Chapter 3

Absolutely Continuous Functions

Undergradese, III: "Hmmm, what do you mean by that?" Translation: "What's the answer so we can all go home?"

— Jorge Cham, www.phdcomics.com

The Cantor function shows that for monotone functions the fundamental theorem of calculus fails for Lebesgue integration. Indeed,

$$\int_0^1 u'(t)\, dt = 0 < u(1) - u(0) = 1.$$

To recover it in the context of Lebesgue integration, we need to restrict ourselves to a subclass of functions of bounded pointwise variation. This leads us to the notion of absolute continuity.

3.1. $AC(I)$ Versus $BPV(I)$

Definition 3.1. Let $I \subset \mathbb{R}$ be an interval. A function $u : I \to \mathbb{R}$ is said to be *absolutely continuous* on I if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{\ell} |u(b_k) - u(a_k)| \le \varepsilon \tag{3.1}$$

for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subset I$ and

$$\sum_{k=1}^{\ell} (b_k - a_k) \le \delta.$$

The space of all absolutely continuous functions $u : I \to \mathbb{R}$ is denoted by $AC(I)$.

Remark 3.2. Note that since ℓ is arbitrary, we can also take $\ell = \infty$, namely, replace finite sums by series.

A function $u : I \to \mathbb{R}$ is *locally absolutely continuous* if it is absolutely continuous in $[a, b]$ for every interval $[a, b] \subset I$. The space of all locally absolutely continuous functions $u : I \to \mathbb{R}$ is denoted by $AC_{\mathrm{loc}}(I)$. Note that

$$AC_{\mathrm{loc}}([a, b]) = AC([a, b]).$$

If $u : I \to \mathbb{R}^d$, then we can define the notion of absolute continuity exactly as in Definition 3.1, with the only difference that the absolute value is now replaced by the norm in $\mathbb{R}^d$. The space of all absolutely continuous functions $u : I \to \mathbb{R}^d$ (respectively, locally absolutely continuous) is denoted by $AC\left(I; \mathbb{R}^d\right)$ (respectively, $AC_{\mathrm{loc}}\left(I; \mathbb{R}^d\right)$).

If $\Omega \subset \mathbb{R}$ is an open set, then we define the notion of absolute continuity for a function $u : \Omega \to \mathbb{R}$ as in Definition 3.1, with the only change that we now require the intervals $[a_k, b_k]$ to be contained in Ω in place of I. The space of all absolutely continuous functions $u : \Omega \to \mathbb{R}$ is denoted by $AC(\Omega)$.

Exercise 3.3. Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$.

(i) Prove that u belongs to $AC(I)$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{k=1}^{\ell} \left(u(b_k) - u(a_k)\right) \right| \leq \varepsilon$$

for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subset I$ and

$$\sum_{k=1}^{\ell} (b_k - a_k) \leq \delta.$$

(ii) Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{k=1}^{\ell} \left(u(b_k) - u(a_k)\right) \right| \leq \varepsilon$$

for every finite number of intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subset I$ and

$$\sum_{k=1}^{\ell} (b_k - a_k) \leq \delta.$$

Prove that u is locally Lipschitz.

Part (ii) of the previous exercise shows that in Definition 3.1 we cannot remove the condition that the intervals (a_k, b_k) are pairwise disjoint.

By taking $\ell = 1$ in Definition 3.1, it follows that an absolutely continuous function $u : I \to \mathbb{R}$ is uniformly continuous. The next exercise shows that the converse is false.

Exercise 3.4. Let $u : (0, 1] \to \mathbb{R}$ be defined by

$$u(x) := x^a \sin \frac{1}{x^b},$$

where $a, b \in \mathbb{R}$. Study to see for which a, b the function u is absolutely continuous. Prove that there exist a, b for which u is uniformly continuous but not absolutely continuous.

Exercise 3.5. Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be differentiable with bounded derivative. Prove that u belongs to $AC(I)$.

Exercise 3.6. Let $u, v \in AC([a, b])$. Prove the following.

(i) $u \pm v \in AC([a, b])$.

(ii) $uv \in AC([a, b])$.

(iii) If $v(x) > 0$ for all $x \in [a, b]$, then $\frac{u}{v} \in AC([a, b])$.

(iv) What happens if the interval $[a, b]$ is replaced by an arbitrary interval $I \subset \mathbb{R}$ (possibly unbounded)?

We now turn to the relation between absolutely continuous functions and functions of bounded pointwise variation. In Corollary 2.23 we have proved that if $u : I \to \mathbb{R}$ has bounded pointwise variation, then u is bounded and u' is Lebesgue integrable. However, the function $u(x) := x$, $x \in \mathbb{R}$, is absolutely continuous, but it is unbounded and $u'(x) = 1$, which is not Lebesgue integrable. Also the function $u(x) := \sin x$, $x \in \mathbb{R}$, is absolutely continuous, bounded, but u' is not Lebesgue integrable. These simple examples show that an absolutely continuous function may not have bounded pointwise variation. Proposition 3.8 below will show that this can happen only on unbounded intervals.

Exercise 3.7. Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be uniformly continuous.

(i) Prove that u may be extended uniquely to $\overline{I}$ in such a way that the extended function is still uniformly continuous.

(ii) Prove that if u belongs to $AC(I)$, then its extension belongs to $AC(\overline{I})$.

(iii) Prove that there exist $A, B > 0$ such that for all $x \in I$,

$$|u(x)| \le A + B|x|.$$

The previous exercise shows that although an absolutely continuous function may be unbounded, it cannot grow faster than linear when $|x| \to \infty$.

Next we show that $AC_{\mathrm{loc}}(I) \subset BPV_{\mathrm{loc}}(I)$.

Proposition 3.8. *Let $I \subset \mathbb{R}$ be an interval and let $u \in AC_{\mathrm{loc}}(I)$ (respectively, $AC(I)$). Then u belongs to $BPV_{\mathrm{loc}}(I)$ (respectively, $BPV(J)$ for every bounded subinterval J of I). In particular, u is differentiable $\mathcal{L}^1$-a.e. in I and u' is locally Lebesgue integrable (respectively, Lebesgue integrable on bounded subintervals of I).*

Proof. Step 1: Assume that $u \in AC_{\mathrm{loc}}(I)$ and let $[a,b] \subset I$. Take $\varepsilon = 1$, and let $\delta > 0$ be as in Definition 3.1. Let n be the integer part of $\frac{2(b-a)}{\delta}$ and partition $[a,b]$ into n intervals $[x_{i-1}, x_i]$ of equal length $\frac{b-a}{n}$,

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Since $\frac{b-a}{n} \leq \delta$, in view of (3.1), on each interval $[x_{i-1}, x_i]$ we have that $\mathrm{Var}_{[x_{i-1},x_i]} u \leq 1$, and so by Remark 2.7,

$$\mathrm{Var}_{[a,b]} u = \sum_{i=1}^{n} \mathrm{Var}_{[x_{i-1},x_i]} u \leq n \leq \frac{2(b-a)}{\delta} < \infty,$$

where we have used the fact that $\frac{b-a}{n} \geq \frac{\delta}{2}$.

Step 2: Assume that $u \in AC(I)$ and let $J \subset I$ be a bounded interval. By Exercise 3.7 we may extend u uniquely to a function $u : \overline{J} \to \mathbb{R}$ such that u belongs to $AC\left(\overline{I}\right)$. The previous step (applied to $\overline{J}$ in place of I) implies that $\mathrm{Var}_{\overline{J}} u < \infty$, and so, in particular, $\mathrm{Var}_J u < \infty$.

This completes the proof. □

Corollary 3.9. *Let $I \subset \mathbb{R}$ be a bounded interval. Then $AC(I) \subset BPV(I)$. In particular, if $u \in AC(I)$, then u is differentiable $\mathcal{L}^1$-a.e. in I and u' is Lebesgue integrable.*

The converse of the previous corollary is false, since absolutely continuous functions are always continuous while monotone functions may not be. Even more, there exist continuous monotone functions that are not absolutely continuous. The Cantor function and the function constructed in Theorem 1.47 are such examples. What is missing for a continuous function in $BPV(I)$ to belong to $AC(I)$ is the so-called (N) property.

We recall the following definition.

Definition 3.10. If $E \subset \mathbb{R}$ is a Lebesgue measurable set and $v : E \to \mathbb{R}$ is a Lebesgue measurable function, then v is *equi-integrable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_F |v(x)| \, dx \leq \varepsilon$$

for every Lebesgue measurable set $F \subset E$, with $\mathcal{L}^1(F) \le \delta$.

Exercise 3.11. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set, let $1 \le p \le \infty$, and let $v \in L^p(E)$. Prove that v is equi-integrable. Prove that if we only assume that $v \in L^1_{\mathrm{loc}}(E)$, then the result may no longer be true.

Theorem 3.12 (Lusin (N) property)**.** *Let $I \subset \mathbb{R}$ be an interval. A function $u : I \to \mathbb{R}$ belongs to $AC_{\mathrm{loc}}(I)$ if and only if*

(i) *u is continuous on I,*

(ii) *u is differentiable $\mathcal{L}^1$-a.e. in I, and $u' \in L^1_{\mathrm{loc}}(I)$,*

(iii) *u maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.*

Property (iii) is called the *Lusin (N) property.* We begin with some preliminary results, which are of interest in themselves.

