CHAPTER 1

The Bernstein Decomposition and the Bernstein Centre

Alan Roche (University of Oklahoma)

Introduction

What is the Bernstein decomposition? In rough terms, it expresses the category $\mathfrak{R}(G)$ of smooth complex representations of a reductive p-adic group G as the product of certain indecomposable full subcategories, often called the components of $\mathfrak{R}(G)$.

The primary reference is [Ber84] where the decomposition forms a crucial step in a description of the centre of the abelian category $\mathfrak{R}(G)$. This is the theory of the Bernstein centre which we also discuss below. The notes [BR92] present a somewhat different approach to both the Bernstein decomposition and the theory of the Bernstein centre. These have very strongly influenced the account presented here. In particular, we directly follow [BR92] in expressing each component of $\mathfrak{R}(G)$ as a module category via the construction of suitable progenerators. (The necessary elementary categorical algebra is reviewed in the body of the chapter.) This breaks into two cases, the cuspidal and the non-cuspidal. The cuspidal case is elementary. The non-cuspidal case, however, is far from elementary. It relies principally on a deep result of Bernstein, often referred to as the Second Adjoint Theorem. I have omitted the long and non-trivial proof of this theorem. A full account, following the approach of [BR92], is to appear in a forthcoming book by David Renard [Ren]. (Note we cannot appeal to Bushnell's alternative proof of the Second Adjoint Theorem [Bus01] because of its dependence on [Ber84] — see Remark 1.8.1.4 below.)

Working within the associated module categories in the two cases, cuspidal and non-cuspidal, we recover Bernstein's description of the centre of $\mathfrak{R}(G)$. Here the treatment of the cuspidal case roughly parallels [BR92] but the precise route we have taken was prompted by [BH03, Section 8]. In the non-cuspidal case, our treatment diverges from [BR92] and is closer in spirit to [Ber84].

The introductions to the individual sections give an overview of the stages (and detours) on the way to the final decomposition and the description of the Bernstein centre. In Section 1.2 and in parts of Sections 1.3 and 1.4, I have borrowed heavily from my notes from a beautiful set of lectures given by Robert Kottwitz at the University of Chicago in the early nineties. Finally, I am grateful to David Renard for alerting me to a gap in a preliminary version of this article.

1.1 Notation and some preliminaries

Let G be a locally profinite group and let $\mathfrak{R}(G)$ denote the category of smooth (complex) representations of G. We write $\mathcal{H}$ for the space $C_c^\infty(G)$ of locally constant compactly supported $\mathbb{C}$-valued functions on G. Under convolution with respect to a left Haar measure on G (which we fix once and for all), $\mathcal{H}$ becomes an associative $\mathbb{C}$-algebra. In the usual way, $\mathfrak{R}(G)$ can be identified with the category of non-degenerate left $\mathcal{H}$-modules and we do this routinely below.

¿From Section 1.4 on, $G = \mathbb{G}(F)$ where F is a non-Archimedean local field and $\mathbb{G}$ is a connected reductive algebraic F-group. In this context, we follow standard abuses of notation and terminology and refer, for example, to parabolic subgroups of G in place of the F-points of F-parabolic subgroups of $\mathbb{G}$.

If R is a ring with identity, then, as usual, we write $\mathfrak{Mod}\text{-}R$ for the category of (unital) right R-modules.

For H a subgroup of G and for $g \in G$, we write ${}^gH = gHg^{-1}$ throughout. Similarly, for any representation ρ of H, we write ${}^g\rho$ for the representation of ${}^gH$ given by ${}^g\rho(x) = \rho(g^{-1}xg)$, for $x \in {}^gH$.

For ease of reference, we collect here some straightforward generalities about compact induction from open subgroups.

1.1.1 Let H be an open subgroup of G and let (ρ, W) be a smooth representation of H. Throughout this Chapter, we write $\mathrm{ind}_H^G\,\rho$ for the resulting compactly induced representation. Thus the space of $\mathrm{ind}_H^G\,\rho$ consists of all functions $f : G \to W$ such that

1. $f(hg) = \rho(h)f(g)$, for all $h \in H$ and all $g \in G$,
2. $\mathrm{supp}\, f$, the support of f, is compact modulo H.

