Preface

When I started graduate school at Harvard in 1960, I knew essentially no abstract algebra. I had seen the definition of a group when I was an undergraduate, but I doubt that I had ever seen a factor group, and I am sure that I had never even heard of modules. Despite my ignorance of algebra (or perhaps because of it), I decided to register for the first half of the graduate algebra sequence, which was being taught that year by Professor Lynn Loomis. I found the course exciting and beautiful, and by the end of the semester I had decided that I wanted to be an algebraist. This decision was reinforced by an equally spectacular second semester.

I have now been teaching mathematics for more than a quarter-century, and I have taught the two-semester first-year graduate algebra course many times. (This has been mostly at the University of Wisconsin, Madison, but I also taught parts of the corresponding courses at Chicago and at Berkeley.) I have never forgotten Professor Loomis's course at Harvard, and in many ways, I try to imitate it. Loomis, for example, used the first semester mostly for noncommutative algebra, and he discussed commutative algebra in the second half of the course. I too divide the year this way, which is reflected in the organization of this book: Part 1 covers group theory and noncommutative rings, and Part 2 deals with field theory and commutative rings.

The course that I took at Harvard "sold" me on algebra, and when I teach it, I likewise try to "sell" the subject. This affects my choice of topics, since I seldom teach a definition, for example, unless it leads to some exciting (or at least interesting) theorem. This philosophy carries over from my teaching into this book, in which I have tried to capture as well as I can the "feel" of my lectures. I would like to make my students and my readers as excited about algebra as I became during my first year of graduate school.

Students in my class are expected to have had an undergraduate algebra course in which they have seen the most basic ideas of group theory, ring theory, and field theory, and they are also assumed to know elementary linear algebra and matrix theory. Most important, they should be comfortable with mathematical proofs; they should know how to read them, invent them, and write them. I do not require, however, that my students actually remember the theorems or even the definitions from their undergraduate algebra courses. Given my own lack of preparation as a first-year graduate student, I am well aware that a few in my audience may be completely innocent of algebra, and I want to conduct the course so that a student such as I was can enjoy it. But this is a graduate course, and it would not be fair to the majority to go on endlessly with "review" material. I resolve this contradiction by making my presentation complete: giving all definitions and basic results, but I do this quickly, and I intersperse the review material with ideas that very few of my audience have seen before. I have attempted to do the same in this book.

By the end of a year-long graduate algebra course, a good student is ready to go more deeply into one or more of the many branches of algebra. She or he might enroll in a course in finite groups, algebraic number theory, ring theory, algebraic geometry, or any of a number of other specialized topics. While I do not pretend that this book would be suitable as a text for any of these second-year courses, I have attempted to include some of the important material from many of them. I hope that this provides a convenient way for interested readers to sample a number of these topics without having to cope with the somewhat inconsistent notations and different assumptions about readers' backgrounds that are found in the various specialized books. No attempt has been made, however, to designate in the text which chapters and sections are first-year material and which are second; this is simply not well defined. Lecturers who teach from this book undoubtedly do not agree on what, precisely, should be the content of a first-year course. In addition to providing opportunities for students to sample advanced topics, the additional material here should provide some flexibility for instructors to construct a course compatible with their own tastes. Also, those with very well-prepared and capable students might elect to leave out much of the "easy stuff" and build a course consisting largely of what I think of as "second-year" topics.

Since it is impossible, in my opinion, to cover all of the material in this book in a two-semester course, some topics must be skipped, and others might be assigned to the students for independent reading. Perhaps it would be useful for me to describe the content of the course as I teach it at Madison.

I cover just about all of the first four chapters on basic group theory, and I do most of Chapter 5 on the Sylow theorems and p-groups, although I omit Theorem 5.27 and Section 5D on Brodkey's theorem. I do Chapter 6 on symmetric and alternating groups except for Section 6D. In Chapter 7, I cover direct products, but I omit Theorem 7.16 and Section 7C on semidirect products. In Chapter 8 on solvable and nilpotent groups, I omit most of Sections 8C and 8D and all of 8F. I do present the Frattini argument (8.10) and the most basic definitions and facts about nilpotent groups. Chapter 9 on transfer theory I skip entirely.

Chapters 10 and 11 on operator groups are a transition between group theory and module theory. I cover Sections 10A and 10B on the Jordan-Hölder theorem for operator groups, but I omit Section 10C on the Krull-Schmidt theorem. I cover

Sections 11A and 11B on chain conditions, but I touch 11C only lightly. I discuss Zorn's lemma (11.17), but I do not present a proof.

Chapter 12 begins the discussion of ring theory, and some readers may feel that there is a downward "jump discontinuity" in the level of sophistication at this point. As in Chapters 1 and 2, where the definitions and most basic properties of groups are presented (reviewed), it seems that here too it is important to give clear definitions and discussions of elementary properties for the sake of those few readers who may not be comfortable with this material. Section 12A could be assigned as independent reading, but I usually go over it quickly in class. I cover all of Chapter 12 in my course. I do Sections 13A and 13B on the Jacobson radical completely, but sometimes I skip Section 13C on the Jacobson density theorem. (I often find that I am running short of time when I get here.) I do Sections 13D and 14A and as much of 14B on the Wedderburn-Artin theorems as I have time for, and I omit the rest of Chapter 14 and all of Chapter 15.