Lemma 3.13. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$. Assume that there exist a set $E \subset I$ (not necessarily measurable) and $M \ge 0$ such that u is differentiable* for all $x \in E$, *with*

$$\left|u'(x)\right| \le M \quad \text{for all } x \in E.$$

Then

$$\mathcal{L}^1_o(u(E)) \le M\mathcal{L}^1_o(E).$$

Proof. Without loss of generality we may assume that $E \subset I^\circ$. Fix $\varepsilon > 0$ and for each $n \in \mathbb{N}$ let E_n be the set of points $x \in E$ such that

$$\mathcal{L}^1_o(u(J)) \le (M+\varepsilon)\,\mathcal{L}^1_o(J) \tag{3.2}$$

for all intervals $J \subset I$ such that $x \in J$ and $0 < \operatorname{length} J < \frac{1}{n}$. Note that $E_n \subset E_{n+1}$. We claim that

$$E = \bigcup_{n=1}^{\infty} E_n. \tag{3.3}$$

Since each E_n is contained in E, to prove the claim, it suffices to prove that each point $x \in E$ belongs to some E_n. Fix $x \in E$. Since $|u'(x)| \le M$ and

$$\lim_{y \to x} \frac{u(y) - u(x)}{y - x} = u'(x),$$

there exists $\delta > 0$ such that

$$|u(y) - u(x)| \le (M+\varepsilon)\,|y - x|$$

for all $y \in I$, with $|y - x| < \delta$. Hence, if $y, y' \in I$, with $|y - y'| < \delta$ and $y < x < y'$,

$$\begin{aligned}\left|u(y) - u\left(y'\right)\right| &\leq |u(y) - u(x)| + \left|u\left(y'\right) - u(x)\right| \\ &\leq (M + \varepsilon)(x - y) + (M + \varepsilon)\left(y' - x\right) \\ &= (M + \varepsilon)\left(y' - y\right).\end{aligned}$$

This implies that $x \in E_n$ for every integer $n > \frac{1}{\delta}$, and so the claim is proved.

We now fix $n \in \mathbb{N}$ and we prove that

$$\mathcal{L}_o^1(u(E_n)) \leq (M + \varepsilon)\left(\mathcal{L}_o^1(E_n) + \varepsilon\right). \tag{3.4}$$

By the definition of $\mathcal{L}_o^1(E_n)$ we may find an open set U_n such that $U_n \supset E_n$ and

$$\mathcal{L}^1(U_n) \leq \mathcal{L}_o^1(E_n) + \varepsilon.$$

By replacing U_n with $U_n \cap I^\circ$, if necessary, we may suppose that $U_n \subset I^\circ$. Decompose U_n as a countable family $\left\{J_k^{(n)}\right\}$ of pairwise disjoint intervals with $0 < \operatorname{length} J_k^{(n)} < \frac{1}{n}$. Define

$$\mathcal{I} := \left\{k : J_k^{(n)} \cap E_n \neq \emptyset\right\}.$$

Note that by (3.2) if $k \in \mathcal{I}$, then

$$\mathcal{L}_o^1\left(u\left(J_k^{(n)}\right)\right) \leq (M + \varepsilon)\,\mathcal{L}_o^1\left(J_k^{(n)}\right),$$

and so

$$\begin{aligned}\mathcal{L}_o^1(u(E_n)) &\leq \mathcal{L}_o^1\left(\bigcup_{k \in \mathcal{I}} u\left(J_k^{(n)}\right)\right) \leq \sum_{k \in \mathcal{I}} \mathcal{L}_o^1\left(u\left(J_k^{(n)}\right)\right) \\ &\leq (M + \varepsilon) \sum_{k \in \mathcal{I}} \mathcal{L}^1\left(J_k^{(n)}\right) \leq (M + \varepsilon)\,\mathcal{L}_o^1\left(\bigcup_k J_k^{(n)}\right) \\ &= (M + \varepsilon)\,\mathcal{L}^1(U_n) \leq (M + \varepsilon)\left(\mathcal{L}_o^1(E_n) + \varepsilon\right),\end{aligned}$$

where we have used the fact that the intervals $J_k^{(n)}$ are pairwise disjoint. Hence, (3.4) holds.

Since $\mathcal{L}_o^1$ is a regular outer measure and $\{E_n\}$ (and in turn $\{u(E_n)\}$) is an increasing sequence, by Proposition B.105 in Appendix B we may let $n \to \infty$ in the previous inequality to obtain

$$\mathcal{L}_o^1(u(E)) \leq (M + \varepsilon)\left(\mathcal{L}_o^1(E) + \varepsilon\right).$$

It now suffices to let $\varepsilon \to 0^+$. □

As a consequence of the previous lemma we have the following result.

Corollary 3.14. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$. Assume that there exists a set $E \subset I$ such that u is differentiable* for all $x \in E$. *If the set E has Lebesgue measure zero or if $u' = 0$ in E, then $\mathcal{L}^1(u(E)) = 0$.*

Proof. Assume that E has Lebesgue measure zero. For every $n \in \mathbb{N}$ write $E_n := \{x \in E : |u'(x)| \le n\}$. Since u is differentiable in E, it follows that

$$E = \bigcup_{n=1}^{\infty} E_n,$$

while by the previous lemma

$$\mathcal{L}_o^1(u(E_n)) \le n\mathcal{L}_o^1(E_n) = 0.$$

By the countable subadditivity of $\mathcal{L}_o^1$ we obtain that $\mathcal{L}^1(u(E)) = 0$.

If $u' = 0$ in E, then we may take $M := 0$ in the previous lemma. □

Remark 3.15. The Cantor function shows that the previous corollary does not hold if we replace everywhere differentiability in E with $\mathcal{L}^1$-a.e. differentiability in E.

Another important consequence of Lemma 3.13 is the following.

Lemma 3.16. *Let $I \subset \mathbb{R}$ be an interval, let $u : I \to \mathbb{R}$ be a Lebesgue measurable function, and let $E \subset I$ be a Lebesgue measurable set on which u is differentiable. Then $u(E)$ is Lebesgue measurable and*

$$\mathcal{L}^1(u(E)) \le \int_E |u'(x)|\, dx. \tag{3.5}$$

Proof. Step 1: By the properties of the Lebesgue measure we may write E as the union of a set of Lebesgue measure zero and countably many compact sets, precisely,

$$E = E_0 \cup \bigcup_n K_n,$$

where K_n is compact and $\mathcal{L}^1(E_0) = 0$. By the previous corollary $\mathcal{L}^1(u(E_0)) = 0$, while $u(K_n)$ is Lebesgue measurable since $u : K_n \to \mathbb{R}$ is continuous (since differentiable). Hence, $u(E)$ is Lebesgue measurable.

Step 2: Assume that $\mathcal{L}^1(E) < \infty$. Fix $n \in \mathbb{N}$. For every $k \in \mathbb{N}$ write

$$E_n^k := \left\{x \in E : \frac{k-1}{2^n} \le |u'(x)| < \frac{k}{2^n}\right\}.$$

Then

$$E = \bigcup_{k=1}^{\infty} E_n^k,$$

and so by the previous lemma

$$\begin{aligned}\mathcal{L}^1(u(E)) &= \mathcal{L}^1\left(\bigcup_{k=1}^{\infty} u\left(E_n^k\right)\right) \le \sum_{k=1}^{\infty} \mathcal{L}^1\left(u\left(E_n^k\right)\right) \\ &\le \sum_{k=1}^{\infty} \frac{k}{2^n}\mathcal{L}^1\left(E_n^k\right) = \sum_{k=1}^{\infty} \frac{k-1}{2^n}\mathcal{L}^1\left(E_n^k\right) + \frac{1}{2^n}\sum_{k=1}^{\infty}\mathcal{L}^1\left(E_n^k\right) \\ &\le \sum_{k=1}^{\infty}\int_{E_n^k}\left|u'(x)\right|\,dx + \frac{\mathcal{L}^1(E)}{2^n} \le \int_E \left|u'(x)\right|\,dx + \frac{\mathcal{L}^1(E)}{2^n},\end{aligned}$$

and it suffices to let $n \to \infty$.

Step 3: If $\mathcal{L}^1(E) = \infty$, for every $k \in \mathbb{Z}$ write

$$E_k := E \cap [k, k+1].$$

Then by the previous step applied to E_k we have

$$\begin{aligned}\mathcal{L}^1(u(E)) &= \mathcal{L}^1\left(u\left(\bigcup_{k=-\infty}^{\infty} E_k\right)\right) \le \sum_{k=-\infty}^{\infty} \mathcal{L}^1(u(E_k)) \\ &\le \sum_{k=-\infty}^{\infty}\int_{E_k}\left|u'(x)\right|\,dx = \int_E \left|u'(x)\right|\,dx,\end{aligned}$$

and the proof is complete. ☐

Remark 3.17. Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$. If u is differentiable on a interval $[a,b] \subset I$, then, in particular, it is continuous on $[a,b]$, and so $u([a,b])$ contains the interval of endpoints $u(a)$ and $u(b)$. Hence by (3.5),

$$|u(b) - u(a)| \le \mathcal{L}^1(u((a,b))) \le \int_a^b \left|u'(x)\right|\,dx.$$

We now turn to the proof of Theorem 3.12.

