S-Mod is faithfully exact if and only if $\mathscr{F}(M) \neq 0$ for every nonzero R-module M, if and only if $\mathscr{F}(\varphi) \neq 0$ for every nonzero morphism φ in R-Mod.

1.25. ¬ Prove that localization (Exercise 1.4) is an *exact* functor.

In fact, prove that localization 'preserves homology': if

$$M_\bullet : \cdots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

is a complex of R-modules and S is a multiplicative subset of R, then the localization $S^{-1}H_i(M_\bullet)$ of the i-th homology of $M_\bullet$ is the i-th homology $H_i(S^{-1}M_\bullet)$ of the localized complex

$$S^{-1}M_\bullet : \cdots \longrightarrow S^{-1}M_{i+1} \xrightarrow{S^{-1}d_{i+1}} S^{-1}M_i \xrightarrow{S^{-1}d_i} S^{-1}M_{i-1} \longrightarrow \cdots$$

[2.12, 2.21, 2.22]

1.26. Prove that localization is faithfully exact in the following sense: let R be a commutative ring, and let

$$(*) \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a sequence of R-modules. Then (*) is exact if and only if the induced sequence of $R_\mathfrak{p}$-modules

$$0 \longrightarrow A_\mathfrak{p} \longrightarrow B_\mathfrak{p} \longrightarrow C_\mathfrak{p} \longrightarrow 0$$

is exact for every prime ideal $\mathfrak{p}$ of R, if and only if it is exact for every maximal ideal $\mathfrak{p}$. (Cf. Exercise V.4.12.)

1.27. ▷ Let R, S be rings. Prove that right-adjoint functors R-Mod $\to$ S-Mod are left-exact and left-adjoint functors are right-exact. [§1.5]

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2. Tensor products and the Tor functors

In the rest of the chapter we will work in the category R-Mod of modules over a *commutative* ring R. Essentially everything we will see can be upgraded to the noncommutative case without difficulty, but a bit of structure is lost in that case. For example, if R is not commutative, then in the category R-Mod of *left-R*-modules the Hom-sets $\mathrm{Hom}_{R\text{-Mod}}(M, N)$ are 'only' abelian groups (cf. the end of §III.5.2). A tensor product $M \otimes_R N$ can only be defined if M is a right-R-module and N is a left-R-module (in a sense, the two module structures annihilate each other, and what is left is an abelian group). By contrast, in the commutative case we will be able to define $M \otimes_R N$ simply as an R-module. In general, the theory goes through as in the commutative case if the modules carry compatible left- *and* right-module structures, except in questions such as the commutativity of tensors, where it would be unreasonable to expect the commutativity of R to have no bearing. All in all, the commutative case is a little leaner, and (we believe) it suffices in terms of conveying the basic intuition on the general features of the theory.

Thus, R will denote a fixed commutative ring, unless stated otherwise.

2.1. Bilinear maps and the definition of tensor product. If M and N are R-modules, we observed in the distant past (§III.6.1) that $M \oplus N$ serves as both the *product* and *coproduct* of M and N: a situation in which a limit coincides with a colimit. As a set, $M \oplus N$ is just $M \times N$; the R-module structure on $M \oplus N$ is defined by componentwise addition and multiplication by scalars. An R-module homomorphism

$$M \oplus N \to P$$

is determined by R-module homomorphisms $M \to P$ and $N \to P$ (this is what makes $M \oplus N$ into a coproduct).

But there is *another* way to map $M \times N$ to an R-module P, compatibly with the R-module structures.

Definition 2.1. Let M, N, P be R-modules. A function $\varphi : M \times N \to P$ is *R-bilinear* if

- $\forall m \in M$, the function $n \mapsto \varphi(m, n)$ is an R-module homomorphism $N \to P$,
- $\forall n \in N$, the function $m \mapsto \varphi(m, n)$ is an R-module homomorphism $M \to P$.⌟

Thus, if $\varphi : M \times N \to P$ is R-bilinear, then $\forall m \in M$, $\forall n_1, n_2 \in N$, $\forall r_1, r_2 \in R$,

$$\varphi(m, r_1 n_1 + r_2 n_2) = r_1 \varphi(m, n_1) + r_2 \varphi(m, n_2),$$

and similarly for $\varphi(\_, n)$.

Note that φ itself is *not* linear, even if we view $M \times N$ as the R-module $M \oplus N$, as recalled above. On the other hand, there ought to be a way to deal with R-bilinear maps 'as if' they were R-linear, because such maps abound in the context of R-modules. For example, the very multiplication on R is itself an R-bilinear map

$$R \times R \to R.$$

Our experience with universal properties suggests the natural way to approach this question. What we need is a new R-module $M \otimes_R N$, with an R-bilinear map

$$\otimes : M \times N \to M \otimes_R N,$$

such that *every* R-bilinear map $M \times N \to P$ factors uniquely through this new module $M \otimes_R N$,

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "\otimes"'] & P \\ M \otimes_R N \arrow[ru, "\exists ! \overline{\varphi}"'] & \end{tikzcd}$$

in such a way that the map $\overline{\varphi}$ is a usual R-module homomorphism.

Thus, $M \otimes_R N$ would be the 'best approximation' to $M \times N$ available in R-Mod, if we want to view R-bilinear maps from $M \times N$ as *R-linear.* The module $M \otimes_R N$ is called the *tensor product* of M and N over R. The subscript R is very important: if M and N are modules over two rings R, S, then S-bilinearity is not the same as R-bilinearity, so $M \otimes_R N$ and $M \otimes_S N$ may be completely different objects. In context, it is not unusual to drop the subscript if the base ring is understood, but we do not recommend this practice.

The prescription given above expresses the tensor product as the solution to a universal problem; therefore we know right away that it will be unique up to isomorphism, if it exists (Proposition I.5.4 once more), and we could proceed to study it by systematically using the universal property.

Example 2.2. For all R-modules N, $R \otimes_R N \cong N$.

Indeed, every R-bilinear $R \times N \to P$ factors through N (as is immediately verified):

$$\begin{tikzcd} R \times N \arrow[r] \arrow[d, "\otimes"'] & P \\ N \arrow[ur, "\exists!"'] & \end{tikzcd}$$

where $\otimes(r, n) = rn$. By the uniqueness property of universal objects, necessarily $N \cong R \otimes_R N$. ⌟

For another example, it is easy to see that there must be a canonical isomorphism

$$M \otimes_R N \xrightarrow{\sim} N \otimes_R M.$$

Indeed, every R-bilinear $\varphi : M \times N \to P$ may be decomposed as[10]

$$\begin{tikzcd} M \times N \arrow[r] \arrow[rr, bend right, "\varphi"'] & N \times M \arrow[r, "\psi"] & P, \end{tikzcd}$$

where $\psi(n, m) = \varphi(m, n)$; ψ is also R-bilinear, so it factors uniquely through $N \otimes_R M$. Therefore, φ factors uniquely through $N \otimes_R M$, and this is enough to conclude that there is a canonical isomorphism $N \otimes_R M \cong M \otimes_R N$.

However, such considerations are a little moot unless we establish that $M \otimes_R N$ exists to begin with. This requires a bit of work.

Lemma 2.3. *Tensor products exist in* R-Mod.