In the second semester, I start with Chapter 16, and I cover almost all of that, except that I go lightly over Section 16D. I construct fraction fields for domains, but I do not discuss localization more generally. Chapters 17 and 18 discuss basic field extension theory and Galois theory; I cover them in their entirety. I discuss Section 19A on separability, but usually I do only a small amount of 19B on purely inseparable extensions, and I skip 19C. I cover Sections 20A and 20B on cyclotomic extensions, but I skip 20C and go very quickly over 20D on compass and straightedge constructions. In Chapter 21 on finite fields, I cover only Sections 21A and 21D: the basic material and the Wedderburn theorem on finite division rings. In Chapter 22, I omit Sections 22C and 22E, but, of course, I cover 22A and 22B on the solvability of polynomials thoroughly, and I present the fundamental theorem of algebra in 22D. In Chapter 23, I do only Sections 23A, 23B, and 23C, discussing norms and traces, Hilbert's Theorem 90, and a very rudimentary introduction to cohomology. I generally skip Chapter 24 on transcendental extensions completely, and I almost completely skip Chapter 25. (I may mention the Artin-Schreier theorem, but I never discuss formally real fields.)

Chapter 26 begins the discussion of the ideal theory of commutative rings. I cover the first two sections, but I skip Section 26C on localization. In Chapter 27 on noetherian rings, I usually cover only the first two sections and seldom get as far as 27C on the uniqueness of primes in the Lasker-Noether theorem. (I wish that I could discuss Krull's results on the heights of prime ideals in Section 27E, but it seems impossible to find the time to do that.) In Chapter 28 on integrality, I cover only the first three sections. I try to cover at least Section 29A, giving the basic properties of Dedekind domains, but often I find that I must skip Chapter 29 entirely because of time pressure. I always leave enough time, however, to prove Hilbert's Nullstellensatz in Section 30A, and that completes the course.

The user of this book will choose what to read (or teach) and what to skip, but I, as the author, was forced to make other choices. For most of these, there were arguments in both directions, and I am certain that very few will agree with all of my decisions, and perhaps I cannot even hope for a majority agreement on each of them separately. I elected not to include tensor products, for example, because there just didn't seem to be much interesting that one could say about them without going deeply either into the theory of simple algebras or into homological algebra. Somewhat similarly, I decided not to discuss injective modules. It would have taken considerable effort just to prove that they exist in most cases, and there did not seem much that one could do with them without going into areas of ring theory beyond what I wished to discuss. Also, I did not discuss fully the characterization of finitely generated modules over PIDs, but I did include what seem to be the two most important special cases: the fundamental theorem for finite abelian groups and the fact that torsion-free, finitely generated modules over PIDs are free.

Some of my inclusions (for example, the Berlekamp factorization algorithm and character theory) may seem too specialized for a book of this sort. I discussed the factorization algorithm for polynomials over finite fields, for instance, mostly because I think it is a really slick idea, but also because computer algebra software has become widely available recently, and it seemed that students of "theoretical" algebra ought to receive at least a glimpse of the sort of algorithms that underlie these programs. I included some character theory (Chapter 15 and part of Chapter 28) partly because it provides nice applications of some of the theory, but also, I must admit, because that is my own primary research interest.

I also had to make decisions about notation. I suspect that the most controversial are those concerning functions and function composition. To me (and I think to most group theorists) it seems more natural to write "fg" rather than "gf" to denote the result of doing first function f and then function g. It did not seem wise to use left-to-right composition in the group theory chapters and then to switch and use right-to-left notation in the rest of the book, and so I decided to make consistency a high priority. Since I am a group theorist (and group theory is the first topic in the book), I elected to use left-to-right composition everywhere.

Customarily, when functions are composed left-to-right, the name of the function is written to the right of the argument. A critic of my left-to-right composition challenged my claim to consistency by betting that I did not write $(x)\sin$ and $(y)(d/dx)$ to denote the sine of x or the derivative of y with respect to x. He was right, of course. Like most mathematicians, I write most functions on the left. Nevertheless, I always (in this book at least) compose left-to-right. By my notational scheme, therefore, the following silly looking equation is technically correct: $f(g(x)) = (gf)(x)$. In order to avoid such strange looking formulas, I take the view that the function name may be written on *either* side of its argument, whichever is most convenient at the moment. In contexts where function composition is important, I nearly always choose to put the function name on the right, but I am perfectly comfortable in writing $\sin(x)$, since we are not generally interested in compositions of trigonometric functions. No information is lost by allowing the same function to appear sometimes on the left and sometimes on the right because the composition rule is always left-to-right and is independent of how the function is written.

Another of my decisions that will not meet with universal approval is that by my definition, rings have unity elements. There are a few places where this is not a good idea: one cannot conveniently state the theorem, for example, that a right artinian "ring" with no nonzero nilpotent ideals must have a unity. Most of the time, however, assuming the existence of a unity is a convenience, and so we have built it into the definition. We have also required in the definition of a module that the unity of the ring act as an identity on the module.

Whatever notational scheme one adopts, it is important that students learn of the existence of common alternatives; how else could they read the literature or attend courses from other lecturers? For this reason, I have attempted to mention competing notations and definitions whenever appropriate in the text.

At the end of each chapter, there is a fairly extensive list of problems. Few of these are routine exercises, and some of them I consider to be quite difficult. The purpose of these problems is not just to give practice with the definitions and with understanding the theorems. My hope is that by working these problems, students will get a feeling of what it is actually like to *do* algebra, and not just to learn it. (I should mention that when I teach my algebra course, I assign five problems per week.)

This is not a "scholarly" book; I have not attempted to trace back to their sources the various definitions, lemmas, theorems, and ideas presented here. I have credited items to individuals in cases such as the "Sylow theorems," the "Jacobson radical," and the "Hilbert basis theorem" where such attribution is standard and well known and in other situations where it seemed appropriate. Even in these cases, however, I have not given bibliographic references to the original sources.

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Let me close this preface by expressing my hope that this book will engender, in some readers at least, the same excitement and love for algebra that I received from Professor Loomis in my first year of graduate school.

I. Martin Isaacs
Madison, WI
1992