Of course, since H is open, (2) simply says that f is supported on a finite union of H-cosets. The group G acts by right translations.

The resulting functor $\mathrm{ind}_H^G : \mathfrak{R}(H) \to \mathfrak{R}(G)$ satisfies a form of Frobenius reciprocity which we now recall. For $w \in W$, we write f_w for the element of $\mathrm{ind}_H^G\,\rho$ such that $\mathrm{supp}\, f_w = H$ and $f_w(1) = w$. Let τ be a smooth representation of G. Consider the maps

$$\begin{aligned} s \mapsto \psi_s &: \mathrm{Hom}_G(\mathrm{ind}_H^G\,\rho, \tau) \to \mathrm{Hom}_H(\rho, \tau \mid H), \\ \psi \mapsto s_\psi &: \mathrm{Hom}_H(\rho, \tau \mid H) \to \mathrm{Hom}_G(\mathrm{ind}_H^G\,\rho, \tau), \end{aligned} \tag{1.1.1.1}$$

given by

$$\begin{aligned} \psi_s(w) &= s(f_w), \quad w \in W, \\ s_\psi(f) &= \sum_{x \in G/H} \tau(x)\psi(f(x^{-1})), \quad f \in \mathrm{ind}_H^G\,\rho. \end{aligned}$$

One checks readily that these are natural inverse isomorphisms. Indeed, by direct calculation, $\psi_{s_\psi} = \psi$. In the same way, $s_{\psi_s}(f_w) = s(f_w)$, for all $w \in W$, whence $s_{\psi_s} = s$ (as the elements f_w, for $w \in W$, generate $\mathrm{ind}_H^G\,\rho$). We omit the straightforward proof that the given maps are natural. Thus $(\mathrm{ind}_H^G, \mathrm{res}_H^G)$ is an adjoint pair where res_H^G denotes the restriction functor from $\mathfrak{R}(G)$ to $\mathfrak{R}(H)$.

CHAPTER 2

Bruhat-Tits Theory and Buildings

Jiu-Kang Yu (Purdue University)

Introduction

This article originated from slides used for a mini-course on Bruhat-Tits theory in a Fields institute workshop in 2004, which were slightly edited and expanded shortly after the workshop, then again in 2008. The informal style of the original lectures is perserved. Some imprecisions and mistakes were fixed but some more probably remain.

Ahead of the workshop we recommended that the participants study Tits' summary of Bruhat-Tits theory in the Corvallis proceedings [Tit79]. This summary has been the portal to Bruhat-Tits theory for most people, and it will continue to be the best user guide. It is very well-written and precise. It does an excellent job of hiding the technicalities and of describing everything in terms of elegant abstract characterizations.

The goal of my lectures is to provide more hints and help to people who want to use Bruhat-Tits theory. I tried to be as orthogonal to Tits' summary as possible. Therefore, I do not give a systematic account of the theory itself. In the first lecture, I provide some background materials that are probably most useful if you learn them before starting to read Tits' article. In the second lecture, I give diverse complements to Tits' article. From my experience, these should be helpful to someone who is currently studying Tits' article. In particular, I try to explain how to go between [Tits] and [BT1-5]. For the third lecture, I discuss more recent developments in representation theory and "functoriality" of buildings that go beyond Tits' article.

2.1 Lecture 1

2.1.1 History and Literature.

Prehistory

— O. Goldman and N. Iwahori: *The theory of p-adic norms*, Acta Math. **109** (1963), 41 pages [GI63].
— N. Iwahori and H. Matsumoto: *On some Bruhat decomposition and the structure of the Hecke ring of* $\mathfrak{p}$*-adic Chevalley groups*, Publ. IHES. **25** (1965), 44 pages [IM65].
— H. Hijikata: *On the arithmetic of* $\mathfrak{p}$*-adic Steinberg groups*, Mimeographed notes at Yale University (1964) [Hij64].

The beginning

It all started from Bruhat's article in the Proceedings of the Boulder conference (1965) "Algebraic groups and discontinuous subgroups"[1]

p-adic Groups

BY

FRANÇOIS BRUHAT

1. **Bounded subgroups.** If G is a real connected Lie group, then the following two statements are well known:

(1) Any compact subgroup of G is contained in a maximal compact subgroup of G.

