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Preface

The five lectures recorded in this volume were first given in January, 1998 as part of the *Miniprogram in Geometry and Duality* at the Institute for Theoretical Physics in Santa Barbara. Shortly thereafter they were repeated at the Institute for Advanced Study (IAS) in Princeton. The aim of the lectures is to introduce basic concepts of supersymmetry to mathematicians. The beautiful applications of supersymmetry in mathematics all require ideas in quantum mechanics and quantum field theory which have heretofore proved difficult to render mathematically rigorous. By contrast, the mathematics which underlies these lectures is finite dimensional geometry and algebra. Our main goal is to expose the classical supersymmetric field theories which are the basis for the applications. Hopefully we provide the reader with enough background to explore the quantum ideas.

I learned much of this material during the 1996–1997 Special Year at the IAS, which is documented in the volumes *Quantum Fields and Strings: A Course for Mathematicians*. Some of the mathematical applications of supersymmetry mentioned above are discussed there. A more detailed treatment of most topics covered in these lectures may also be found in those books.

It is a pleasure to thank Pierre Deligne, with whom I wrote the detailed texts on which these lectures are based, and who contributed many comments about these notes. My understanding of this subject was shaped by many lectures and discussions. I single out Joseph Bernstein, David Kazhdan, John Morgan, Nathan Seiberg, Edward Witten. Sheldon Katz and Jeff Rabin made useful suggestions after reading a preliminary version, as did an anonymous referee. Finally, Arthur Greenspoon did his usual stellar job proofreading and making an index. I warmly thank them all.

I applaud the Institute for Theoretical Physics, in particular David Gross, for encouraging the interaction between mathematics and theoretical high energy physics. Thanks also to Karen Uhlenbeck for enlisting me to repeat these lectures in Princeton. My stay at the IAS, where these notes were written, was supported by the National Science Foundation and the J. Seward Johnson Sr. Charitable Trust. I am grateful for the support of these organizations.

Dan Freed
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Introduction

Symmetry principles play a large role in both classical and quantum mechanics. In mechanics symmetries give rise to the conservation laws. Indeed, the conservation of energy and momentum in a classical system follow from invariance under basic motions of time and space. In a relativistic quantum system these motions are part of the Poincaré group, and elementary particles are classified according to its irreducible unitary representations of positive energy. Particles fall into two classes—bosons and fermions—and traditional symmetry principles do not mix the two classes. *Supersymmetries* are symmetries which exchange bosons and fermions. They were introduced into physics in the mid-'70s. A supersymmetric theory is more constrained than a nonsupersymmetric one, and in recent years that has led to great advances in the understanding of supersymmetric quantum field theories and string theories, particularly theories with a large amount of supersymmetry where the constraints are most stringent.

These advances in theoretical physics have had important mathematical consequences as well. Perhaps most spectacular was the advent of the Seiberg-Witten invariants in four dimensional geometry. These invariants contain all of the information encoded in Donaldson invariants, but are much easier to work with and exhibit many new features. The connection between the Seiberg-Witten invariants and Donaldson invariants has yet to be made mathematically rigorous, but in physics the precise link is derived by knowing the behavior of a certain supersymmetric gauge theory. Physics ideas played a particularly important role here since the mathematical ideas which developed independently of physics did not produce the Seiberg-Witten equations. In a different direction, new invariants arising from supersymmetric physics in two dimensions have led to advances in symplectic topology and enumerative algebraic geometry. Applications of supersymmetry to mathematics have impacted other parts of algebraic geometry, as well as differential geometry, representation theory, topology, and other fields. To be sure, there are mathematical ramifications of quantum field theory which do not involve supersymmetry, but it is safe to say that supersymmetry will continue to play an important role in the interaction with theoretical physics.

The five lectures recorded here are offered as background to understanding the physics behind these mathematical applications. We stop far short of explaining the quantum ideas. Rather, we restrict ourselves to material which is easily accessible—that is to say, provable—using present-day mathematics. In part, this is because an account of the quantum ideas aimed at mathematicians has just appeared (see References). But mostly we feel that the basic ideas of classical supersymmetric field theory are unfamiliar to most mathematicians, and one cannot go deeply into the quantum theory without a firm grasp of these elementary ideas.

The first few lectures begin with some very general ideas, but then later lectures move on to more specific examples so that by the end the mathematical reader may feel lost in a zoo of unfriendly barely distinguishable rare beasts. We mathematicians love general concepts which we can apply to a wide range of examples. Think, for example, of groups, topological spaces, and vector bundles, just to name a few. Supersymmetric field theories are the antithesis: they are tied to specific dimensions, they depend on special properties of low dimensional Lie groups, and there are few examples. We have tried to emphasize the general ideas as much as possible, but a bit of taxonomy is inevitable. In a few places we give an expository overview of some physical ideas. Readers who have struggled through physics articles or seminars may benefit from these descriptions, but the reader unfamiliar with quantum field theory is well-advised to skip them at first reading.

Lecture 1, *What Are Fermions?*, explains a mathematical framework which includes bosons and fermions. While they exist quite democratically in the quantum world, the classical "odd" fields whose quantization give fermionic particles have a very different geometric significance than usual even fields. We adopt an algebro-geometric approach which we feel gives the right geometric intuition, allows the development of effective computational algorithms, and matches the existing physics literature. References to other approaches appear at the end of the lectures.

The space of states of a classical mechanical system is a symplectic manifold. Observables are functions on the state space, and via the symplectic structure an observable generates motion. For many important classical field theories that structure is summarized in a single expression, the lagrangian. In Lecture 2, *Lagrangians and Symmetries*, we explain how to pass from the lagrangian to the state space and how to construct the observables which correspond to symmetries. When we incorporate the odd fields of Lecture 1, the same framework applies to supersymmetries. The examples in this lecture are all taken from mechanics; only in subsequent lectures do we move on to fields.

Some general features of supersymmetric field theories are laid out in Lecture 3, *Supersymmetry in Various Dimensions*. We define the super Poincaré group and discuss its positive energy irreducible unitary representations, which are the particle multiplets of a supersymmetric quantum field theory. In this lecture we also survey the landscape of supersymmetric theories, introducing the reader to the specific features of different dimensions and different types of theories. At the end of the lecture we explain how to pass from free classical fields to the corresponding quantum particles.

In Lecture 4, *Theories with Two Supersymmetries*, we turn from a general description to an in-depth look at one particular example. Here we meet a familiar friend from geometry—the gradient flow equation—which is now reinterpreted in the context of supersymmetry.

We encounter more old geometric friends in Lecture 5, *Theories with More Supersymmetry*—Kähler manifolds and their quotients, hyperkähler manifolds and their quotients—and also make new friends—for example, special Kähler manifolds. This last lecture is aimed at the reader who aims to seriously study the physics. It highlights features of more complicated theories than those considered previously, and includes a detailed computation illustrative of the many which we omit.

As befits a series of lecture notes, there are no references in the main text and no comprehensive bibliography. Rather, we include a brief guide for further reading and encourage the reader to consult the bibliographies in those texts.