Proof of Theorem 3.12. Step 1: Assume that u satisfies (i)–(iii) and fix $[a,b] \subset I$. We claim that u belongs to $AC([a,b])$. Let $\{(a_k,b_k)\}_k$ be a finite number of nonoverlapping intervals of $[a,b]$ and let

$$E_k := \left\{x \in (a_k,b_k) : \ u'(x) \ \text{exists}\right\}.$$

By (ii), $\mathcal{L}^1((a_k,b_k) \setminus E_k) = 0$, and so by (iii), $\mathcal{L}^1(u((a_k,b_k) \setminus E_k)) = 0$. Since u is continuous, by the intermediate value theorem it assumes all values between $u(a_k)$ and $u(b_k)$, and so the open interval of endpoints $u(a_k)$ and

$u(b_k)$ is contained in $u((a_k, b_k))$. Therefore by Lemma 3.16,

$$\sum_k |u(b_k) - u(a_k)| \le \sum_k \mathcal{L}^1(u((a_k, b_k))) = \sum_k \mathcal{L}^1(u(E_k))$$
$$\le \sum_k \int_{E_k} |u'(x)|\, dx = \sum_k \int_{a_k}^{b_k} |u'(x)|\, dx.$$

Since, by (ii), u' is Lebesgue integrable in $[a,b]$, in view of Exercise 3.11, u' is equi-integrable in $[a,b]$. Hence, given $\varepsilon > 0$, we may find $\delta > 0$ such that if $\sum_k (b_k - a_k) \le \delta$, then $\sum_k \int_{a_k}^{b_k} |u'(x)|\, dx \le \varepsilon$. The absolute continuity of u in $[a,b]$ follows from the previous inequality.

Step 2: Conversely, assume that $u \in AC_{\text{loc}}(I)$. Fix $[a,b] \subset I$. By Proposition 3.8 and Corollary 2.23, u is differentiable $\mathcal{L}^1$-a.e. in $[a,b]$ and u' is Lebesgue integrable in $[a,b]$. It remains to show that u satisfies property (iii). Thus, fix a Lebesgue measurable set $E \subset [a,b]$ with $\mathcal{L}^1(E) = 0$. Fix $\varepsilon > 0$ and let $\delta > 0$ be as in Definition 3.1. Since $\mathcal{L}^1(E) = 0$, we may find an open set $A \supset E$ such that $\mathcal{L}^1(A) \le \delta$. Decompose $A \cap [a,b]$ into a countable family $\{J_k\}$ of pairwise disjoint intervals. Since u is continuous, for every k we may find $a_k, b_k \in \overline{J_k}$ such that

$$u(a_k) = \min_{x \in \overline{J_k}} u(x), \quad u(b_k) = \max_{x \in \overline{J_k}} u(x),$$

so that $\overline{u(J_k)} = [u(a_k), u(b_k)]$. Using the fact that $\mathcal{L}^1(A) \le \delta$, we have that

$$\sum_k |b_k - a_k| \le \delta,$$

and so, by the absolute continuity of u and Remark 3.2,

$$\sum_k |u(b_k) - u(a_k)| \le \varepsilon.$$

Hence,

$$\mathcal{L}^1(u(E)) \le \mathcal{L}^1(u(A \cap [a,b]))$$
$$\le \sum_k \mathcal{L}^1(u(J_k)) \le \sum_k \mathcal{L}^1\left(\overline{u(J_k)}\right)$$
$$= \sum_k \mathcal{L}^1([u(a_k), u(b_k)]) = \sum_k |u(b_k) - u(a_k)| \le \varepsilon.$$

Given the arbitrariness of $\varepsilon > 0$, we conclude that $\mathcal{L}^1(u(E)) = 0$. □

Remark 3.18. Note that since the sets E_k defined in the first part of the proof are Borel sets (why?), the previous theorem continues to hold if in place of the (N) property we only require that u map Borel sets of Lebesgue measure zero into sets of Lebesgue measure zero.

Remark 3.19. Note that the Cantor function does not satisfy the (N) property since it sends a set of Lebesgue measure zero, the Cantor set $\mathbb{D}$, into the full interval $[0,1]$.

Exercise 3.20. Let $u : [a,b] \to \mathbb{R}$ be continuous and strictly increasing. Prove that u is absolutely continuous if and only if it maps the set

$$E := \left\{x \in [a,b] : u'(x) = \infty\right\}$$

into a set of Lebesgue measure zero.

Exercise 3.21. Let $u : [a,b] \to \mathbb{R}$ be continuous and strictly increasing. Prove that its inverse $u^{-1} : [u(a), u(b)] \to \mathbb{R}$ is absolutely continuous if and only if the set

$$E := \left\{x \in [a,b] : u'(x) = 0\right\}$$

has Lebesgue measure zero.

In view of Remark 3.17 and Step 1 of the proof of Theorem 3.12, we have the following.

Corollary 3.22. *Let $I \subset \mathbb{R}$ be an interval. If $u : I \to \mathbb{R}$ is* everywhere *differentiable in I and $u' \in L^1_{\mathrm{loc}}(I)$, then u belongs to $AC_{\mathrm{loc}}(I)$.*

Exercise 3.23. Prove that if $u : [a,b] \to \mathbb{R}$ is continuous, differentiable on $[a,b]$ except for at most a countable number of points, and if u' is Lebesgue integrable, then u belongs to $AC([a,b])$.

Corollary 3.24. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be such that*

(i) *u is continuous on I,*

(ii) *u is differentiable $\mathcal{L}^1$-a.e. in I, and u' belongs to $L^1_{\mathrm{loc}}(I)$ and is equi-integrable,*

(iii) *u maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.*

Then u belongs to $AC(I)$.

Proof. It is enough to repeat Step 1 of the proof of Theorem 3.12 word for word, with the only differences that $[a,b]$ should be replaced by I and that Exercise 3.11 is no longer needed, since, by (ii), u' is assumed to be equi-integrable. □

Remark 3.25. Note that if I is a bounded interval, then $u : I \to \mathbb{R}$ belongs to $AC(I)$ if and only if (i)–(iii) of the previous corollary hold. Indeed, if $u \in AC(I)$, then by Corollary 3.9, u' is integrable, and so equi-integrable by Exercise 3.11. Thus, property (ii) of the previous corollary holds. Properties (i) and (iii) follow from Theorem 3.12.

We will prove later on (see Corollary 3.41) that conditions (i)–(iii) in the previous corollary are actually necessary and sufficient for $u : I \to \mathbb{R}$ to be absolutely continuous.

The next corollary will be useful in the study of Sobolev spaces in Chapter 7.

Corollary 3.26. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be such that*

(i) *u is continuous on I,*

(ii) *u is differentiable $\mathcal{L}^1$-a.e. in I, and $u' \in L^p(I)$ for some $1 \le p \le \infty$,*

(iii) *u maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.*

Then u belongs to $AC(I)$.

Proof. In view of the previous corollary, it remains to show that u' is locally Lebesgue integrable and equi-integrable. The fact that u' is locally Lebesgue integrable follows by Hölder's inequality, while equi-integrability follows from Exercise 3.11. □

As a consequence of Theorem 3.12 we can characterize those functions with locally bounded pointwise variation that are locally absolutely continuous functions. Precisely, we have the following result.

Corollary 3.27. *Let $I \subset \mathbb{R}$ be an interval. A function $u : I \to \mathbb{R}$ belongs to $AC_{\text{loc}}(I)$ if and only if*

(i) *u is continuous on I,*

(ii) *$u \in BPV_{\text{loc}}(I)$,*

(iii) *u maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.*

Proof. In view of Corollary 2.23 we are in a position to apply Theorem 3.12. □

The next exercise gives an example of a function that satisfies the (N) property, but it is not of bounded pointwise variation in any interval of $(0,1)$.

Exercise 3.28. Let $\{(a_n, b_n)\}$ be a base for the topology of $(0,1)$ and let $\{r_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} r_n < \infty$. We construct inductively a sequence of functions $u_n : [0,1] \to \mathbb{R}$ and a sequence of intervals (c_n, d_n) as follows. Let $u_0(x) := x$. Assume that $u_{n-1} : [0,1] \to \mathbb{R}$ has been defined and that u_n is a continuous piecewise affine function. Let $(c_n, d_n) \subset (a_n, b_n)$ be an interval such that u_{n-1} restricted to (c_n, d_n) is linear and $\mathcal{L}^1(u_{n-1}((c_n,d_n))) < r_n$. We define the continuous function u_n to be u_{n-1} outside (c_n,d_n), while in (c_n,d_n) we define it as a continuous piecewise affine function such that $\mathcal{L}^1(u_n((c_n,d_n))) < r_n$ and $\operatorname{Var}_{(c_n,d_n)} u_n > n$.

(i) Prove that $\{u_n\}$ converges uniformly to a continuous function $u : [0,1] \to \mathbb{R}$.

(ii) Prove that for every interval $[a,b] \subset (0,1)$ with $a < b$, $\operatorname{Var}_{[a,b]} u = \infty$.

(iii) Prove that u has the (N) property. Hint: Let

$$E_0 := [0,1] \setminus \bigcup_{i=1}^{\infty} (c_i,d_i), \quad E_n := (c_n,d_n) \setminus \bigcup_{i=n+1}^{\infty} (c_i,d_i),$$

and

$$E_\infty := \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} (c_i,d_i).$$

For every set $E \subset [0,1]$, with $\mathcal{L}^1(E) = 0$, write

$$E = (E \cap E_\infty) \cup \bigcup_{n=0}^{\infty} (E \cap E_n).$$

Exercise 3.29 (The Cantor set and the (N) property)**.** Let $\mathbb{D}$ be the Cantor set.