Proof. Given R-modules M and N, we construct 'by hand' a module satisfying the universal requirement. Let $F^R(M \times N) = R^{\oplus(M \times N)}$ be the free R-module on $M \times N$ (§III.6.3). This module comes equipped with a *set*-map

$$j : M \times N \to F^R(M \times N),$$

universal with respect to all set-maps from $M \times N$ to any R-module P; the main task is to make this into an *R-bilinear* map. For example, we have to identify elements in $F^R(M \times N)$ of the form $j(m, n_1 + n_2)$ with elements $j(m, n_1) + j(m, n_2)$, etc. Thus, let K be the R-submodule of $F^R(M \times N)$ generated by all elements

$$j(m, r_1 n_1 + r_2 n_2) - r_1 j(m, n_1) - r_2 j(m, n_2)$$

and

$$j(r_1 m_1 + r_2 m_2, n) - r_1 j(m_1, n) - r_2 j(m_2, n)$$

[10]Here is one situation in which the commutativity of R does play a role: if R is not commutative, then this decomposition becomes problematic, even if M and N carry bimodule structures. One can therefore not draw the conclusion $M \otimes_R N \cong N \otimes_R M$ in that case.

as m, m_1, m_2 range in M, n, n_1, n_2 range in N, and r_1, r_2 range in R. Let

$$M \otimes_R N := \frac{F^R(M \times N)}{K},$$

endowed with the map $\otimes : M \times N \to M \otimes_R N$ obtained by composing j with the natural projection:

$$\otimes : M \times N \xrightarrow{j} F^R(M \times N) \longrightarrow M \otimes_R N = F^R(M \times N)/K$$

The element $\otimes(m, n)$ (that is, the class of $j(m, n)$ modulo K) is denoted $m \otimes n$.

It is evident that $(m, n) \to m \otimes n$ defines an R-bilinear map. We have to check that $M \otimes_R N$ satisfies the universal property, and this is also straightforward. If $\varphi : M \times N \to P$ is any R-bilinear map, we have a unique induced R-linear map $\tilde{\varphi}$ from the free R-module, by the universal property of the latter:

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[d, "j"'] & P \\
F^R(M \times N) \arrow[ur, "\exists ! \tilde{\varphi}"'] &
\end{tikzcd}$$

We claim that $\tilde{\varphi}$ restricts to 0 on K. Indeed, to verify this, it suffices to verify that $\tilde{\varphi}$ sends to zero every generator of K, and this follows from the fact that φ is R-bilinear. For example,

$$\begin{aligned}
\tilde{\varphi}(j(m,&r_1 n_1 + r_2 n_2) - r_1 j(m, n_1) - r_2 j(m, n_2)) \\
&= \tilde{\varphi}(j(m, r_1 n_1 + r_2 n_2)) - r_1 \tilde{\varphi}(j(m, n_1)) - r_2 \tilde{\varphi}(j(m, n_2)) \\
&= \varphi(m, r_1 n_1 + r_2 n_2) - r_1 \varphi(m, n_1) - r_2 \varphi(m, n_2) \\
&= 0.
\end{aligned}$$

It follows (by the universal property of quotients!) that $\tilde{\varphi}$ factors uniquely through the quotient by K:

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[d, "j"'] \arrow[dd, bend right=60, "\otimes"'] & P \\
F^R(M \times N) \arrow[ur, "\tilde{\varphi}"] \arrow[d] & \\
M \otimes_R N = F^R(M \times N)/K \arrow[uur, "\exists ! \overline{\varphi}"'] &
\end{tikzcd}$$

and we are done. $\square$

As is often the case with universal objects, the explicit construction used to prove the existence of $M \otimes_R N$ is almost never invoked. It is however good to keep in mind that elements of $M \otimes_R N$ arise from elements of the free R-module on $M \times N$, and therefore an arbitrary element of $M \otimes_R N$ is a *finite linear combination*

$$(*) \qquad \sum_i r_i (m_i \otimes n_i)$$

with $r_i \in R$, $m_i \in M$, and $n_i \in N$. The R-bilinearity of $\otimes : M \times N \to M \otimes_R N$ amounts to the rules:

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$$
$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$$
$$m \otimes (rn) = (rm) \otimes n = r(m \otimes n),$$

for all $m, m_1, m_2 \in M$, $n_1, n_2, n \in N$, and $r \in R$. In particular, note that the coefficients r_i in (*) are *not* necessary, since they can be absorbed into the corresponding terms $m_i \otimes n_i$:

$$\sum_i r_i(m_i \otimes n_i) = \sum_i (r_i m_i) \otimes n_i.$$

Elements of the form $m \otimes n$ (that is, needing only one summand in the expression) are called *pure tensors. Dear reader,* please remember that pure tensors are special: usually, not every element of the tensor product is a pure tensor. See Exercise 2.1 for one situation in which every tensor happens to be pure, and appreciate how special that is.

Pure tensors are nevertheless very useful, as a set of generators for the tensor product. For example, if two homomorphisms $\alpha, \beta : M \otimes_R N \to P$ coincide on *pure* tensors, then $\alpha = \beta$. Frequently, computations involving tensor products are reduced to simple verifications for pure tensors.

2.2. Adjunction with Hom and explicit computations. *The tensor product is left-adjoint to* Hom. Once we parse what this rough statement means, it will be a near triviality; but as we have found out in §1.5, the mere fact that $\otimes_R$ is left-adjoint to *any* functor is enough to draw interesting conclusions about it.

First, we note that every R-module N defines, via $\otimes_R$, a new covariant functor $R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$, defined on objects by

$$M \mapsto M \otimes_R N.$$

To see how this works on morphisms, let

$$\alpha : M_1 \to M_2$$

be an R-module homomorphism. Crossing with N and composing with $\otimes$ defines an R-bilinear map

$$M_1 \times N \to M_2 \times N \to M_2 \otimes N,$$

and hence an induced R-linear map

$$\alpha \otimes N : M_1 \otimes N \to M_2 \otimes N.$$

On pure tensors, this map is simply given by $m \otimes n \mapsto \alpha(m) \otimes n$, and functoriality follows immediately: if $\beta : M_0 \to M_1$ is a second homomorphism, then $(\alpha \otimes N) \circ (\beta \otimes N)$ and $(\alpha \circ \beta) \otimes N$ both map pure tensors $m \otimes n$ to $\alpha(\beta(m)) \otimes n$, so they must agree on all tensors.

The adjunction statement given at the beginning of this subsection compares this functor with the covariant functor $P \mapsto \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P)$; cf. §1.2. Let's see more precisely how it works.

We have defined $M \otimes_R N$ so that giving an R-linear map $M \otimes_R N \to P$ to an R-module P is 'the same as' giving an R-bilinear map $M \times N \to P$. Now recall the definition of R-bilinear map: $\varphi : M \times N \to P$ is R-bilinear if both $\varphi(m, \_)$ and $\varphi(\_, n)$ are R-linear maps, for all $m \in M$ and $n \in N$. The first part of this prescription says that φ determines a function

$$M \to \mathrm{Hom}_R(N, P);$$

the second part says that this is an R-module homomorphism. Therefore, an R-bilinear map is 'the same as' an element of

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

These simple considerations should be enough to make the following seemingly complicated statement rather natural:

Lemma 2.4. *For all R-modules M, N, P, there is an isomorphism of R-modules*

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \cong \mathrm{Hom}_R(M \otimes_R N, P).$$

Proof. As noted before the statement, every $\alpha \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ determines an R-bilinear map $\varphi : M \times N \to P$, by

$$(m, n) \mapsto \alpha(m)(n).$$

By the universal property, φ factors uniquely through an R-linear map $\overline{\varphi} : M \otimes_R N \to P$. Therefore, α determines a well-defined element $\overline{\varphi} \in \mathrm{Hom}_R(M \otimes_R N, P)$.

The reader will check (Exercise 2.11) that this map $\alpha \mapsto \overline{\varphi}$ is R-linear and construct an inverse. □

Corollary 2.5. *For every R-module N, the functor $\_ \otimes_R N$ is left-adjoint to the functor* $\mathrm{Hom}_R(N, \_)$.

Proof. The claim is that the isomorphism found in Lemma 2.4 is natural in the sense hinted at, but not fully explained, in §1.5; the interested reader should have no problems checking this naturality. □

By Lemma 1.17 (or rather its co-version), we can conclude that for each R-module N, the functor $\_ \otimes_R N$ preserves colimits, and so does $M \otimes_R \_$, by the basic commutativity of tensor products verified in §2.1. In particular, and this is good material for another Pavlovian reaction,

$$M \otimes_R \_ \text{ and } \_ \otimes_R N \text{ are right-exact functors}$$

(cf. Example 1.18).