(2) Two maximal compact subgroups are conjugate by an inner automorphism.

[skipping to the end of the article]

Added in November 1965. During the conference, considerable progress was made towards an affirmative solution of the conjectures above. It also appeared that the properties thus established have interesting applications; for instance, they provide a simplified approach to Kneser's theorem on H^1 of simply connected groups over the p-adics. A joint paper on this subject is in preparation, by F. Bruhat and J. Tits.

These results were exposed orally by J. Tits at the conference. The precise form on which they are given in the mimeographed notes of his talk must however be somewhat modified; in particular, it is not true that minimal k-parahoric subgroups of a group G—as defined in these notes—are conjugate by elements of G_k. In fact, the notion of k-parahoric subgroup given there does not appear to be "the good one" when G does not split over an unramified extension of k.

On the other hand, the methods sketched there turn out to give further results. For instance, it can be shown that the Conjecture (II) (iv) above is essentially a consequence of the other parts of that conjecture and, in particular, is true in the split case.

BIBLIOGRAPHY

1. F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des representations des groupes p-adiques*, Bull. Soc. Math. France **89** (1961), 43–75.

Announcements by Bruhat-Tits

- *Groupes algébriques simples sur un corps local*, Proceedings of a Conference on Local Fields (Driebergen, 1966), 14 pages [BT67].
- *BN-paires de type affine et données radicielles*, C.R. Acad. Sci. Paris **263** (1966), 4 pages [BT66a].
- *Groupes simples résiduellement déployés sur un corps local*, ibid., 3 pages [BT66b].
- *Groupes algébriques simples sur un corps local*, ibid., 4 pages [BT66c].
- *Groupes algébriques simples sur un corps local: cohomologie galoisienne, décompositions d'Iwasawa et de Cartan*, ibid., 3 pages [BT66d].

Tits' summary

Tits It is [Tit79] in the Corvallis proceedings (1977) [BC79]. A must read.

[1]Reprinted from [Bru66], with permission.

CHAPTER 4

Character Theory of Reductive p-adic Groups

Paul J. Sally, Jr. (University of Chicago) and
Loren Spice (University of Michigan)

Introduction

There are three basic topics in the harmonic analysis on reductive p-adic groups. They are:

1. representation theory;
2. character theory and asymptotics; and
3. Fourier analysis.

The goal is to learn enough about (1) and (2) to get to (3). It seems that people often lose their way.

In this lecture, we address (2). We get some help from [HC99] and the matching theorem (Theorem 4.5). In Sections 4.1–4.3, we discuss the history of character computations. In Sections 4.4 and 4.5, we give two explicit examples.

Throughout, F is a p-adic field of characteristic zero — that is, a finite extension of $\mathbb{Q}_p$ — and G is the set of F-rational points of a connected reductive F-group. (The characteristic-zero restriction is not always necessary, but is convenient for our purposes.) For example, G could be $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$. We write dg for a Haar measure on G, and $C_c^\infty(G)$ for the space of locally constant, compactly supported, complex-valued functions on G.

We recall some basic terminology. A *representation* of G is a pair (π, V) of a complex vector space V and a homomorphism π from G into $\mathrm{GL}(V)$. We do not equip V with a topology. Thus, for example, when we speak of irreducible representations, we mean those that have no non-zero, proper invariant subspaces at all, not just no closed such subspaces.

It turns out to be useful for many purposes to study the restrictions of representations of G to compact, open subgroups. If K is such a subgroup, then we write V^K for the space of K-fixed vectors in V. We say that the representation is *smooth* if every vector in V lies in some such V^K. Smoothness is so essential that we will usually forget to mention it, collapsing "smooth representation" to just "representation".

The (smooth) representation (π, V) is said to be *admissible* if V^K is finite dimensional for every compact, open subgroup K of G. It is an important theorem of Harish-Chandra and Jacquet that every irreducible (smooth) representation of G is admissible.

Let (π, V) be an irreducible admissible representation of G. If $f \in C_c^\infty(G)$, then

$$\pi(f) := \int_G f(g)\pi(g)dg \in \mathrm{End}_{\mathbb{C}}(V) \tag{4.1}$$

is of finite rank and hence of trace class.