(i) Prove that every number $x \in [0,2]$ can be written as

$$x = 2\sum_{n=1}^{\infty} \frac{c_n}{3^n},$$

where $c_n \in \{0,1,2\}$, and deduce that every element in $[0,2]$ can be written as the sum of two elements of $\mathbb{D}$.

(ii) Prove that if $v : \mathbb{D} \to \mathbb{R}$ is continuous and $\mathcal{L}^1(v(\mathbb{D})) = 0$, then v can be extended to a continuous function on $[0,1]$ that has the (N) property. Hint: Make v differentiable outside $\mathbb{D}$.

(iii) For each $x \in \mathbb{D}$ write

$$x = 2\sum_{n=1}^{\infty} \frac{c_n(x)}{3^n},$$

where $c_n \in \{0,1\}$ and prove that there exist two continuous functions $u_1 : [0,1] \to \mathbb{R}$ and $u_2 : [0,1] \to \mathbb{R}$ with the (N) property and such that for all $x \in \mathbb{D}$,

$$u_1(x) = \sum_{n=1}^{\infty} \frac{c_{2n}(x)}{3^n}, \quad u_2(x) = \sum_{n=1}^{\infty} \frac{c_{2n+1}(x)}{3^n}.$$

(iv) Prove that $u_1 + u_2$ does not have the (N) property.

(v) Why is this example important for absolute continuity?

We now show that absolutely continuous functions can be characterized as the family of all functions for which the fundamental theorem of calculus holds (for the Lebesgue integral).

Theorem 3.30 (Fundamental theorem of calculus). *Let $I \subset \mathbb{R}$ be an interval. A function $u : I \to \mathbb{R}$ belongs to $AC_{\mathrm{loc}}(I)$ if and only if*

(i) *u is continuous in I,*

(ii) *u is differentiable $\mathcal{L}^1$-a.e. in I, and u' belongs to $L^1_{\mathrm{loc}}(I)$,*

(iii) *the fundamental theorem of calculus is valid; that is, for all $x, x_0 \in I$,*

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(t)\, dt.$$

The proof hinges upon on a preliminary result, which is of independent interest.

Lemma 3.31. *Let $I \subset \mathbb{R}$ be an interval and let $v : I \to \mathbb{R}$ be a Lebesgue integrable function. Fix $x_0 \in I$ and let*

$$u(x) := \int_{x_0}^{x} v(t)\, dt, \quad x \in I.$$

Then the function u is absolutely continuous in I and $u'(x) = v(x)$ for $\mathcal{L}^1$-a.e. $x \in I$.

Proof. The facts that u is absolutely continuous and differentiable for $\mathcal{L}^1$-a.e. $x \in I$ follow from Exercise 3.11 and Proposition 3.8, respectively. In the remainder of the proof we show that $u'(x) = v(x)$ for $\mathcal{L}^1$-a.e. $x \in I$.

Step 1: Assume first that $v = \chi_E$ for some Lebesgue measurable set $E \subset \mathbb{R}$. Fix a bounded open interval $J \subset I$ containing x_0 in its closure. We claim that

$$u'(x) = 1 \text{ for } \mathcal{L}^1\text{-a.e. } x \in E \cap J.$$

By the definition of Lebesgue outer measure we may find a decreasing sequence $\{U_n\}$ of open sets such that $U_n \supset E \cap J$ and

$$\mathcal{L}^1(U_\infty \setminus (E \cap J)) = 0, \quad \text{where } U_\infty := \bigcap_{n=1}^{\infty} U_n. \tag{3.6}$$

By replacing U_n with $U_n \cap J$, we may assume that $U_n \subset J$. Define

$$u_n(x) := \int_{x_0}^{x} \chi_{U_n}(t)\, dt, \quad x \in J.$$

Since $U_1 \subset J$, which is bounded, we are in a position to apply Lebesgue's dominated convergence theorem and (3.6) to conclude that for all $x \in J$,

$$\begin{aligned}\lim_{n\to\infty} u_n(x) &= \lim_{n\to\infty} \int_{x_0}^{x} \chi_{U_n}(t)\,dt = \int_{x_0}^{x} \chi_{U_\infty}(t)\,dt \\ &= \int_{x_0}^{x} \chi_{E\cap J}(t)\,dt = \int_{x_0}^{x} \chi_{E}(t)\,dt = u(x),\end{aligned}$$

where in the fourth equality we have used the fact that the open interval of endpoints x and x_0 is contained in J.

Hence, in the interval J we may write u in terms of the telescopic series

$$u = u_1 + \sum_{n=1}^{\infty} (u_{n+1} - u_n).$$

Note that since $U_n \supset U_{n+1}$,

$$(u_n - u_{n+1})(x) = \int_{x_0}^{x} \chi_{U_n \setminus U_{n+1}}(t)\,dt, \quad x \in J,$$

and since $\chi_{U_n \setminus U_{n+1}} \geq 0$, the function $u_n - u_{n+1}$ is monotone in the intervals $(-\infty, x_0) \cap J$ and $[x_0, \infty,) \cap J$. By Fubini's theorem (applied in each interval) we get that for $\mathcal{L}^1$-a.e. $x \in J$,

$$u'(x) = u_1'(x) + \sum_{n=1}^{\infty} \left(u_{n+1}'(x) - u_n'(x)\right) = \lim_{n\to\infty} u_n'(x).$$

On the other hand, if $x \in U_n$, then $u_n'(x) = 1$ (why?), and so, if $x \in U_\infty$, then $u_n'(x) = 1$ for all $n \in \mathbb{N}$. Hence, we have proved that $u'(x) = 1$ for $\mathcal{L}^1$-a.e. $x \in U_\infty$, and so, in particular, for $\mathcal{L}^1$-a.e. $x \in E \cap J$.

Next we show that $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in J \setminus E$. Since for all $x \in J$,

$$u(x) = \int_{x_0}^{x} \chi_E(t)\,dt = \int_{x_0}^{x} \left(1 - \chi_{J\setminus E}\right)(t)\,dt,$$

by applying what we just proved to $\chi_{J\setminus E}$, we conclude that $u'(x) = 1 - 1 = 0$ for $\mathcal{L}^1$-a.e. $x \in J \setminus E$. Thus, $u'(x) = \chi_E(x)$ for $\mathcal{L}^1$-a.e. $x \in J$. By letting $J \nearrow I$, we obtain the same result in I.

Step 2: By the linearity of the derivatives and Step 1 we conclude that if v is a simple function, then $u'(x) = v(x)$ for $\mathcal{L}^1$-a.e. $x \in I$. If v is a nonnegative Lebesgue measurable function, then we may construct an increasing sequence $\{s_n\}$ of nonnegative simple functions such that $s_n(x) \nearrow v(x)$ for $\mathcal{L}^1$-a.e. $x \in I$. Then by Lebesgue's dominated convergence theorem for all $x \in I$,

$$\lim_{n\to\infty} \int_{x_0}^{x} s_n(t)\,dt = \int_{x_0}^{x} v(t)\,dt = u(x),$$

and so we may proceed as in the first step (using telescopic series) to show that $u'(x) = v(x)$ for $\mathcal{L}^1$-a.e. $x \in I$.

In the general case, it suffices to write $v = v^+ - v^-$. □

We are now ready to prove Theorem 3.30.

Proof of Theorem 3.30. Assume that $u \in AC_{\text{loc}}(I)$. In view of Theorem 3.12, it remains to prove (iii). Let $[a, b] \subset I$ be so large that $x_0 \in [a, b]$ and define

$$w(x) := u(x) - \left(u(a) + \int_{x_0}^{x} u'(t)\, dt\right), \quad x \in [a, b].$$

By Lemma 3.31 and Theorem 3.12, there exists a Lebesgue measurable set $E \subset [a, b]$, with $\mathcal{L}^1(E) = 0$, such that for all $x \in [a, b] \setminus E$ the function w is differentiable at x and $w'(x) = 0$. By Corollary 3.14 we have that $\mathcal{L}^1(w([a, b] \setminus E)) = 0$. On the other hand, since w is absolutely continuous in $[a, b]$ (see Exercise 3.6 and Lemma 3.31), by Theorem 3.12 it sends sets of Lebesgue measure zero into sets of Lebesgue measure zero, and so $\mathcal{L}^1(w(E)) = 0$. Thus, we have shown that $\mathcal{L}^1(w([a, b])) = 0$. But since w is a continuous function, by the intermediate value theorem $w([a, b])$ is either a point or a proper interval. Thus, it has to be a point. In conclusion, we have proved that $w(x) \equiv \text{const}$. Since $w(x_0) = 0$, it follows that $w = 0$, and so (iii) holds for all $x \in [a, b]$. Given the arbitrariness of $[a, b]$, we have that (iii) holds for all $x \in I$.

Conversely, assume that (i)–(iii) are satisfied. Then again by Lemma 3.31, u belongs to $AC_{\text{loc}}(I)$ and the proof is complete. □

The next corollary follows from the previous theorem and Corollary 3.22.

Corollary 3.32. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be* everywhere *differentiable. If $u' \in L^1_{\text{loc}}(I)$, then for all $x, x_0 \in I$,*

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(t)\, dt.$$

Using the previous corollary, we are in a position to complete the proof of the Katznelson–Stromberg theorem. We begin with some well-known results on Riemann integration.