These observations have several consequences, which make 'computations' with tensor products more reasonable. Here is a sample:

Corollary 2.6. *For all R-modules M_1, M_2, N,*

$$(M_1 \oplus_R M_2) \otimes N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N).$$

(Moreover, by commutativity, $M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$ just as well.) Indeed, coproducts are colimits. In fact, $\otimes$ must then commute with *arbitrary* (possibly infinite) direct sums:

$$(\bigoplus_{\alpha \in A} M_\alpha) \otimes_R N \cong \bigoplus_{\alpha \in A} (M_\alpha \otimes_R N).$$

This computes all tensors for *free* R-modules:

Corollary 2.7. *For any two sets A, B:*

$$R^{\oplus A} \otimes_R R^{\oplus B} \cong R^{\oplus A \times B}.$$

Indeed, 'distributing' the direct sum identifies the left-hand side with the direct sum $(R^{\oplus A})^{\oplus B}$, which is isomorphic to the right-hand side (Exercise III.6.5). For *finitely generated* free modules, this simply says that $R^{\oplus m} \otimes R^{\oplus n} \cong R^{\oplus mn}$.

Note that if $e_1, \dots, e_m$ generate M and $f_1, \dots, f_n$ generate N, then the pure tensors $e_i \otimes f_j$ must generate $M \otimes_R N$. In the free case, if the e_i's and f_j's form bases of $R^{\oplus m}$, $R^{\oplus n}$, resp., then the mn elements $e_i \otimes f_j$ must be a basis for $R^{\oplus m} \otimes R^{\oplus n}$. Indeed they generate it; hence they must be linearly independent since this module is free of rank mn. In particular, this is all that can happen if R is a field k and the modules are, therefore, just k-vector spaces (Proposition VI.1.7). Tensor products are more interesting over more general rings.

Corollary 2.8. *For all R-modules N and all ideals I of R,*

$$\frac{R}{I} \otimes_R N \cong \frac{N}{IN}.$$

Indeed, $\_ \otimes_R N$ is right-exact; thus, the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow \frac{R}{I} \longrightarrow 0$$

induces an exact sequence

$$I \otimes_R N \longrightarrow R \otimes_R N \longrightarrow \frac{R}{I} \otimes_R N \longrightarrow 0.$$

The image of $I \otimes_R N$ in $R \otimes_R N \cong N$ is generated by the image of the pure tensors $a \otimes n$ with $a \in I$, $n \in N$; this is IN. Thus, the second sequence identifies $N/(IN)$ with $(R/I) \otimes_R N$, as needed.

Corollary 2.9. *For all ideals I, J of R,*

$$\frac{R}{I} \otimes_R \frac{R}{J} \cong \frac{R}{I+J}.$$

This follows immediately from Corollary 2.8 and the 'third isomorphism theorem', Proposition III.5.17. Indeed, $IR/J = (I+J)/J$.

Example 2.10. $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

Indeed, $(m) + (n) = (\gcd(m,n))$ in $\mathbb{Z}$. For instance,

$$\frac{\mathbb{Z}}{2\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{3\mathbb{Z}} \cong 0,$$

a favorite on qualifying exams (cf. Exercise 2.2). ⌟

Corollary 2.8 is a template example for a basic application of $\otimes$: tensor products may be used to transfer constructions involving R (such as quotienting by an ideal I) to constructions involving R-modules (such as quotienting by a corresponding submodule). There are several instances of this operation; the reader will take a look at *localization* in Exercise 2.5.

2.3. Exactness properties of tensor; flatness. It is important to remember that the tensor product is *not* an exact functor: left-exactness may very well fail. This can already be observed in the sequence appearing in the discussion following Corollary 2.8: for an ideal I of R and an R-module N, the map

$$I \otimes_R N \to N$$

induced by the inclusion $I \subseteq R$ after tensoring by N may not be injective.

Example 2.11. Multiplication by 2 gives an inclusion

$$\mathbb{Z} \xhookrightarrow{\cdot 2} \mathbb{Z}$$

identifying the first copy of $\mathbb{Z}$ with the ideal (2) in the second copy. Tensoring by $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$ (and keeping in mind that $R \otimes_R N \cong N$), we get the homomorphism

$$\frac{\mathbb{Z}}{2\mathbb{Z}} \xrightarrow{\cdot 2} \frac{\mathbb{Z}}{2\mathbb{Z}},$$

which sends both [0] and [1] to zero. This is the zero-morphism, and in particular it is not injective. ⌟

On the other hand, if $N \cong R^{\oplus A}$ is *free,* then $\_ \otimes_R N$ *is* exact. Indeed, every inclusion

$$M_1 \subseteq M_2$$

is mapped to $M_1 \otimes_R R^{\oplus A} \to M_2 \otimes_R R^{\oplus A}$, which is identified (via Corollary 2.6) with the *inclusion*

$$M_1^{\oplus A} \subseteq M_2^{\oplus A}.$$

Example 2.12. Since vector spaces are free (Proposition VI.1.7), tensoring is exact in k-Vect: if

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

is an exact sequence of k-vector spaces and W is a k-vector space, then the induced sequence

$$0 \longrightarrow V_1 \otimes_k W \longrightarrow V_2 \otimes_k W \longrightarrow V_3 \otimes_k W \longrightarrow 0$$

is exact on both sides. ⌟

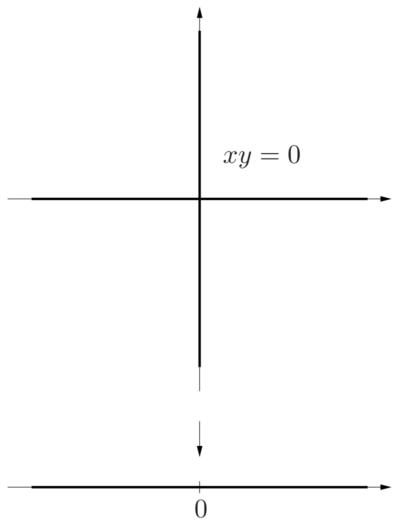
The reader should now wonder whether it is useful to study a condition on an R-module N, guaranteeing that the functor $\_ \otimes_R N$ *is* left-exact as well as right-exact.

Definition 2.13. An R-module N is *flat* if the functor $\_ \otimes_R N$ is exact. ⌟

In the exercises the reader will explore easy properties of this notion and useful equivalent formulations in particular cases.

We have already checked that $\mathbb{Z}/2\mathbb{Z}$ is *not* a flat $\mathbb{Z}$-module, while free modules are flat. Flat modules are hugely important: in algebraic geometry, 'flatness' is the condition expressing the fact that the objects in a family vary 'continuously', preserving certain key invariants.

Example 2.14. Consider the affine algebraic set $\mathscr{V}(xy)$ in the plane $\mathbb{A}^2$ (over a fixed field k) and the 'projection on the first coordinate' $\mathscr{V}(xy) \to \mathbb{A}^1$, $(x, y) \mapsto x$:



In terms of coordinate rings (cf. §VII.2.3), this map corresponds to the homomorphism of k-algebras:

$$k[x] \to \frac{k[x,y]}{(xy)}$$

defined by mapping x to the coset $x + (xy)$ (this will be completely clear to the reader who has worked out Exercise VII.2.12!). This homomorphism defines a $k[x]$-module structure on $k[x,y]/(xy)$, and we can wonder whether the latter is *flat* in the sense of Definition 2.13. From the geometric point of view, clearly something 'not flat' is going on over the point $x = 0$, so we consider the inclusion of the ideal (x) in $k[x]$:

$$k[x] \xhookrightarrow{\cdot x} k[x]$$

Tensoring by $k[x,y]/(xy)$, we obtain

$$\frac{k[x,y]}{(xy)} \xrightarrow{\cdot x} \frac{k[x,y]}{(xy)}$$

which is *not* injective, because it sends to zero the nonzero coset $y+(xy)$. Therefore $k[x,y]/(xy)$ is not flat as a $k[x]$-module.