Theorem 4.1 (Harish-Chandra [HC70a]) *There exists a locally integrable class function Θ_π on G, locally constant on the regular set, such that*

$$\hat{f}(\pi) := \mathrm{tr}\,\pi(f) = \int_G f(g)\Theta_\pi(g)dg.$$

Moreover, $g \mapsto |D(g)|^{1/2}\Theta_\pi(g)$ is bounded on G.

The function Θ_π is called the *character* of (π, V). Here, D is the usual discriminant on G. The *regular (semisimple) set* G^{reg} is the set of $x \in G$ for which $D(x) \neq 0$. For example, if $G = \mathrm{GL}_n(F)$, then $D(x)$ is a non-zero multiple of the product of the differences of the eigenvalues of x, and the regular set is the set of matrices with n distinct non-zero eigenvalues.

Our basic philosophy is that *characters tell all.* Such was the case for real groups, but, for p-adic groups, who knows?

First, note that the space $C_c^\infty(G)$ is invariant under conjugation by elements of G, in the following sense. For $f \in C_c^\infty(G)$ and $g \in G$, we define $f^g(x) = f(gxg^{-1})$ for $x \in G$. Then $f^g \in C_c^\infty(G)$. A *distribution* on G is a linear functional on $C_c^\infty(G)$. The distribution Λ is *invariant* if $\Lambda(f^g) = \Lambda(f)$ for all $f \in C_c^\infty(G)$ and $g \in G$.

Let $\mathcal{E}(G)$ be the space of (equivalence classes of) irreducible unitary representations of G. To quote Harish-Chandra [HC70b],

> The central question in harmonic analysis may now be formulated as follows. *Given an invariant distribution Λ on G, how to express it as a "linear combination" of the characters Θ_π ($\pi \in \mathcal{E}(G)$)?* In fact, to every such Λ, we would like to associate a "distribution" $\hat{\Lambda}$ on $\mathcal{E}(G)$ in such a way that $\Lambda(f) = \hat{\Lambda}(\hat{f})$ for all $f \in C_c^\infty(G)$. (Here by a "distribution" on $\mathcal{E}(G)$, we mean some sort of linear functional on a suitable space of functions on $\mathcal{E}(G)$.) If δ is the Plancherel measure, such that $\delta(f) = f(1)$ ($f \in C_c^\infty(G)$), the determination of $\hat{\delta}$ is just the problem of the explicit Plancherel formula for G.

The distribution $\hat{\Lambda}$ is called the *Fourier transform* of Λ. One of the principal goals of harmonic analysis on reductive p-adic groups is the determination of these Fourier transforms.

It now appears that it is possible to write down the Plancherel formula without knowing the characters, even though the characters appear in the Plancherel formula. Recall that the Plancherel formula may be written as follows:

$$f(1) = \sum_{\pi \in \mathcal{E}_2(G)} d(\pi)\hat{f}(\pi) + \begin{pmatrix} \text{integral over the} \\ \text{rest of the} \\ \text{tempered dual} \end{pmatrix},$$

where $\mathcal{E}_2(G)$ is the discrete series of G and the rest of the tempered spectrum consists of subquotients of representations parabolically induced from the discrete series of Levi subgroups. Recall that $d(\pi)$ is the formal degree of the discrete-series representation π.

CHAPTER 6

Notes on the Local Langlands Program

Paul Mezo (Carleton University)

Preface

A previous version of these notes served as a companion to the mini-course on the local Langlands program given in the Fields Institute workshop on the representation theory of p-adic groups at the University of Ottawa in 2004. Some errors have been removed from this version, but there are doubtlessly others which have not been detected. Many thanks to the organizers and participants of the workshop for such a rewarding experience.