Exercise 3.33 (Riemann integration, I)**.** Let $u : [a, b] \to \mathbb{R}$ be a bounded function and for $x \in [a, b]$ define

$$\omega(x) := \lim_{\delta \to 0^+} \sup \{|u(x_1) - u(x_2)| : x_1, x_2 \in [a, b], \\ |x_1 - x| \le \delta,\ |x_2 - x| \le \delta\}.$$

(i) Prove that u is continuous at $x \in [a, b]$ if and only if $\omega(x) = 0$.

(ii) Prove that if the set

$$E := \{x \in [a,b] : \ u \text{ is discontinuous at } x\}$$

has positive Lebesgue outer measure, then there exists a constant $\alpha > 0$ such that the set

$$E_1 := \{x \in E : \ \omega(x) \geq \alpha\}$$

has positive Lebesgue outer measure.

(iii) Let $t := \mathcal{L}_o^1(E_1) > 0$ and prove that for every partition P the intervals containing points of E_1 in their interior have total length greater than or equal to t.

(iv) Deduce that if u is Riemann integrable, then E has Lebesgue measure zero.

Exercise 3.34 (Riemann integration, II)**.** Let $u : [a,b] \to \mathbb{R}$ be a bounded function.

(i) Prove that if $g : [a,b] \to \mathbb{R}$ is Lipschitz and $g'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in [a,b]$, then g is constant.

(ii) Let

$$g(x) := \overline{\int_a^x} u(t)\, dt - \underline{\int_a^x} u(t)\, dt, \quad a < x \leq b, \quad g(a) := 0,$$

where $\overline{\int_a^x}$ and $\underline{\int_a^x}$ are the upper and lower Riemann integrals. Prove that g is Lipschitz.

(iii) Prove that if u is continuous at x, then g is differentiable at x and $g'(x) = 0$.

(iv) Deduce that if u is continuous $\mathcal{L}^1$-a.e. in $[a,b]$, then u is Riemann integrable.

Proof of Theorem 2.26, continued. We claim that u' cannot be Riemann integrable on any closed interval $[a,b]$. Indeed, by the previous exercises, this would imply that u' is continuous except for a set of Lebesgue outer measure zero. But, since u is nowhere monotone, if u' is continuous at x, then necessarily $u'(x) = 0$. Hence $u' = 0$ $\mathcal{L}^1$-a.e. on $[a,b]$. By the previous corollary, we would get that u is a constant in $[a,b]$, which is a contradiction. □

Remark 3.35. Note that the function constructed in Theorem 2.26 satisfies the hypotheses of Corollary 3.32. Hence, it provides an example of a differentiable function for which the fundamental theorem of calculus holds for Lebesgue integration but not for Riemann integration.

Exercise 3.36. Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a convex function.

(i) Prove that the left and right derivatives $f'_-(x)$ and $f'_+(x)$ exist in $\mathbb{R}$ for all $x \in I^\circ$.

(ii) Prove that the functions f'_- and f'_+ are increasing.

(iii) Prove that

$$f(y) - f(x) = \int_x^y f'_-(t)\,dt = \int_x^y f'_+(t)\,dt \tag{3.7}$$

for all $x, y \in I$ with $x < y$.

As a corollary of Theorem 3.30 we recover the formula for integration by parts.

Corollary 3.37 (Integration by parts). *Let $I \subset \mathbb{R}$ be an interval and let $u, v \in AC_{\text{loc}}(I)$. Then for all $x, x_0 \in I$,*

$$\int_{x_0}^x uv'\,dt = u(x)\,v(x) - u(x_0)\,v(x_0) - \int_{x_0}^x u'v\,dt.$$

Proof. Since $u, v \in AC_{\text{loc}}(I)$, by Exercise 3.6, $uv \in AC_{\text{loc}}(I)$, and thus, by part (iii) of Theorem 3.30,

$$u(x)\,v(x) - u(x_0)\,v(x_0) = \int_{x_0}^x (uv)'\,dt.$$

Since the functions u, v, and uv are differentiable $\mathcal{L}^1$-a.e. in I, the standard calculus rule for a product now gives the desired result. $\square$

Another consequence of Theorem 3.30 is the following result, which says that for monotone functions, to prove absolute continuity in a fixed interval $[a, b]$, it is sufficient to test the fundamental theorem of calculus only at the endpoints a and b.

Corollary 3.38 (Tonelli). *Let $I \subset \mathbb{R}$ be an interval, let $u : I \to \mathbb{R}$ be a monotone function, and let $[a, b] \subset I$. Then u belongs to $AC([a, b])$ if and only if*

$$\int_a^b |u'(x)|\,dx = |u(b) - u(a)|.$$

Moreover, if u is bounded, then u belongs to $AC(I)$ if and only if

$$\int_I |u'(x)|\,dx = \sup_I u - \inf_I u.$$

Proof. Without loss of generality we may assume that u is increasing.

Step 1: If $u \in AC([a,b])$, then by Theorem 3.30 we have

$$\int_a^b u'(x)\,dx = u(b) - u(a). \tag{3.8}$$

Conversely, assume that (3.8) holds and let

$$w(x) := \int_a^x u'(t)\,dt, \quad x \in [a,b].$$

If $a \le x < y \le b$, then by Corollary 1.37,

$$w(y) - w(x) = \int_x^y u'(t)\,dt \le u(y) - u(x), \tag{3.9}$$

and so the function $u - w$ is increasing in $[a,b]$. But by (3.8),

$$u(b) - u(a) = \int_a^b u'(x)\,dx = w(b) - w(a),$$

and so $(u-w)(b) = (u-w)(a)$. This implies that the increasing function $u - w$ must be constant in $[a,b]$. Since $w \in AC([a,b])$ by Lemma 3.31, it follows that $u \in AC([a,b])$.

Step 2: If $u \in AC(I)$, then by the previous step for every $\inf I < a < b < \sup I$, we have

$$\int_a^b u'(x)\,dx = u(b) - u(a).$$

Letting $a \to (\inf I)^+$ and $b \to (\sup I)^-$ and using the Lebesgue monotone convergence theorem and the fact that u is continuous and increasing gives

$$\int_I u'(x)\,dx = \sup_I u - \inf_I u. \tag{3.10}$$

Conversely, assume that (3.10) holds. Then u' is Lebesgue integrable. Moreover, by Exercise 1.39, u is continuous in I. Fix $c \in \mathbb{R}$ and define

$$w(x) := c + \int_{\inf I}^x u'(t)\,dt, \quad x \in I.$$

As in the previous step we have that $u - w$ is increasing (and bounded) and that (3.9) holds for all $\inf I < x < y < \sup I$. Letting $x \to (\inf I)^+$ and $y \to (\sup I)^-$ in (3.9) and using the Lebesgue monotone convergence theorem, the monotonicity and the continuity of u and w, we obtain that

$$\begin{aligned}\lim_{x \to (\sup I)^-} u(x) - \lim_{x \to (\inf I)^+} u(x) &= \int_{\inf I}^{\sup I} u'(x)\,dx \\ &= \lim_{x \to (\sup I)^-} w(x) - \lim_{x \to (\inf I)^+} w(x),\end{aligned}$$

and so, since $u-w$ is increasing and continuous,

$$\begin{aligned}\sup_{x\in I}(u(x)-w(x)) &= \lim_{x\to(\sup I)^-}(u(x)-w(x))\\ &= \lim_{x\to(\inf I)^+}(u(x)-w(x))\\ &= \inf_{x\in I}(u(x)-w(x)),\end{aligned}$$

which implies as before that $u-w$ must be constant. By choosing c appropriately, we have that $u=w$, and so u belongs to $AC(I)$, since w does (see Lemma 3.31). ☐

Since a function with bounded pointwise variation is the difference of two bounded monotone functions, the previous corollary implies the following result.

Theorem 3.39 (Tonelli). *Let $I\subset\mathbb{R}$ be an interval, let $u\in BPV_{\mathrm{loc}}(I)$, and let $[a,b]\subset I$. Then u belongs to $AC([a,b])$ if and only if*

$$\int_a^b |u'|\,dx = \operatorname{Var}_{[a,b]} u. \tag{3.11}$$

In addition, if u belongs to $BPV(I)$, then u belongs to $AC(I)$ if and only if

$$\int_I |u'|\,dx = \operatorname{Var} u. \tag{3.12}$$

Proof. Step 1: Let $\boldsymbol{V}$ be the increasing function defined in (2.2). By (2.12),

$$\int_a^b |u'|\,dx \le \int_a^b |\boldsymbol{V}'|\,dx \le \boldsymbol{V}(b)-\boldsymbol{V}(a) = \operatorname{Var}_{[a,b]} u. \tag{3.13}$$

Hence, if

$$\int_a^b |u'|\,dx = \operatorname{Var}_{[a,b]} u,$$

then all the previous inequalities are equalities, and so

$$\int_a^b |\boldsymbol{V}'|\,dx = \boldsymbol{V}(b)-\boldsymbol{V}(a).$$

Since $\boldsymbol{V}$ is increasing, it follows by Corollary 3.38 that the function $\boldsymbol{V}$ belongs to $AC([a,b])$. In view of (2.3),

$$\sum_k |u(b_k)-u(a_k)| \le \sum_k |\boldsymbol{V}(b_k)-\boldsymbol{V}(a_k)|,$$

and so $u\in AC([a,b])$.