The term *flat* was inspired precisely by such 'geometric' examples. ⌟

2.4. The Tor functors. The 'failure of exactness' of the functor $\_ \otimes_R N$ is measured by another functor R-Mod $\to$ R-Mod, called $\mathrm{Tor}_1^R(\_, N)$: if N is flat (for example, if it is free), then $\mathrm{Tor}_1^R(M,N) = 0$ for all modules M. In fact (amazingly) if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of R-modules, one obtains a new exact sequence after tensoring by any N:

$$\mathrm{Tor}_1^R(C,N) \longrightarrow A \otimes_R N \longrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow 0,$$

so if $\mathrm{Tor}_1^R(C,N) = 0$, then the module on the left vanishes; thus *every* short exact sequence ending in C remains exact after tensoring by N in this case. *In fact* (astonishingly) for all N one can continue this sequence with more Tor-modules, obtaining a longer exact complex:

$$\mathrm{Tor}_1^R(A,N) \to \mathrm{Tor}_1^R(B,N) \to \mathrm{Tor}_1^R(C,N) \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.$$

This is not the end of the story: the complex may be continued even further by invoking new functors $\mathrm{Tor}_2^R(\_, N)$, $\mathrm{Tor}_3^R(\_, N)$, etc. These are the *derived functors* of tensor. To 'compute' these functors, one may apply the following procedure: given an R-module M, find a free resolution (§VI.4.2)

$$\cdots \longrightarrow R^{\oplus S_2} \longrightarrow R^{\oplus S_1} \longrightarrow R^{\oplus S_0} \longrightarrow M \longrightarrow 0;$$

throw M away, and tensor the free part by N, obtaining a complex $M_\bullet \otimes_R N$:

$$\cdots \longrightarrow N^{\oplus S_2} \longrightarrow N^{\oplus S_1} \longrightarrow N^{\oplus S_0} \longrightarrow 0$$

(recall again that tensor commutes with colimits, hence with direct sums, therefore $R^{\oplus m} \otimes_R N \cong N^{\oplus m}$); then take the *homology* of this complex (cf. §III.7.3). Astoundingly, this *will not depend* (up to isomorphism) on the chosen free resolution, so we can define

$$\mathrm{Tor}_i^R(M,N) := H_i(M_\bullet \otimes N).$$

For example, according to this definition $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$ (Exercise 2.14), and $\mathrm{Tor}_i^R(M,N) = 0$ for all $i > 0$ and all M if N is flat (because then tensoring by N is an exact functor, so tensoring the resolution of M returns an exact sequence, thus with no homology). In fact, this proves a remarkable property of the Tor functors: if $\mathrm{Tor}_1^R(M,N) = 0$ for all M, then $\mathrm{Tor}_i^R(M,N) = 0$ for all $i > 0$ for all modules M. Indeed, N is then flat.

At this point you may feel that something is a little out of balance: why focus on the functor $\_ \otimes_R N$, rather than $M \otimes_R \_$? Since $M \otimes_R N$ is canonically isomorphic to $N \otimes_R M$ (in the commutative case; cf. Example 2.2), we could expect the same to apply to every Tor_i^R: $\mathrm{Tor}_i^R(M,N)$ ought to be canonically isomorphic to $\mathrm{Tor}_i^R(N,M)$ for all i. Equivalently, we should be able to compute $\mathrm{Tor}_i^R(M,N)$ as the homology of $M \otimes_R N_\bullet$, where $N_\bullet$ is a free resolution of N. This is indeed the case.

In due time (§§IX.7 and 8) we will prove this and all the other wonderful facts we have stated in this subsection. For now, we are asking the reader to believe that the Tor functors can be defined as we have indicated, and the facts reviewed here will suffice for simple computations (see for example Exercises 2.15 and 2.17) and applications.

In fact, we know enough about finitely generated modules over PIDs to get a preliminary sense of what is involved in proving such general facts. Recall that we have been able to establish that every finitely generated module M over a PID R has a free resolution of length 1:

$$0 \longrightarrow R^{\oplus m_1} \longrightarrow R^{\oplus m_0} \longrightarrow M \longrightarrow 0\ .$$

This property *characterizes* PIDs (Proposition VI.5.4). If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of R-modules, it is not hard to see that one can produce 'compatible' resolutions, in the sense that the rows of the following diagram will be exact as well as the columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R^{\oplus a_1} \arrow[r] \arrow[d, "\alpha"] & R^{\oplus b_1} \arrow[r] \arrow[d, "\beta"] & R^{\oplus c_1} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & R^{\oplus a_0} \arrow[r] \arrow[d] & R^{\oplus b_0} \arrow[r] \arrow[d] & R^{\oplus c_0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(This will be proven in gory detail in §IX.7.) Tensor the two 'free' rows by N; they remain exact (tensoring commutes with direct sums):

$$\begin{tikzcd}
0 \arrow[r] & N^{\oplus a_1} \arrow[r] \arrow[d, "\alpha\otimes N"] & N^{\oplus b_1} \arrow[r] \arrow[d, "\beta\otimes N"] & N^{\oplus c_1} \arrow[r] \arrow[d, "\gamma\otimes N"] & 0 \\
0 \arrow[r] & N^{\oplus a_0} \arrow[r] & N^{\oplus b_0} \arrow[r] & N^{\oplus c_0} \arrow[r] & 0
\end{tikzcd}$$

Now the columns (preceded and followed by 0) are precisely the complexes $A_\bullet \otimes_R N$, $B_\bullet \otimes_R N$, $C_\bullet \otimes_R N$ whose homology 'computes' the Tor modules. Applying the snake lemma (Lemma III.7.8; cf. Remark III.7.10) gives the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H_1(A_\bullet \otimes_R N) \arrow[r] & H_1(B_\bullet \otimes_R N) \arrow[r] & H_1(C_\bullet \otimes_R N) \arrow[dll, "\delta"] & \\
 & H_0(A_\bullet \otimes_R N) \arrow[r] & H_0(B_\bullet \otimes_R N) \arrow[r] & H_0(C_\bullet \otimes_R N) \arrow[r] & 0,
\end{tikzcd}$$

which is precisely the sequence of Tor modules conjured up above,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1^R(A,N) \arrow[r] & \mathrm{Tor}_1^R(B,N) \arrow[r] & \mathrm{Tor}_1^R(C,N) \arrow[dll, "\delta"] & \\ & A\otimes_R N \arrow[r] & B\otimes_R N \arrow[r] & C\otimes_R N \arrow[r] & 0 \end{tikzcd}$$

with a 0 on the left for good measure (due to the fact that Tor_2^R vanishes if R is a PID; cf. Exercise 2.17).

Note that Tor_i^k vanishes for $i > 0$ if k is a field, as vector spaces are flat, and Tor_i^R vanishes for $i > 1$ if R is a PID (Exercise 2.17). These facts are not surprising, in view of the procedure described above for computing Tor and of the considerations at the end of §VI.5.2: a bound on the length of free resolutions for modules over a ring R will imply a bound on nonzero Tor's. For particularly nice rings (such as the rings corresponding to 'smooth' points in algebraic geometry) this bound agrees with the Krull dimension; but precise results of this sort are beyond the scope of this book.

Exercises

R denotes a fixed commutative ring.