6.1 The Galois group

6.1.1 Definitions and examples. Let F be a perfect field, so that every extension field of F is separable. An extension field of F is called *Galois* if it is algebraic and normal. Another way of looking at a Galois extension is as a splitting field of countably many polynomials over F. Given a Galois extension K of F, one defines the *Galois group* $\mathrm{Gal}(K/F)$ as the group of automorphisms of K which fix the elements of F. There are a few examples we should be familiar with:

1. Suppose E is a finite normal extension of F. Then E is Galois and $\mathrm{Gal}(E/F)$ is the usual Galois group of order $[E:F]$.
2. The algebraic closure $\overline{F}$ of F is a Galois extension of infinite order. Its Galois group $\mathrm{Gal}(\overline{F}/F)$ is called the *absolute Galois group* of F. Observe that $\overline{K} = \overline{F}$ and $\mathrm{Gal}(\overline{F}/K)$ is a subgroup of $\mathrm{Gal}(\overline{F}/F)$ for any finite extension K of F.
3. An *abelian extension* is a Galois extension field whose Galois group is abelian. Suppose K_1 and K_2 are abelian extensions of F. Then the Galois group of their compositum $\mathrm{Gal}(K_1K_2/F)$ is a subgroup of $\mathrm{Gal}(K_1/F) \times \mathrm{Gal}(K_2/F)$ and is hence also abelian. This allows one to define a *maximal abelian extension* F^{ab} of F.
4. Let p be a prime number, $n \geq 1$ and $\mathbb{F}_{p^n}$ be the finite field of order p^n. It is the splitting field of $X^{p^n} - X$ over $\mathbb{F}_p$. Clearly, the characteristic of $\mathbb{F}_{p^n}$ is p and

$$|\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)| = [\mathbb{F}_{p^n} : \mathbb{F}_p] = n.$$

   A simple application of the binomial theorem shows that the *Frobenius automorphism* defined by

$$\mathrm{Fr}_p(x) = x^p, \ x \in \mathbb{F}_{p^n},$$

is indeed a field automorphism of $\mathbb{F}_{p^n}$. Since the order of the Frobenius automorphism is n, the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is the cyclic group generated by Fr_p.

5. Let $\mathbb{Q}_p$ be the field of p-adic numbers. Then the unique *unramified* extension K of degree n is the splitting field of the polynomial $X^{p^n} - X$ (see [Cas86, Ch. 8, §2]). $\mathrm{Gal}(K/\mathbb{Q}_p)$ is a cyclic group of order n and one of its generators acts as the Frobenius automorphism on the residue field $\mathbb{F}_p$. This determines a unique isomorphism $\mathrm{Gal}(K/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Obviously, K is an abelian extension of $\mathbb{Q}_p$.
6. We refer to a finite extension of $\mathbb{Q}_p$ as a $\mathfrak{p}$-*adic* field. The notions of the previous example generalize to arbitrary $\mathfrak{p}$-adic fields [Cas86, Ch. 8, §2]. Since the compositum of two unramified extensions is again unramified ([Lan94, II, §4, Prop. 8]) it makes sense to define the *maximal unramified extension* E_u of E. It is an abelian extension of infinite degree. In particular $E^{ab} \supset E_u$.

6.1.2 Profinite realizations of $\mathrm{GL}(\overline{F}/F)$**.** Suppose K and E are Galois extensions of F, and $K \supset E \supset F$. Then there the restriction map defined by

$$\sigma \mapsto \sigma_{|E}, \ \ \sigma \in \mathrm{Gal}(K/F)$$

is a homomorphism of $\mathrm{Gal}(K/F)$ onto $\mathrm{Gal}(E/F)$. The *inverse limit* (or *projective limit*) over the finite Galois extensions $K \supset F$ is defined as

$$\varprojlim_K \mathrm{Gal}(K/F) = \left\{ (\sigma_K) \in \prod_K \mathrm{Gal}(K/F) : (\sigma_K)_{|E} = \sigma_E \text{ whenever } K \supset E \right\}.$$

Clearly, this inverse limit is a group and for each finite Galois extension E of F there is a surjective projection $p_E : \varprojlim \mathrm{Gal}(K/F) \to \mathrm{Gal}(E/F)$.

We may define a map from $\mathrm{Gal}(\overline{F}/F)$ to $\varprojlim \mathrm{Gal}(K/F)$ by sending each $\sigma \in \mathrm{Gal}(\overline{F}/F)$ to the collection of restrictions $(\sigma_{|K})$ to finite Galois extensions. This map is a group isomorphism ([RZ00, Th. 2.11.1]). We shall often identify these two groups.

