z)$$

where  $\epsilon_2$  is another dual number. Now let us calculate the result when first we deform infinitesimally in the direction of  $X$  then deform infinitesimally in the direction of  $Y$ :

$$\begin{aligned} &g_{\lambda\mu}(\xi_\mu, z + \epsilon_2 Y) + \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda + \epsilon_2 m_{\lambda\mu}, z + \epsilon_2 Y) \\ &= g_{\lambda\mu}(\xi_\mu, z) + \epsilon_2 m_{\lambda\mu}(\xi_\lambda, z) + \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda, z) + \epsilon_1 \epsilon_2 (Y(\ell_{\lambda\mu}(\xi_\lambda, z)) + m_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda}). \end{aligned}$$

If we deform in the opposite order, we have

$$g_{\lambda\mu}(\xi_\mu, z) + \epsilon_1 \ell_{\lambda\mu}(\xi_\lambda, z) + \epsilon_2 m_{\lambda\mu}(\xi_\lambda, z) + \epsilon_1 \epsilon_2 (X(m_{\lambda\mu}(\xi_\lambda, z)) + \ell_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda}).$$

Therefore, if we put

$$\rho([X, Y]) = \{\tilde{\vartheta}_{\lambda\mu}\}$$

we have

$$\begin{aligned} \tilde{\vartheta}_{\lambda\mu} &= \left\{ \ell_{\lambda\mu} \frac{dm_{\lambda\mu}}{d\xi_\lambda} - m_{\lambda\mu} \frac{d\ell_{\lambda\mu}}{d\xi_\lambda} + X(m_{\lambda\mu}) - Y(\ell_{\lambda\mu}) \right\} \frac{d}{d\xi_\lambda} \\ &= [\theta_{\lambda\mu}, \tau_{\lambda\mu}] + \{X(\tau_{\lambda\mu}) - Y(\ell_{\lambda\mu})\} \frac{d}{d\xi_\lambda} \\ &= [\rho(X), \rho(Y)] + X(\rho(Y)) - Y(\rho(X)). \end{aligned}$$

□