Conversely, if $u \in AC([a,b])$, then by Theorem 3.30 for every partition $\{x_0, \dots, x_n\}$ of $[a,b]$,

$$\sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| = \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} u' \, dx \right| \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |u'| \, dx = \int_a^b |u'| \, dx.$$

Taking the supremum over all partitions, we get

$$\operatorname{Var}_{[a,b]} u \le \int_a^b |u'| \, dx,$$

which, together with (3.13) yields the desired equality.

Step 2: Assume that $u \in BPV(I)$. Then by (2.10),

$$\int_I |u'(x)| \, dx \le \int_I |\boldsymbol{V}'(x)| \, dx \le \sup_I \boldsymbol{V} - \inf_I \boldsymbol{V} = \operatorname{Var} u,$$

and so if (3.12) holds, then

$$\int_I |\boldsymbol{V}'(x)| \, dx = \sup_I \boldsymbol{V} - \inf_I \boldsymbol{V}.$$

In turn, by the previous corollary, $\boldsymbol{V}$ is absolutely continuous. As in the previous step we conclude that u is absolutely continuous.

Conversely, if $u \in AC(I)$, then by the previous step (3.11) holds for every $[a,b] \subset I$. If $\inf I \in I$, define $a_n :\equiv \inf I$, and otherwise construct a sequence $a_n \searrow \inf I$. Similarly, if $\sup I \in I$, define $b_n :\equiv \sup I$, and otherwise construct a sequence $b_n \nearrow \sup I$. It suffices to apply the previous step in $[a_n, b_n]$ and then to let $n \to \infty$ using Proposition 2.6 and the Lebesgue monotone convergence theorem. □

Remark 3.40. From the previous proof it follows that if $I \subset \mathbb{R}$ is an interval, $[a,b] \subset I$, and $u \in BPV_{\mathrm{loc}}(I)$ (respectively, $BPV(I)$), then $u \in AC([a,b])$ (respectively, $AC(I)$) if and only if $\boldsymbol{V} \in AC([a,b])$ (respectively, $\boldsymbol{V} \in AC(I)$).

As a consequence of Tonelli's theorem we can prove the converse of Corollary 3.24.

Corollary 3.41. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$. Then u belongs to $AC(I)$ if and only if*

(i) *u is continuous on I,*

(ii) *u is differentiable $\mathcal{L}^1$-a.e. in I, and u' belongs to $L^1_{\mathrm{loc}}(I)$ and is equi-integrable,*

(iii) *u maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.*

Proof. In view of Theorem 3.12 and of Corollary 3.24, it remains to show that if $u \in AC(I)$, then u' is equi-integrable. By Exercise 3.7 we may assume, without loss of regularity, that I is closed. Fix $\varepsilon > 0$ and let $\delta > 0$ be as in Definition 3.1. Consider a Lebesgue measurable set $E \subset I$, with $\mathcal{L}^1(E) \leq \frac{\delta}{2}$. By the outer regularity of the Lebesgue measure we may find an open set $A \supset E$ such that $\mathcal{L}^1(A) < \delta$. Decompose A into a countable family $\{J_k\}$ of pairwise disjoint intervals. By replacing each J_k with $J_k \cap I$, we may assume that $J_k \subset I$. Let $\overline{J_k} = [a_k, b_k]$. Using the fact that $\mathcal{L}^1(A) < \delta$, we have that

$$\sum_k |b_k - a_k| < \delta.$$

Consider a partition $P_k = \left\{x_0^{(k)}, \ldots, x_{m_k}^{(k)}\right\}$ of $[a_k, b_k]$. Since

$$\sum_k \sum_{i=1}^{m_k} \left| x_i^{(k)} - x_{i-1}^{(k)} \right| = \sum_k |b_k - a_k| < \delta,$$

it follows from the fact that $u \in AC(I)$ that

$$\sum_k \sum_{i=1}^{m_k} \left| u\left(x_i^{(k)}\right) - u\left(x_{i-1}^{(k)}\right) \right| \leq \varepsilon.$$

Taking the supremum over every partition P_k of $[a_k, b_k]$ for each k, we have that

$$\sum_k \operatorname{Var}_{[a_k,b_k]} u \leq \varepsilon.$$

Hence, by Tonelli's theorem applied to each interval $[a_k, b_k]$,

$$\int_E |u'| \, dx \leq \int_{A \cap I} |u'| \, dx = \sum_k \int_{[a_k,b_k]} |u'| \, dx = \sum_k \operatorname{Var}_{[a_k,b_k]} u \leq \varepsilon.$$

This concludes the proof. □

In the discussion before Exercise 3.7 we have shown that an absolutely continuous function defined in an unbounded interval may not have bounded pointwise variation. As a corollary of the previous theorem we can now characterize the absolutely continuous functions that have bounded pointwise variation.

Corollary 3.42. *Let $I \subset \mathbb{R}$ be an interval and let $u \in AC_{\mathrm{loc}}(I)$. Then u belongs to $BPV(I)$ if and only if u' belongs to $L^1(I)$. In this case u belongs to $AC(I)$.*

Proof. In view of Corollary 2.23, it remains to show that if u' is Lebesgue integrable, then u belongs to $BPV(I)$. By Proposition 3.8 we have that

$u \in BPV_{\mathrm{loc}}(I)$. Hence by the Theorem 3.39, for every $[a,b] \subset I$,

$$\int_a^b |u'| \, dx = \operatorname{Var}_{[a,b]} u.$$

Taking a_n and b_n as in Step 2 of the proof of Theorem 3.39 and using Proposition 2.6 (note that in that proposition the function u is completely arbitrary), Exercise 2.8, and the Lebesgue monotone convergence theorem, we have that

$$\int_I |u'| \, dx = \operatorname{Var}_I u.$$

Since u' is Lebesgue integrable, it follows that $\operatorname{Var}_I u < \infty$.

The last statement follows from Corollary 3.26. □

Exercise 3.43. Let $p \geq 1$ and let $AC_p([a,b])$ be the class of all functions $u : [a,b] \to \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left(\sum_{k=1}^{\ell} |u(b_k) - u(a_k)|^p \right)^{1/p} \leq \varepsilon$$

for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subset I$ and

$$\left(\sum_{k=1}^{\ell} (b_k - a_k)^p \right)^{1/p} \leq \delta.$$

(i) Prove that if $u \in AC_p([a,b])$, then $\operatorname{Var}_p u < \infty$ (see Exercise 2.29).

(ii) Prove that the function

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n/p}} \cos 2^n \pi x, \quad x \in [0,1],$$

is such that $\operatorname{Var}_p u < \infty$, but it does not belong to $AC_p([0,1])$.

3.2. Chain Rule and Change of Variables

Next we discuss the validity of the chain rule and of the change of variables for absolutely continuous functions. The next result establishes the validity of the chain rule under very weak hypotheses.

Theorem 3.44 (Chain rule)**.** *Let $I, J \subset \mathbb{R}$ be two intervals and let $f : J \to \mathbb{R}$ and $u : I \to J$ be such that f, u, and $f \circ u$ are differentiable $\mathcal{L}^1$-a.e. in their respective domains. If f maps sets of Lebesgue measure zero into sets of Lebesgue measure zero, then for $\mathcal{L}^1$-a.e. $x \in I$,*

$$(f \circ u)'(x) = f'(u(x))\, u'(x), \tag{3.14}$$

where $f'(u(x))\, u'(x)$ is interpreted to be zero whenever $u'(x) = 0$ (even if f is not differentiable at $u(x)$).

In the proof we will show that $(f \circ u)'(x) = 0$ and $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in I$ such that f is not differentiable at $u(x)$.

To prove Theorem 3.44, we need an auxiliary result, which is a converse of Corollary 3.14.

Lemma 3.45. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$. Assume that u has derivative (finite or infinite) on a set $E \subset I$ (not necessarily measurable), with $\mathcal{L}^1(u(E)) = 0$. Then $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in E$.*

Proof. Let $E^* := \{x \in E : |u'(x)| > 0\}$. We claim that $\mathcal{L}^1(E^*) = 0$. For every integer $k \in \mathbb{N}$ let

$$E_k^* := \left\{ x \in E^* : |u(x) - u(y)| \geq \frac{|x-y|}{k} \text{ for all } y \in \left(x - \tfrac{1}{k}, x + \tfrac{1}{k}\right) \cap I \right\}.$$

Noting that

$$E^* = \bigcup_{k=1}^{\infty} E_k^*,$$

we fix k and we let $F := J \cap E_k^*$, where J is an interval of length less than $\frac{1}{k}$. To prove that $\mathcal{L}^1(E^*) = 0$, it suffices to show that $\mathcal{L}^1(F) = 0$. Since $\mathcal{L}^1(u(E)) = 0$ and $F \subset E$, for every $\varepsilon > 0$ we may find a sequence of intervals $\{J_n\}$ such that

$$u(F) \subset \bigcup_{n=1}^{\infty} J_n, \quad \sum_{n=1}^{\infty} \mathcal{L}^1(J_n) < \varepsilon.$$

Let $E_n := u^{-1}(J_n) \cap F$. Since $\{E_n\}$ covers F, we have

$$\begin{aligned} \mathcal{L}_o^1(F) &\leq \sum_{n=1}^{\infty} \mathcal{L}_o^1(E_n) \leq \sum_{n=1}^{\infty} \sup_{x,y \in E_n} |x-y| \\ &\leq \sum_{n=1}^{\infty} k \sup_{x,y \in E_n} |u(x) - u(y)| =: \mathcal{I}, \end{aligned}$$

where we have used the fact that $E_n \subset J \cap E_k^*$. Since $u(E_n) \subset J_n$, we have

$$\sup_{x,y \in E_n} |u(x) - u(y)| \leq \mathcal{L}^1(J_n),$$

and so

$$\mathcal{I} \leq k \sum_{n=1}^{\infty} \mathcal{L}^1(J_n) < k\varepsilon.$$

It now suffices to let $\varepsilon \to 0^+$. □

Corollary 3.46. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$, $v : I \to \mathbb{R}$. Assume that there exists a set $E \subset I$ such that u and v are differentiable for all $x \in E$ and $u(x) = v(x)$ for all $x \in E$. Then $u'(x) = v'(x)$ for $\mathcal{L}^1$-a.e. $x \in E$.*

Proof. Let $w := u - v$. Then $w(E) = \{0\}$, and so $\mathcal{L}^1(w(E)) = 0$. By the previous lemma, $u'(x) - v'(x) = w'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in E$. □

We turn to the proof of Theorem 3.44.