It is instructive to work out some of these ideas for the absolute Galois group of $\mathbb{F}_p$. The finite extensions of $\mathbb{F}_p$ are the fields $\mathbb{F}_{p^n}$ where n runs over the positive integers. We know that $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is generated by Fr_p and has order n. Therefore

$$\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \varprojlim_n \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}. \qquad (6.1.2.1)$$

The image of the Fr_p under this isomorphism is $(1, 1, 1, \ldots)$. The group $\hat{\mathbb{Z}}$ is isomorphic to the direct product $\prod_p \mathbb{Z}_p$ of the p-adic integers (Exercise 6.1.4.2). We may therefore think of the Frobenius automorphism as the element $(1, 1, 1, \ldots)$ of $\prod_p \mathbb{Z}_p$.

6.1.3 Topologies. Regarding $\mathrm{Gal}(\overline{F}/F)$ as an inverse limit, it is apparent that it is a subset of the direct product over the finite Galois extensions. The obvious choice of topology for each of the finite Galois groups appearing in this direct product is the discrete topology and the obvious choice of topology for the direct product is the product topology. We topologize $\mathrm{Gal}(\overline{F}/F)$ using the relative topology induced by the product topology.

It is left as an exercise to show that $\mathrm{Gal}(\overline{F}/F)$ is a closed subset with respect to the ambient product topology. In fact it is a *compact* subset as it is a closed set of a compact space (by Tychonoff's theorem).

There is another way to define a topology on $\mathrm{Gal}(\overline{F}/F)$ which does not rely on inverse limits. As $\mathrm{Gal}(\overline{F}/F)$ is a group, any topology is determined uniquely by a neighbourhood base at the identity. We take this neighbourhood base to be the set of Galois groups $\mathrm{Gal}(\overline{F}/K)$, where K runs over the finite Galois extensions of F. The resulting topology is called the *Krull topology.*

We now have two topologies on the absolute Galois group: the Krull topology and the relative topology. The former topology has an algebraic flavour and the latter a set-theoretic flavour. The two topologies are identical ([RZ00, Th. 2.11.1]) despite their dissimilar definitions. We will henceforth refer to both topologies as the Krull topology.

There are distinct advantages to the perspectives of both definitions of the Krull topology. The algebraic perspective is more favourable in the proof of the following generalization of the Fundamental Theorem of Galois Theory to infinite extensions ([RZ00, Th. 2.11.3]).

Theorem 6.1.3.1 (Krull) *The map sending each* closed *subgroup of* $\mathrm{Gal}(\overline{F}/F)$ *to the subfield of elements fixed by the group defines a bijection onto the subfields of* $\overline{F}$ *containing* F*. Moreover a closed subgroup is normal if and only if its fixed field is a Galois extension.*

When proving results of a topological nature, the set-theoretic perspective is preferable. We have already seen this when showing that $\mathrm{Gal}(\overline{F}/F)$ is compact. The set-theoretic perspective is also more convenient when determining how the Krull topology transfers through the isomorphisms $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$. One may define a neighbourhood base of the identity in $\prod_n \mathbb{Z}/n\mathbb{Z}$ by

$$\left\{(x_i) \in \prod_n \mathbb{Z}/n\mathbb{Z} : x_i = 0 \bmod n,\ i \in I\right\},\ I \text{ finite.}$$

Restricting these sets to $\hat{\mathbb{Z}}$ furnishes a neighbourhood base for the identity in $\hat{\mathbb{Z}}$. Under the isomorphism $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$, this neighbourhood base maps to the sets

$$\left\{(y_p) \in \prod_p \mathbb{Z}_p : y_p = 0 \bmod p^{\alpha_p},\ p \in J\right\},\ J \text{ finite},\ \alpha_p \geq 0. \tag{6.1.3.1}$$

This neighbourhood base determines a topology which is the product topology on $\prod_p \mathbb{Z}_p$ with the usual p-adic topology on each $\mathbb{Z}_p$.