2.1. ▷ Let M, N be R-modules, and assume that N is *cyclic.* Prove that every element of $M \otimes_R N$ may be written as a pure tensor. [§2.1]

2.2. ▷ Prove 'by hand' (that is, without appealing to the right-exactness of tensor) that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong 0$ if m, n are relatively prime integers. [§2.2]

2.3. Prove that $R[x_1, \dots, x_n] \otimes_R R[y_1, \dots, y_m] \cong R[x_1, \dots, x_n, y_1, \dots, y_m]$.

2.4. ¬ Let S, T be commutative R-algebras. Verify the following:

- The tensor product $S \otimes_R T$ has an operation of multiplication, defined on pure tensors by $(s_1 \otimes t_1) \cdot (s_2 \otimes t_2) := s_1 s_2 \otimes t_1 t_2$ and making it into a commutative R-algebra.
- With respect to this structure, there are R-algebra homomorphisms $i_S : S \to S \otimes T$, resp., $i_T : T \to S \otimes T$, defined by $i_S(s) := s \otimes 1$, $i_T(t) := 1 \otimes t$.
- The R-algebra $S \otimes_R T$, with these two structure homomorphisms, is a coproduct of S and T in the category of commutative R-algebras: if U is a commutative R-algebra and $f_S : S \to U$, $f_T : T \to U$ are R-algebra homomorphisms, then there exists a unique R-algebra homomorphism $f_S \otimes f_T$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[dr, "i_S"] \arrow[drr, bend left, "f_S"] & & \\ & S\otimes_R T \arrow[r, "f_S\otimes f_T"] & U \\ T \arrow[ur, "i_T"'] \arrow[urr, bend right, "f_T"'] & & \end{tikzcd} .$$

In particular, if S and T are simply commutative rings, then $S \otimes_{\mathbb{Z}} T$ is a coproduct of S and T in the category of commutative rings. This settles an issue left open at the end of §III.2.4. [2.10]

2.5. $\triangleright$ (Cf. Exercises V.4.7 and V.4.8.) Let S be a multiplicative subset of R, and let M be an R-module. Prove that $S^{-1}M \cong M \otimes_R S^{-1}R$ as R-modules. (Use the universal property of the tensor product.)

Through this isomorphism, $M \otimes_R S^{-1}R$ inherits an $S^{-1}R$-module structure. [§2.2, 2.8, 2.12, 3.4]

2.6. $\neg$ (Cf. Exercises V.4.7 and V.4.8.) Let S be a multiplicative subset of R, and let M be an R-module.

- Let N be an $S^{-1}R$-module. Prove that $(S^{-1}M) \otimes_{S^{-1}R} N \cong M \otimes_R N$.
- Let A be an R-module. Prove that $(S^{-1}A) \otimes_R M \cong S^{-1}(A \otimes_R M)$.

(Both can be done 'by hand', by analyzing the construction in Lemma 2.3. For example, there is a homomorphism $M \otimes_R N \to (S^{-1}M) \otimes_{S^{-1}R} N$ which is surjective because, with evident notation, $\frac{m}{s} \otimes n = m \otimes \frac{n}{s}$ in $(S^{-1}M) \otimes_{S^{-1}R} N$; checking that it is injective amounts to easy manipulation of the relations defining the two tensor products.

Both isomorphisms will be easy consequences of the associativity of tensor products; cf. Exercise 3.4.) [2.21, 3.4]

2.7. Changing the base ring in a tensor may or may not make a difference:

- Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$.
- Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \ncong \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$.

2.8. Let R be an integral domain, with field of fractions K, and let M be a finitely generated R-module. The tensor product $V := M \otimes_R K$ is a K-vector space (Exercise 2.5). Prove that $\dim_K V$ equals the rank of M as an R-module, in the sense of Definition VI.5.5.

2.9. Let G be a finitely generated abelian group of rank r. Prove that $G \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r$. Prove that for infinitely many primes p, $G \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^r$.

2.10. Let $k \subseteq k(\alpha) = F$ be a finite simple field extension. Note that $F \otimes_k F$ has a natural ring structure; cf. Exercise 2.4.

- Prove that α is separable over k if and only if $F \otimes_k F$ is *reduced* as a ring.
- Prove that $k \subseteq F$ is Galois if and only if $F \otimes_k F$ is isomorphic to $F^{[F:k]}$ as a ring.

(Use Corollary 2.8 to 'compute' the tensor. The CRT from §V.6.1 will likely be helpful.)

2.11. $\triangleright$ Complete the proof of Lemma 2.4. [§2.2]

2.12. Let S be a multiplicative subset of R (cf. Exercise V.4.7). Prove that $S^{-1}R$ is flat over R. (Hint: Exercises 2.5 and 1.25.)

2.13. Prove that direct sums of flat modules are flat.

2.14. ▷ Prove that, according to the definition given in §2.4, $\mathrm{Tor}_0^R(M,N)$ is isomorphic to $M \otimes_R N$. [§2.4]

2.15. ▷ Prove that for $r \in R$ a non-zero-divisor and N an R-module, the module $\mathrm{Tor}_1^R(R/(r),N)$ is isomorphic to the r-torsion of N, that is, the submodule of elements $n \in N$ such that $rn = 0$ (cf. §VI.4.1). (This is the reason why Tor is called Tor.) [§2.4, 6.21]

2.16. Let I, J be ideals of R. Prove that $\mathrm{Tor}_1^R(R/I,R/J) \cong (I \cap J)/IJ$. (For example, this Tor_1^R vanishes if $I+J=R$, by Lemma V.6.2.) Prove that $\mathrm{Tor}_i^R(R/I,R/J)$ is isomorphic to $\mathrm{Tor}_{i-1}^R(I,R/J)$ for $i>1$.

2.17. ▷ Let M, N be modules over a PID R. Prove that $\mathrm{Tor}_i^R(M,N)=0$ for $i \geq 2$. (Assume M, N are finitely generated, for simplicity.) [§2.4]

2.18. Let R be an integral domain. Prove that a cyclic R-module is flat if and only if it is free.

2.19. ¬ The following criterion is quite useful.

- Prove that an R-module M is flat if and only if every monomorphism of R-modules $A \hookrightarrow B$ induces a monomorphism of R-modules $A \otimes_R M \hookrightarrow B \otimes_R M$.
- Prove that it suffices to verify this condition for all *finitely generated* modules B. (Hint: For once, refer back to the construction of tensor products given in Lemma 2.3. An element $\sum_i a_i \otimes m_i \in A \otimes_R M$ goes to zero in $B \otimes_R M$ if the corresponding element $\sum_i (a_i, m_i)$ equals a combination of the relations defining $B \otimes_R M$ in the free R-module $F^R(B \times M)$. This will be an identity involving only finitely many elements of B; hence....)
- Prove that it suffices to verify this condition when $B = R$ and $A = I$ is an ideal of R. (Hint: We may now assume that B is finitely generated. Find submodules B_j such that $A = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_r = B$, with each B_j/B_{j-1} cyclic. Reduce to verifying that $A \otimes_R M$ injects in $B \otimes_R M$ when B/A is cyclic, hence $\cong R/I$ for some ideal I. Conclude by a Tor_1^R argument or—but this requires a little more stamina—by judicious use of the snake lemma.)
- Deduce that an R-module M is flat if and only if the natural homomorphism $I \otimes_R M \to IM$ is an isomorphism for every ideal I of R.

If you believe in Tor's, now you can also show that an R-module M is flat if and only if $\mathrm{Tor}_1^R(R/I,M)=0$ for all ideals I of R. [2.20]

2.20. Let R be a PID. Prove that an R-module M is flat if and only if it is torsion-free. (If M is finitely generated, the classification theorem of §VI.5.3 makes this particularly easy. Otherwise, use Exercise 2.19.)

Geometrically, this says roughly that an algebraic set fails to be 'flat' over a nonsingular curve if and only if some component of the set is contracted to a point. This phenomenon is displayed in the picture in Example 2.14.

2.21. ¬ (Cf. Exercise V.4.11.) Prove that *flatness is a local property:* an R-module M is flat if and only if $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p}$, if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$. (Hint: Use Exercises 1.25 and 2.6.

The $\Longrightarrow$ direction will be straightforward. For the converse, let $A \subseteq B$ be R-modules, and let K be the kernel of the induced homomorphism $A \otimes_R M \to B \otimes_R M$. Prove that the kernel of the localized homomorphism $A_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m} \to B_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m}$ is isomorphic to $K_\mathfrak{m}$, and use Exercise V.4.12.) [2.22]

2.22. $\neg$ Let M, N be R-modules, and let S be a multiplicative subset of R. Use the definition of Tor given in §2.4 to show $S^{-1}\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N)$. (Use Exercise 1.25.) Use this fact to give a leaner proof that flatness is a local property (Exercise 2.21). [2.25]

2.23. $\triangleright$ Let

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

be an exact sequence of R-modules, and assume that P is flat.