Proof of Theorem 3.44. Let

$$G := \{z \in J : \ f \text{ is not differentiable at } z\},$$
$$F := \{x \in I : \ u \text{ or } f \circ u \text{ is not differentiable at } x\}.$$

By hypothesis $\mathcal{L}^1(G) = \mathcal{L}^1(F) = 0$. Let

$$E := \{x \in I^\circ \setminus F : \ u(x) \in G\}.$$

Since $u(E) \subset G$, we have $\mathcal{L}^1(u(E)) = 0$, and since f maps sets of Lebesgue measure zero into sets of Lebesgue measure zero, we obtain that

$$\mathcal{L}^1((f \circ u)(E)) = 0.$$

By Lemma 3.45 applied to u and to $f \circ u$, we conclude that

$$u'(x) = (f \circ u)'(x) = 0$$

for $\mathcal{L}^1$-a.e. $x \in E$.

On the other hand, if $x \in I^\circ \setminus F$ and $u(x) \notin G$, then we may apply the standard chain rule to conclude that $f \circ u$ is differentiable at x with $(f \circ u)'(x) = f'(u(x))\, u'(x)$. □

The next example shows the importance of the (N) property.

Example 3.47. Let $u : [0,1] \to \mathbb{R}$ be a strictly increasing function such that $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in [0,1]$ (see Theorem 1.47) and let $f := u^{-1}$. Note that f is strictly increasing, and so by Lebesgue's theorem it is differentiable for $\mathcal{L}^1$-a.e. $x \in [0,1]$, despite the fact that $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in [0,1]$. Moreover, $(f \circ u)(x) = x$ for all $x \in [0,1]$, and so $(f \circ u)'(x) = 1$ for all $x \in [0,1]$, while $f'(u(x))\, u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in [0,1]$, since $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in [0,1]$.

Corollary 3.48. *Let $I, J \subset \mathbb{R}$ be two intervals and let $f : J \to \mathbb{R}$ and $u : I \to J$ be such that f and u are differentiable $\mathcal{L}^1$-a.e. in their respective domains. Suppose that u' is zero at most on a set of Lebesgue measure zero. Then $f \circ u$ is differentiable $\mathcal{L}^1$-a.e. in I and the chain rule (3.14) holds.*

Proof. Let G and F be as in the proof of Theorem 3.44 and let

$$E := \{x \in I^\circ : \ u(x) \in G\}.$$

Since $u(E) \subset G$, we have $\mathcal{L}^1(u(E)) = 0$, and so by Lemma 3.45 we conclude that $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in E$. But then, according to our assumption that u' is zero at most on a set of Lebesgue measure zero, E must have

Lebesgue measure zero. If $x \in I^\circ \setminus E$, then f is differentiable at $u(x)$, and so the chain rule (3.14) holds $\mathcal{L}^1$-a.e. in $I^\circ \setminus E$. □

Corollary 3.49. *Let $I, J \subset \mathbb{R}$ be two intervals and let $f : J \to \mathbb{R}$ and $u : I \to J$ be such that u and $f \circ u$ are differentiable $\mathcal{L}^1$-a.e. in their respective domains. If $f \in AC_{\mathrm{loc}}(J)$, then the chain rule (3.14) holds.*

Proof. In view of Theorem 3.12, all the hypotheses of Theorem 3.44 are satisfied. □

A less trivial consequence of Theorem 3.44 is the following result.

Corollary 3.50. *Let $I, J \subset \mathbb{R}$ be two intervals, let $f \in AC_{\mathrm{loc}}(J)$, and let $u : I \to J$ be monotone. Then $f \circ u$ is differentiable $\mathcal{L}^1$-a.e. in I and the chain rule (3.14) holds.*

Proof. Note that the composite $f \circ u$ belongs to $BPV_{\mathrm{loc}}(I)$ by Exercise 2.21, and so by Corollary 2.23, $f \circ u$ is differentiable $\mathcal{L}^1$-a.e. in I. We are now in a position to apply the previous corollary. □

Exercise 3.51. Let $I, J \subset \mathbb{R}$ be two intervals, let $f \in AC_{\mathrm{loc}}(J)$, and let $u : I \to J$ be monotone and $AC_{\mathrm{loc}}(I)$. Prove that $f \circ u$ belongs to $AC_{\mathrm{loc}}(I)$.

Corollary 3.52. *Let $I, J \subset \mathbb{R}$ be two intervals, let $f : J \to \mathbb{R}$ be locally Lipschitz, and let $u \in BPV_{\mathrm{loc}}(I)$. Then $f \circ u$ is differentiable $\mathcal{L}^1$-a.e. in I and the chain rule (3.14) holds.*

Proof. By Theorem 2.31, we have that $f \circ u \in BPV_{\mathrm{loc}}(I)$. Hence we can apply Corollary 2.23 to conclude that $f \circ u$ is differentiable $\mathcal{L}^1$-a.e. in I. Since f is locally Lipschitz, it is locally absolutely continuous, and so the result follows from Corollary 3.49. □

Before moving to the next topic, we observe that while all the results proved before Theorem 3.44 continue to hold in $AC\left(I; \mathbb{R}^d\right)$ (respectively, $AC_{\mathrm{loc}}\left(I; \mathbb{R}^d\right)$) (see Chapter 4), this is not the case for Theorem 3.44 and its corollaries. Indeed, if $f : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function with $d > 1$ and if $u : I \to \mathbb{R}^d$ is absolutely continuous, then $f \circ u \in AC(I)$ (see Step 1 of the proof of Theorem 3.68 below), but the analog of (3.14), which is

$$(f \circ u)'(x) = \sum_{i=1}^{d} \frac{\partial f}{\partial u_i}(u(x))\, u_i'(x)\,, \tag{3.15}$$

where $\frac{\partial f}{\partial u_i}(u(x))\, u_i'(x)$ is interpreted to be zero whenever $u_i'(x) = 0$, may fail. This is illustrated by the next example.

Example 3.53. Let $d = 2$, and consider the functions

$$f(z) = f(z_1, z_2) := \max\{z_1, z_2\}, \quad z \in \mathbb{R}^2,$$

and $u(x) := (x, x)$ for $x \in \mathbb{R}$. Then $v(x) := (f \circ u)(x) = x$ so that $v'(x) = 1$, while the right-hand side of (3.15) is *nowhere defined, since* $u'(x) = (1, 1)$.

We will discuss this problem in more detail at the end of Section 4.3.

As a corollary of Theorem 3.44 we have the following change of variables formula.

Theorem 3.54 (Change of variables). *Let* $g : [c, d] \to \mathbb{R}$ *be an integrable function and let* $u : [a, b] \to [c, d]$ *be differentiable* $\mathcal{L}^1$*-a.e. in* $[a, b]$*. Then* $(g \circ u) u'$ *is integrable and the change of variables*

$$\int_{u(\alpha)}^{u(\beta)} g(t)\, dt = \int_{\alpha}^{\beta} g(u(x))\, u'(x)\, dx \tag{3.16}$$

holds for all $\alpha, \beta \in [a, b]$ *if and only if the function* $f \circ u$ *belongs to* $AC([a, b])$*, where*

$$f(z) := \int_c^z g(t)\, dt, \quad z \in [c, d].$$

Proof. If $f \circ u \in AC([a, b])$, then since f is absolutely continuous (see Lemma 3.31), we can apply Corollary 3.49 to obtain the chain rule formula

$$(f \circ u)'(x) = g(u(x))\, u'(x) \tag{3.17}$$

for $\mathcal{L}^1$-a.e. $x \in [a, b]$. Since $f \circ u \in AC([a, b])$, it follows from Corollary 3.9 and (3.17) that $(g \circ u) u'$ is integrable, and by the fundamental theorem of calculus (see Theorem 3.30), for all $\alpha, \beta \in [a, b]$,

$$\begin{aligned} \int_{u(\alpha)}^{u(\beta)} g(t)\, dt &= (f \circ u)(\beta) - (f \circ u)(\alpha) \\ &= \int_{\alpha}^{\beta} (f \circ u)'(x)\, dx = \int_{\alpha}^{\beta} g(u(x))\, u'(x)\, dx. \end{aligned}$$

Conversely, if $(g \circ u) u'$ is integrable and the identity

$$(f \circ u)(\beta) - (f \circ u)(\alpha) = \int_{\alpha}^{\beta} g(u(x))\, u'(x)\, dx$$

holds for all $\alpha, \beta \in I$, then, since the right-hand side is absolutely continuous by Lemma 3.31, it follows that $f \circ u$ is absolutely continuous. □

Remark 3.55. The previous proof shows, in particular, that the function $x \in I \mapsto g(u(x))\, u'(x)$ is measurable (see also Exercise 1.41), since it is the derivative of the function $f \circ u$, but this does not imply that the function $x \in I \mapsto g(u(x))$ is measurable (see the next exercise).

