Preface

This monograph contains my lecture notes for a graduate course on function theory at the Fields Institute in Toronto, Ontario, Canada, that met January through March 2008. In three months of classes, 4 hours a week, we covered Chapters 2, 3 and most of 4, and reviewed roughly half of the material in the appendices.

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Our goal in these lectures is to investigate that part of the theory of spaces of holomorphic functions on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ arising from interpolation and corona problems, with of course special attention paid to the disk $\mathbb{D} = \mathbb{B}_1$. Our approach will be to introduce the diverse array of techniques used in the theory in the simplest settings possible, rather than to produce an encyclopedic summary of the achievements to date. The reader we have in mind is a relative novice to these topics who wishes to learn more about the field. Basic real and complex analysis is assumed as well as the theory of the Poisson integral in the unit disk (see e.g. Chapter 11 in [37]). Preliminary material in

- functional analysis (open mapping theorem, closed graph theorem, Hahn-Banach and Banach-Alaoglu Theorems, spectral theory for compact operators and the Fredholm alternative),
- the theory of Sobolev spaces (weak derivatives and embedding theorems) and maximal functions (Lebesgue's Differentiation Theorem and the John-Nirenberg inequality),
- the theory of H^p spaces on the unit disk (F. and M. Riesz Theorem, factorization theorems and Beurling's Theorem on the shift operator), and
- spectral theory of a normal operator on a Hilbert space

is included in Appendices A, B, C and D respectively.

The main topics reached in the lectures themselves include

- interpolating sequences for classical function spaces and their multiplier algebras originating with Carleson's characterization for $H^2(\mathbb{D})$ and its multiplier algebra $H^\infty(\mathbb{D})$, and continuing with the characterization for the Dirichlet space $\mathcal{D}(\mathbb{B}_n)$ and its multiplier algebra in several dimensions,
- corona problems for classical function algebras originating with Carleson's Corona Theorem for $H^\infty(\mathbb{D})$, continuing with those for $H^\infty(\mathbb{D}) \cap \mathcal{D}(\mathbb{D})$ and the multiplier algebra for $\mathcal{D}(\mathbb{D})$, and concluding with the Toeplitz Corona Theorem and some partial results toward corona theorems for certain function spaces on the unit ball $\mathbb{B}_n$,
- an introduction to the theory of Toeplitz and Hankel operators, Fefferman's duality of H^1 and BMO, and the problem of best approximation by analytic functions in the uniform norm, and
- Hilbert space methods and the Nevanlinna-Pick property.

There are four main threads interwoven in these lecture notes. The first two follow the development of *interpolation* and *corona* theorems respectively in the past half century, beginning with the pioneering works of Lennart Carleson [13], [14]. We progress through Carleson's original proof of *interpolation* using Blaschke products and duality, followed by the constructive proof of Peter Jones, and ending with a purely Hilbert space proof. We give Gamelin's variation on Wolff's proof of Carleson's *Corona* Theorem, followed by corona theorems for other algebras using the theory of best approximation in the L^∞ norm, the boundedness of the Beurling transform, and estimates on solutions to the $\overline{\partial}$ problem.

The third thread developed here is the use of *trees* in the analysis of spaces of holomorphic functions [4], [5], [6], [7]. In the disk, trees are related to the well known Haar basis of $L^2(\mathbb{T})$ on the circle $\mathbb{T}$. In higher dimensions a "dirty" construction is required and then used to characterize Carleson measures and interpolation in some cases.

The fourth thread is the (complete) *Nevanlinna-Pick property*, a property shared by many classical Hilbert function spaces including the Dirichlet and Drury-Arveson spaces. The magic weaved by this property is evident in both interpolation and corona problems: a sequence is interpolating for a Hilbert space with the NP property if and only if it is interpolating for its multiplier algebra; a Hilbert space with the *complete* NP property has the baby corona property if and only if its multiplier algebra has no corona.

We will touch on only a small portion of the present literature on interpolation and corona problems. Rather than listing the vast number of currently relevant papers not considered here, we urge the reader to conduct an online search.

In order to limit the complexity of notation and proofs we will keep to the Hilbert space case $p = 2$, with only occasional comments on extensions to $1 < p < \infty$. However, much of the material presented here can be extended to $p \neq 2$ and there is a striking similarity between the results in the Hilbert and Banach space cases, despite the sometimes very different techniques used when $p \neq 2$ and the NP property is unavailable. This raises an important problem:

Problem 0.1 Find an analogue of the complete Nevanlinna-Pick machinery in the world of Banach spaces.

Finally, we have attempted to present the main body of the lectures, appearing here as Chapters 2, 3 and 4, in complete detail referencing only the appendices. Chapters 1, 5 and 6 will occasionally reference the literature as well.

Remark 0.2 Two results related to these lectures have been obtained by participants during and shortly after the program at the Fields Institute. First, N. Arcozzi, R. Rochberg, E. Sawyer and B. Wick [8] obtained a Nehari theorem for the Dirichlet space $\mathcal{D}$ on the disk by showing that the bilinear form $T_b(f, g) = \langle fg, b \rangle_{\mathcal{D}}$ is bounded on $\mathcal{D} \times \mathcal{D}$ if and only if $|b'(z)|^2 \, dxdy$ is a Carleson measure for $\mathcal{D}$, thus resolving an old conjecture of R. Rochberg. Second, S. Costea, E. Sawyer and B. Wick [18] proved the corona theorem for the Drury-Arveson space H_n^2 when $n > 1$.