- Prove that M is flat if and only if N is flat.
- Prove that for all R-modules Q, the induced sequence

$$0 \longrightarrow M \otimes_R Q \longrightarrow N \otimes_R Q \longrightarrow P \otimes_R Q \longrightarrow 0$$

  is exact.

[2.24, §5.4]

2.24. $\neg$ Let R be a commutative Noetherian local ring with (single) maximal ideal $\mathfrak{m}$, and let M be a finitely generated flat R-module.

- Choose elements $m_1, \dots, m_r \in M$ whose cosets mod $\mathfrak{m}M$ are a basis of $M/\mathfrak{m}M$ as a vector space over the field $R/\mathfrak{m}$. By Nakayama's lemma, $M = \langle m_1, \dots, m_r \rangle$ (Exercise VI.3.10).
- Obtain an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus r} \longrightarrow M \longrightarrow 0,$$

  where N is finitely generated.
- Prove that this sequence induces an exact sequence

$$0 \longrightarrow N/\mathfrak{m}N \longrightarrow (R/\mathfrak{m})^{\oplus r} \longrightarrow M/\mathfrak{m}M \longrightarrow 0.$$

  (Use Exercise 2.23.)
- Deduce that $N = 0$. (Nakayama.)
- Conclude that M is free.

Thus, a finitely generated module over a (Noetherian[11]) local ring is flat if and only if it is free. Compare with Exercise VI.5.5. [2.25, 6.8, 6.12]

2.25. Let R be a commutative Noetherian ring, and let M be a finitely generated R-module. Prove that

$$M \text{ is flat} \iff \operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0 \text{ for every maximal ideal } \mathfrak{m} \text{ of } R\,.$$

[11] The Noetherian hypothesis is actually unnecessary, but it simplifies the proof by allowing the use of Nakayama's lemma.

(Use Exercise 2.21, and refine the argument you used in Exercise 2.24; remember that Tor localizes, by Exercise 2.22. The Noetherian hypothesis is actually unnecessary, but the proofs are harder without it.)

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3. Base change

We have championed several times the viewpoint that deep properties of a ring R are encoded in the category R-Mod of R-modules; one extreme position is to simply replace R with R-Mod as the main object of study. The question then arises as to how to deal with ring homomorphisms from this point of view, or more generally how the categories R-Mod, S-Mod of modules over two (commutative) rings R, S may relate to each other. The reader should expect this to happen by way of *functors* between the two categories and that the situation at the categorical level will be substantially richer than at the ring level.

3.1. Balanced maps. Before we can survey the basic definitions, we must upgrade our understanding of tensor products. It turns out that $M \otimes_R N$ satisfies a more encompassing universal property than the one examined in §2.1. Let M, N be modules over a commutative ring R, as in §2.1, and let G be an abelian group, i.e., a $\mathbb{Z}$-module.

Definition 3.1. A $\mathbb{Z}$-bilinear map $\varphi : M \times N \to G$ is *R-balanced* if $\forall m \in M$, $\forall n \in N$, $\forall r \in R$,

$$\varphi(rm, n) = \varphi(m, rn).$$

⌟

If G is an R-module and $\varphi : M \times N \to G$ is R-bilinear, then it is R-balanced[12]. But in general the notion of 'R-balanced' appears to be quite a bit more general, since G is not even required to be an R-module. This may lead the reader to suspect that a solution to the universal problem of factoring balanced maps may be a different gadget than the 'ordinary' tensor product, but we are in luck in this case, and the ordinary tensor product does the universal job for balanced maps as well.

To understand this, recall that we constructed $M \otimes_R N$ as a quotient

$$M \otimes_R N = \frac{R^{\oplus(M\times N)}}{K},$$

where K is generated by the relations necessary to imply that the map

$$M \times N \to R^{\oplus(M\times N)} \to \frac{R^{\oplus(M\times N)}}{K}$$

is R-bilinear. We have observed that every element of $M \otimes_R N$ may be written as a linear combination of pure tensors:

$$\sum_i m_i \otimes n_i;$$

[12] Also note that if R is not commutative and M, resp., N, carries a right-, resp., left-, R module structure, then the notion of 'balanced map' makes sense. This leads to the definition of tensor (as an abelian group) in the noncommutative case.

it follows that the group homomorphism

$$\mathbb{Z}^{\oplus(M\times N)} \twoheadrightarrow M \otimes_R N$$

defined on generators by $(m,n) \mapsto m\otimes n$ is *surjective;* its kernel K_B consists of the combinations

$$\sum_i (m_i,n_i) \in \mathbb{Z}^{\oplus(M\times N)} \quad \text{such that} \quad \sum_i (m_i,n_i) \in K,$$

where the sum on the right is viewed in $R^{\oplus(M\times N)}$. The reader will verify (Exercise 3.1) that K_B is generated by elements of the form

$$\begin{gathered}(m, n_1+n_2) - (m,n_1) - (m,n_2),\\ (m_1+m_2, n) - (m_1,n) - (m_2,n),\\ (rm,n) - (m,rn)\end{gathered}$$

(with $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $r \in R$). Therefore, we have an induced isomorphism of abelian groups

$$(*) \qquad \frac{\mathbb{Z}^{\oplus(M\times N)}}{K_B} \cong \frac{R^{\oplus(M\times N)}}{K},$$

which amounts to an alternative description of $M\otimes_R N$. The point of this observation is that the group on the left-hand side of (*) is manifestly a solution to the universal problem of factoring $\mathbb{Z}$-bilinear, R-balanced maps. Therefore, we have proved

Lemma 3.2. *Let R be a commutative ring; let M, N be R-modules, and let G be an abelian group. Then every $\mathbb{Z}$-bilinear, R-balanced map $\varphi : M\times N \to G$ factors through $M\otimes_R N$; that is, there exists a unique group homomorphism $\overline{\varphi} : M\otimes_R N \to G$ such that the diagram*

$$\begin{tikzcd} M\times N \arrow[r, "\varphi"] \arrow[d, "\otimes"'] & G \\ M\otimes_R N \arrow[ur, "\exists!\overline{\varphi}"'] & \end{tikzcd}$$

commutes.

The universal property explored in §2.1 is recovered as the statement that *if* G is an R-module and φ is R-bilinear, *then* the induced group homomorphism $M\otimes_R N \to G$ is in fact an R-linear map.

Remark 3.3. Balanced maps $\varphi : M\times N \to G$ may be defined as soon as M is a *right*-R-module and N is a *left*-R-module, even if R is not commutative: require $\varphi(mr,n) = \varphi(m,rn)$ for all $m\in M$, $n\in N$, $r\in R$. The abelian group defined by the left-hand side of (*) still makes sense and is taken as the definition of the tensor product $M\otimes_R N$; but note that this does not carry an R-module structure in general. This structure is recovered if, e.g., M is a two-sided R-module. ⌟

3.2. Bimodules; adjunction again. The enhanced universal property for the tensor will allow us to upgrade the adjunction formula given in Lemma 2.4. This requires the introduction of yet another notion.

Definition 3.4. Let R, S be two commutative[13] rings. An (R,S)-*bimodule* is an abelian group N endowed with compatible R-module and S-module structures, in the sense that $\forall n \in N$, $\forall r \in R$, $\forall s \in S$,

$$r(sn) = s(rn).$$ ⌟

For example, as R is commutative, every R-module N is an (R,R)-bimodule: $\forall r_1, r_2 \in R$ and $\forall n \in N$,

$$r_1(r_2 n) = (r_1 r_2)n = (r_2 r_1)n = r_2(r_1 n).$$

If M is an R-module and N is an (R,S)-bimodule, then the tensor product $M \otimes_R N$ acquires an S-*module* structure: define the action of $s \in S$ on pure tensors $m \otimes n$ by

$$s(m \otimes n) := m \otimes (sn),$$

and extend to all tensors by linearity. In fact, this gives $M \otimes_R N$ an (R,S)-bimodule structure.

Similarly, if N is an (R,S)-bimodule and P is an S-module, then the abelian group $\mathrm{Hom}_S(N,P)$ is an (R,S)-bimodule: the R-module structure is defined by setting $(r\alpha)(n) = \alpha(rn)$ for all $r \in R$, $n \in N$, and $\alpha \in \mathrm{Hom}_S(N,P)$.