Lemma 6.1.3.2 *The cyclic subgroup of* $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ *generated by the Frobenius automorphism is a proper dense subgroup.*

Proof In view of the preceding observations, we shall identify $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ with $\prod_p \mathbb{Z}_p$ as topological groups. Under this identification, the Frobenius element is $(1, 1, 1, \ldots)$. To prove the lemma, it therefore suffices to show that for a fixed set in the neighbourhood base (6.1.3.1) there exists $m \in \mathbb{Z}$ such that $m = 0 \bmod p^{\alpha_p}$, for all $p \in J$. This is a simple consequence of the Chinese Remainder Theorem. This proves the density of the subgroup. It must be proper as $\prod_p \mathbb{Z}_p$ is not isomorphic to $\mathbb{Z} \cong \langle(1, 1, 1, \ldots)\rangle$. □

6.1.4 Exercises.

1. Prove that the positive integers are partially ordered by divisibility.
2. Prove that the groups $\hat{\mathbb{Z}}$ and $\prod_p \mathbb{Z}_p$ are isomorphic. (*Hint:* If n has prime factorization $p_1^{a_1} \cdots p_k^{a_k}$ then $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}$.)
3. Prove that $\mathrm{Gal}(F^{ab}/F)$ is isomorphic to the inverse limit of the Galois groups of the finite abelian extensions of F
4. Suppose F is a $\mathfrak{p}$-adic field and prove that $\mathrm{Gal}(F_u/F)$ is isomorphic to the inverse limit of the Galois groups of the finite unramified extensions of F.
5. Suppose that F is a $\mathfrak{p}$-adic field whose residue field has order q. Prove that $\mathrm{Gal}(F_u/F) \cong \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ as topological groups. Conclude that $\mathrm{Gal}(F_u/F) \cong \prod_p \mathbb{Z}_p$ as topological groups.
6. Suppose K is a Galois extension of F. Prove that the restriction map $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(K/F)$ has kernel $\mathrm{Gal}(\overline{F}/K)$.
7. Prove the following:
   (a) Every open subgroup of a topological group is also closed.
   (b) Every open subgroup of $\mathrm{Gal}(\overline{F}/F)$ has finite index.
   (c) The fixed field of an open subgroup of $\mathrm{Gal}(\overline{F}/F)$ is a finite extension of F whose degree is equal to the index of the subgroup.

6.2 Local abelian class field theory

The central result of abelian class field theory is Artin reciprocity. The reciprocal relationship of this result is between a field and the Galois groups of its abelian extensions. There is a global as well as a local formulation of Artin reciprocity. They are complementary, but we shall only describe the local $\mathfrak{p}$-adic formulation. From now on we take F to be a $\mathfrak{p}$-adic field, that is, a finite extension of $\mathbb{Q}_p$.

6.2.1 Local Artin reciprocity. We begin with a version of Artin reciprocity stated in terms of finite abelian extensions.

Theorem 6.2.1.1 *Suppose L is a finite abelian extension of F. Then there is a surjective homomorphism*

$$(\cdot\,, L/F) : F^\times \to \mathrm{Gal}(L/F)$$

with kernel equal to the norm $N_{L/F}(L^\times)$ of $L^\times$ in $F^\times$. In addition, the map defined by $L \mapsto N_{L/F}(L^\times)$ is a bijection between the finite abelian extensions of F and the open subgroups of finite index in $F^\times$.

The homomorphism $(\cdot\,, L/F)$ of this theorem is called the *local (Artin) reciprocity map.* Its definition for an arbitrary finite abelian extension is somewhat roundabout as it has either a cohomological ([Ser67, §2.2, Th. 2]) or global ([Lan94, XI, §4]) foundation. In the specific case that L is an *unramified* finite abelian extension, the local reciprocity map is trivial on the group of units $\mathcal{O}_F^*$, and $(\varpi, L/F)$ is the unique element of $\mathrm{Gal}(L/F)$ which acts as the Frobenius automorphism on the residue field for any uniformizer ϖ of F ([Ser67, §2.5]). For an arbitrary finite abelian extension L we have

$$(\mathcal{O}_F^*, L/F) = \mathrm{Gal}(L/K), \qquad (6.2.1.1)$$

where K is the largest unramified extension of F contained in L ([Ser67, Th. 3 (d)]).