This mess is needed to even make sense of the promised upgrade of adjunction. As with most such results, the proof is not difficult once one understands what the statement says.

Lemma 3.5. *Suppose M is an R-module, N is an (R,S)-bimodule, and P is an S-module. Then there is a canonical isomorphism of abelian groups*

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P)) \cong \mathrm{Hom}_S(M \otimes_R N, P).$$

Proof. Every element $\alpha \in \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))$ determines a map

$$\varphi : M \times N \to P,$$

via $\varphi(m, \_) := \alpha(m)$; φ is clearly $\mathbb{Z}$-bilinear. Further, for all $r \in R$, $m \in M$, $n \in N$:

$$\varphi(rm, n) = \alpha(rm)(n) \overset{1}{=} r\alpha(m)(n) \overset{2}{=} \alpha(m)(rn) = \varphi(m, rn),$$

where $\overset{1}{=}$ holds by the R-linearity of α and $\overset{2}{=}$ holds by the definition of the R-module structure on $\mathrm{Hom}_S(N,P)$. Thus φ is R-balanced. By Lemma 3.2, such a map determines (and is determined by) a homomorphism of abelian groups

$$\overline{\varphi} : M \otimes_R N \to P,$$

[13] Once more, the noncommutative case would be very worthwhile pursuing, but (not without misgivings) we have decided otherwise. In this more general case one requires N to be both a *left-R*-module and a *right-S*-module with the compatibility expressed by $(rn)s = r(ns)$ for all choices of $n \in N$, $r \in R$, $s \in S$. This type of bookkeeping is precisely what is needed in order to extend the theory to the noncommutative case.

such that $\varphi(m,n) = \overline{\varphi}(m \otimes n)$. We claim that $\overline{\varphi}$ is S-linear. Indeed, $\forall s \in S$ and for all pure tensors $m \otimes n$,

$$\begin{aligned}\overline{\varphi}(s(m \otimes n)) &= \overline{\varphi}(m \otimes (sn)) = \varphi(m, sn) = \alpha(m)(sn) = s\alpha(m)(n) = s\varphi(m,n) \\ &= s\overline{\varphi}(m \otimes n),\end{aligned}$$

where we have used the S-linearity of $\alpha(m)$. Thus,

$$\overline{\varphi} \in \mathrm{Hom}_S(M \otimes_R N, P).$$

Tracing the argument backwards, every element of $\mathrm{Hom}_S(M \otimes_R N, P)$ determines an element of $\mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))$, and these two correspondences are clearly inverses of each other. $\square$

If $R = S$, we recover the adjunction formula of Lemma 2.4; note that in this case the isomorphism is clearly R-linear.

3.3. Restriction and extension of scalars. Coming back to the theme mentioned at the beginning of this section, consider the case in which we have a homomorphism $f : R \to S$ of (commutative) rings. It is natural to look for functors between the categories R-Mod and S-Mod of modules over R, S, respectively. There is a rather simple-minded functor from S-Mod to R-Mod ('restriction of scalars'), while tensor products allow us to define a functor from R-Mod to S-Mod ('extension of scalars'). A third important functor R-$\mathsf{Mod} \to S$-Mod may be defined, also 'extending scalars', but for which we do not know a good name.

Restriction of scalars. Let $f : R \to S$ be a ring homomorphism, and let N be an S-module. Recall (§III.5.1) that this means that we have chosen an action of the ring S on the abelian group N, that is, a ring homomorphism

$$\sigma : S \to \mathrm{End}_{\mathsf{Ab}}(N).$$

Composing with f,

$$\sigma \circ f : R \to S \to \mathrm{End}_{\mathsf{Ab}}(N)$$

defines an action *of* R on the abelian group N, and hence an R-module structure on N.

(Even) more explicitly, if $r \in R$ and $n \in N$, define the action of r on n by setting

$$rn := f(r)n.$$

Since S is commutative, this defines in fact an (R,S)-bimodule structure on N. Further, S-linear homomorphisms are in particular R-linear; this assignment is (covariantly) functorial S-$\mathsf{Mod} \to R$-Mod.

If f is injective, so that R may be viewed as a subring of S, then all we are doing is viewing N as a module on a 'restricted' range of scalars, hence the terminology. For example, this is how we view a complex vector space as a *real* vector space, in the simplest possible way.

We will denote by f_* this functor S-$\mathsf{Mod} \to R$-Mod induced from f by restriction of scalars. Note that f_* is trivially exact, because the kernels and images of a homomorphism of modules are the same regardless of the base ring. In view of the considerations in Example 1.18, this hints that f_* may have *both* a left-adjoint and a right-adjoint functor, and this will be precisely the case (Proposition 3.6).

Extension of scalars is defined from R-Mod to S-Mod, by associating to an R-module M the tensor product $f^*(M) := M \otimes_R S$, which (as we have seen in §3.2) carries naturally an S-module structure. This association is evidently covariantly functorial. If

$$R^{\oplus B} \to R^{\oplus A} \to M \to 0$$

is a presentation of M (cf. §VI.4.2), tensoring by S gives (by the right-exactness of tensor) a presentation of $f^*(M)$:

$$S^{\oplus B} \to S^{\oplus A} \to M \otimes_R S \to 0.$$

Intuitively, this says that $f^*(M)$ is the module defined by 'the same generators and relations' as M, but with coefficients in S.

The third functor, denoted $f^!$, also acts from R-Mod to S-Mod and is yet another natural way to combine the ingredients we have at our disposal: if M is an R-module, we have pointed out[14] in §3.2 that

$$f^!(M) := \mathrm{Hom}_R(S, M)$$

may be given a natural S-module structure (by setting $s\alpha(s') := \alpha(ss')$). This is again evidently a covariantly functorial prescription.

Proposition 3.6. *Let $f : R \to S$ be a homomorphism of commutative rings. Then, with notation as above, f_* is right-adjoint to f^* and left-adjoint to $f^!$. In particular, f_* is exact, f^* is right-exact, and $f^!$ is left-exact.*

Proof. Let M, resp., N, be an R-module, resp., an S-module. Note that, trivially, $\mathrm{Hom}_S(S, N)$ is canonically isomorphic to N (as an S-module) and to $f_*(N)$ (as an R-module). Thus[15]

$$\begin{aligned}\mathrm{Hom}_R(M, f_*(N)) &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N)) \\ &\cong \mathrm{Hom}_S(M \otimes_R S, N) = \mathrm{Hom}_S(f^*(M), N)\end{aligned}$$

where we have used Lemma 3.5. These bijections are canonical[16], proving that f^* is left-adjoint to f_*.

Similarly, there is a canonical isomorphism $N \cong N \otimes_S S$ (Example 2.2); thus $N \otimes_S S \cong f_*(N)$ as R-modules, and for every R-module M

$$\begin{aligned}\mathrm{Hom}_R(f_*(N), M) &\cong \mathrm{Hom}_R(N \otimes_S S, M) \\ &\cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_S(N, f^!(M))\end{aligned}$$

again by Lemma 3.5. This shows that $f^!$ is right-adjoint to f_*, concluding the proof. □

[14] The roles of R and S were reversed in §3.2.

[15] These are isomorphisms of abelian groups, and in fact isomorphisms of R-modules if one applies f_* to the Hom_S terms.

[16] The reader should check this. . . .

Remark 3.7 (Warning)**.** Our choice of notation, f_*, etc., is somewhat nonstandard, and the reader should not take it too literally. It is inspired by analogs in the context of sheaf theory over schemes, but some of the properties reviewed above require crucial adjustments in that wider context: for example f_* is *not* exact as a sheaf operation on schemes. ⌟

Exercises

In the following exercises, R, S denote commutative rings.

3.1. ▷ Verify that a combination of pure tensors $\sum_i (m_i \otimes n_i)$ is zero in the tensor product $M \otimes_R N$ if and only if $\sum_i (m_i, n_i) \in \mathbb{Z}^{\oplus(M\times N)}$ is a combination of elements of the form

$$(m, n_1 + n_2) - (m, n_1) - (m, n_2),$$
$$(m_1 + m_2, n) - (m_1, n) - (m_2, n),$$
$$(rm, n) - (m, rn),$$

with $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $r \in R$. [§3.1]

3.2. If $f : R \to S$ is a ring homomorphism and M, N are S-modules (hence R-modules by restriction of scalars), prove that there is a canonical homomorphism of R-modules $M \otimes_R N \to M \otimes_S N$.

3.3. ▷ Let R, S be commutative rings, and let M be an R-module, N an (R, S)-bimodule, and P as S-module. Prove that there is an isomorphism of R-modules

$$M \otimes_R (N \otimes_S P) \cong (M \otimes_R N) \otimes_S P.$$

In this sense, $\otimes$ is 'associative'. [3.4, §4.1]

3.4. ▷ Use the associativity of the tensor product (Exercise 3.3) to prove again the formulas given in Exercise 2.6. (Use Exercise 2.5.) [2.6]

3.5. Let $f : R \to S$ be a ring homomorphism. Prove that $f^!$ commutes with limits, f^* commutes with colimits, and f_* commutes with both. In particular, deduce that these three functors all preserve finite direct sums.

3.6. Let $f : R \to S$ be a ring homomorphism, and let $\varphi : N_1 \to N_2$ be a homomorphism of S-modules. Prove that φ is an isomorphism if and only if $f_*(\varphi)$ is an isomorphism. (Functors with this property are said to be *conservative*.) In fact, prove that f_* is *faithfully* exact: a sequence of S-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is exact *if and only if* the sequence of R-modules

$$0 \longrightarrow f_*(L) \longrightarrow f_*(M) \longrightarrow f_*(N) \longrightarrow 0$$

is exact. In particular, a sequence of R-modules is exact if and only if it is exact as a sequence of abelian groups. (This is completely trivial but useful nonetheless.)

3.7. Let $i : k \subseteq F$ be a finite field extension, and let W be an F-vector space of finite dimension n. Compute the dimension of $i_*(W)$ as a k-vector space (where i_* is restriction of scalars; cf. §3.3).

3.8. Let $i : k \subseteq F$ be a finite field extension, and let V be a k-vector space of dimension n. Compute the dimension of $i^*(V)$ and $i^!(V)$ as F-vector spaces.

3.9. Let $f : R \to S$ be a ring homomorphism, and let M be an R-module. Prove that the extension $f^*(M)$ satisfies the following universal property: if N is an S-module and $\varphi : M \to N$ is an R-linear map, then there exists a unique S-linear map $\tilde{\varphi} : f^*(M) \to S$ making the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\iota"'] & N \\ f^*(M) \arrow[ur, "\exists!\tilde{\varphi}"'] & \end{tikzcd}$$

commute, where $\iota : M \to f^*(M) = M \otimes_R S$ is defined by $m \mapsto m \otimes 1$. (Thus, $f^*(M)$ is the 'best approximation' to the R-module M in the category of S-modules.)

3.10. Prove the following *projection formula:* if $f : R \to S$ is a ring homomorphism, M is an R-module, and N is an S-module, then $f_*(f^*(M) \otimes_S N) \cong M \otimes_R f_*(N)$ as R-modules.

3.11. Let $f : R \to S$ be a ring homomorphism, and let M be a *flat* R-module. Prove that $f^*(M)$ is a flat S-module.

3.12. In 'geometric' contexts (such as the one hinted at in Remark 3.7), one would actually work with categories which are *opposite* to the category of commutative rings; cf. Example 1.9. A ring homomorphism $f : R \to S$ corresponds to a morphism $f^\circ : S^\circ \to R^\circ$ in the opposite category, and we can simply define $f^\circ{}_*$, etc., to be f_*, etc.

For morphisms $f^\circ : S^\circ \to R^\circ$ and $g^\circ : T^\circ \to S^\circ$ in the opposite category, prove that

- $(f^\circ \circ g^\circ)_* \cong f^\circ{}_* \circ g^\circ{}_*$,
- $(f^\circ \circ g^\circ)^* \cong g^{\circ *} \circ f^{\circ *}$,
- $(f^\circ \circ g^\circ)^! \cong g^{\circ !} \circ f^{\circ !}$,

where $\cong$ stands for 'naturally isomorphic'. (These are the formulas suggested by the notation: a $*$ in the subscript invariably suggests a basic 'covariance' property of the notation, while modifiers in the superscript usually suggest contravariance. The switch to the opposite category is natural in the algebro-geometric context.)

3.13. Let $p > 0$ be a prime integer, and let $\pi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the natural projection. Compute $\pi^*(A)$ and $\pi^!(A)$ for all finitely generated abelian groups A, as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Compute $\iota^*(A)$ and $\iota^!(A)$ for all finitely generated abelian groups A, where $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is the natural inclusion.

3.14. Let $f : R \to S$ be an *onto* ring homomorphism; thus, $S \cong R/I$ for some ideal I of R.

- Prove that, for all R-modules M, $f^!(M) \cong \{m \in M \mid \forall a \in I,\, am = 0\}$, while $f^*(M) \cong M/IM$. (Exercise III.7.7 may help.)
- Prove that, for all S-modules N, $f^! f_*(N) \cong N$ and $f^* f_*(N) \cong N$.
- Prove that f_* is fully faithful (Definition 1.6).
- Deduce that if there is an onto homomorphism $R \to S$, then S-Mod is equivalent to a full subcategory of R-Mod.

3.15. Let $f : R \to S$ be a ring homomorphism, and assume that the functor $f_* : S\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$ is an equivalence of categories.

- Prove that there is a homomorphism of rings $\overline{g} : S \to \mathrm{End}_{\mathsf{Ab}}(R)$ such that the composition $R \to S \to \mathrm{End}_{\mathsf{Ab}}(R)$ is the homomorphism realizing R as a module over itself (that is, the homomorphism studied in Proposition III.2.7).
- Use the fact that S is commutative to deduce that $\overline{g}(S)$ is isomorphic to R. (Refine the result of Exercise III.2.17.) Deduce that f has a left-inverse $g : S \to R$.
- Therefore, $f_* \circ g_*$ is naturally isomorphic to the identity; in particular, $f_* \circ g_*(S) \cong S$ as an R-module. Prove that this implies that g is injective. (If $a \in \ker g$, prove that a is in the annihilator of $f_* \circ g_*(S)$.)
- Conclude that f is an isomorphism.

Two rings are *Morita equivalent* if their category of left-modules are equivalent. The result of this exercise is a (very) particular case of the fact that two *commutative* rings are Morita equivalent if and only if they are isomorphic. The commutativity is crucial in this statement: for example, it can be shown that any ring R is Morita equivalent to the ring of matrices[17] $\mathcal{M}_{n,n}(R)$, for all $n > 0$.

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4. Multilinear algebra

4.1. Multilinear, symmetric, alternating maps. *Multilinear* maps may be defined similarly to bilinear maps: if $M_1, \dots, M_\ell, P$ are R-modules, a function

$$\varphi : M_1 \times \cdots \times M_\ell \to P$$

is *R-multilinear* if it is R-linear in each factor, that is, if the function obtained by arbitrarily fixing all but the i-th component is R-linear in the i-th factor, for $i = 1, \dots, r$.

Again it is natural to ask whether R-multilinear maps may be turned into R-linear maps: whether there exists an R-module $M_1 \otimes_R \cdots \otimes_R M_\ell$ through which every R-multilinear map must factor. Luckily, this module is already available to us:

Claim 4.1. *Every R-multilinear map $M_1 \times \cdots \times M_\ell \to P$ factors uniquely through*

$$((\cdots(M_1 \otimes_R M_2) \otimes_R \cdots) \otimes_R M_{\ell-1}) \otimes_R M_\ell.$$

[17]The author was once told that $\mathcal{M}_{n,n}(\mathbb{C})$ is ‘not seriously noncommutative’ since it is Morita equivalent to $\mathbb{C}$, which is commutative.