Exercise 3.56. Let $I, J \subset \mathbb{R}$ be intervals. Prove that there exist a continuous increasing function $u : I \to J$ and a Lebesgue measurable function $g : J \to \mathbb{R}$ such that $g \circ u = I \to \mathbb{R}$ is not Lebesgue measurable. Hint: See Exercise 1.45.

Corollary 3.57. *Assume that $g : [c, d] \to \mathbb{R}$ is an integrable function and that $u : [a, b] \to [c, d]$ is monotone and absolutely continuous. Then $(g \circ u)\, u'$ is integrable and the change of variables formula (3.16) holds.*

Exercise 3.58. Prove that under the hypotheses of the previous corollary the function $f \circ u$ is absolutely continuous and then prove the corollary.

Corollary 3.59. *Assume that $g : [c, d] \to \mathbb{R}$ is a measurable, bounded function and that $u : [a, b] \to [c, d]$ is absolutely continuous. Then $(g \circ u)\, u'$ is integrable and the change of variables formula (3.16) holds.*

Exercise 3.60. Prove that under the hypotheses of the previous corollary the function $f \circ u$ is absolutely continuous and then prove the corollary.

Corollary 3.61. *Assume that $g : [c, d] \to \mathbb{R}$ is an integrable function, that $u : [a, b] \to [c, d]$ is absolutely continuous, and that $(g \circ u)\, u'$ is integrable. Then the change of variables formula (3.16) holds.*

Proof. Let

$$g_n(z) := \begin{cases} n & \text{if } g(z) > n, \\ g(z) & \text{if } -n \le g(z) \le n, \\ -n & \text{if } g(z) < -n. \end{cases}$$

Applying the Lebesgue dominated convergence theorem and the previous corollary, we obtain

$$\begin{aligned} \int_{u(\alpha)}^{u(\beta)} g(z)\, dz &= \lim_{n\to\infty} \int_{u(\alpha)}^{u(\beta)} g_n(z)\, dz \\ &= \lim_{n\to\infty} \int_\alpha^\beta g_n(u(x))\, u'(x)\, dx = \int_\alpha^\beta g(u(x))\, u'(x)\, dx. \end{aligned}$$

∎

To extend the previous results to arbitrary intervals, we consider two functions $g : J \to \mathbb{R}$ and $u : I \to J$ and we assume that there exist in $\mathbb{R}$ the limits

$$\lim_{x\to(\inf I)^+} u(x) = \ell, \qquad \lim_{x\to(\sup I)^-} u(x) = L.$$

In this case, the analog of (3.16) becomes

$$\int_\ell^L g(z)\, dz = \int_{\inf I}^{\sup I} g(u(x))\, u'(x)\, dx. \tag{3.18}$$

Indeed, we have the following result.

Theorem 3.62. *Let $I, J \subset \mathbb{R}$ be two intervals, let $g : J \to \mathbb{R}$ be an integrable function, and let $u : I \to J$ be differentiable $\mathcal{L}^1$-a.e. in I and such that there exist in $\mathbb{R}$ the limits*

$$\lim_{x\to(\inf I)^+} u(x) = \ell, \qquad \lim_{x\to(\sup I)^-} u(x) = L. \tag{3.19}$$

Then $(g \circ u)\, u'$ is integrable on I and the changes of variables (3.16) and (3.18) hold for all $[\alpha, \beta] \subset I$ if and only if the function $f \circ u$ belongs to $AC(I) \cap BPV(I)$, where

$$f(z) := \int_{\inf J}^{z} g(t)\, dt, \quad z \in J.$$

Proof. Assume that $f \circ u \in AC(I) \cap BPV(I)$. Then, we can proceed as in the proof of Theorem 3.54 to show that (3.16) holds for all $[\alpha, \beta] \subset I$. Since $f \circ u \in BPV(I)$, by Corollary 2.23 and (3.17) its derivative $(g \circ u)\, u'$ is integrable on I. Hence, by taking limits as α and β approach the endpoints of I and using (3.19) and the Lebesgue dominated convergence theorem, we conclude that (3.18) holds.

Conversely, if $(g \circ u)\, u'$ is integrable and (3.16) and (3.18) hold for all $[\alpha, \beta] \subset I$, then

$$(f \circ u)(\beta) - (f \circ u)(\alpha) = \int_{\alpha}^{\beta} g(u(x))\, u'(x)\, dx$$

for all $\alpha, \beta \in I$. As in Theorem 3.54 we deduce that $f \circ u \in AC_{\mathrm{loc}}(I)$. In turn, by Corollaries 3.41 and 3.42, it follows that $f \circ u \in AC(I) \cap BPV(I)$. □

Closely related to the change of variables formula is the area formula, which will be discussed next.

Definition 3.63. Let X be a nonempty set and let $\psi : X \to [0, \infty]$ be a function. We define the *infinite sum* of ψ over X as

$$\sum_{t\in X} \psi(t) := \sup \left\{ \sum_{t\in Y} \psi(t) : Y \subset X \text{ finite} \right\}.$$

Exercise 3.64. Let X be a nonempty set and let $\psi : X \to [0, \infty]$ be a function. Prove that if $\sum_{t\in X} \psi(t) < \infty$, then the set

$$\{t \in X : \psi(t) > 0\}$$

is countable.

Theorem 3.65 (Area formula)**.** *Let $I \subset \mathbb{R}$ be an interval, let $\psi : I \to [0, \infty]$ be a Borel function, and let $u : I \to \mathbb{R}$ be differentiable $\mathcal{L}^1$-a.e. in I and* *such that u maps sets of Lebesgue measure zero into sets of Lebesgue measure zero. Then*

$$\int_{\mathbb{R}} \sum_{t \in u^{-1}(\{y\})} \psi(t)\, dy = \int_I \psi(x) \left| u'(x) \right| dx. \tag{3.20}$$

Proof. Step 1: Assume first that $I = (a, b)$ and that $u \in C_c^1(I)$. Consider the open set $A := \{x \in I : u'(x) \neq 0\}$ and let $\{(a_k, b_k)\}$ be the countable family of connected components of A. If $u' > 0$ in (a_k, b_k) and $\psi \in L^\infty((a_k, b_k))$, then by Corollary 3.59 (applied in (a_k, b_k) to the functions u and $g := \psi \circ v$, where $v := \left(u|_{(a_k,b_k)} \right)^{-1}$) we get

$$\int_{a_k}^{b_k} \psi(x)\, u'(x)\, dx = \int_{u(a_k)}^{u(b_k)} \psi\left(\left(u|_{(a_k,b_k)} \right)^{-1}(y) \right) dy. \tag{3.21}$$

On the other hand, since $u|_{(a_k,b_k)}$ is strictly decreasing and continuous, for every $y \in (u(a_k), u(b_k))$ there is one and only one $t \in (a_k, b_k)$ such that $u(t) = y$, so that $(a_k, b_k) \cap u^{-1}(\{y\}) = \{t\}$, while, if $y \in \mathbb{R} \setminus (u(a_k), u(b_k))$, then $(a_k, b_k) \cap u^{-1}(\{y\}) = \emptyset$. This shows that

$$\begin{aligned} \int_{\mathbb{R}} \sum_{t \in (a_k,b_k) \cap u^{-1}(\{y\})} \psi(t)\, dy &= \int_{u(a_k)}^{u(b_k)} \sum_{t \in (a_k,b_k) \cap u^{-1}(\{y\})} \psi(t)\, dy \\ &= \int_{u(a_k)}^{u(b_k)} \psi\left(\left(u|_{(a_k,b_k)} \right)^{-1}(y) \right) dy. \end{aligned}$$

Combining this equality with (3.21) gives

$$\int_{a_k}^{b_k} \psi(x)\, u'(x)\, dx = \int_{\mathbb{R}} \sum_{t \in (a_k,b_k) \cap u^{-1}(\{y\})} \psi(t)\, dy. \tag{3.22}$$

To remove the additional assumption that $\psi \in L^\infty((a_k, b_k))$, it suffices to apply (3.22) to $\psi_n := \min\{\psi, n\}$ and to use the Lebesgue monotone convergence theorem.

A similar argument shows that if $u' < 0$ in (a_k, b_k), then

$$\int_{a_k}^{b_k} \psi(x) \left| u'(x) \right| dx = \int_{\mathbb{R}} \sum_{t \in (a_k,b_k) \cap u^{-1}(\{y\})} \psi(t)\, dy.$$

Adding over k, we obtain

$$\int_A \psi(x) \left| u'(x) \right| dx = \int_{\mathbb{R}} \sum_{t \in A \cap u^{-1}(\{y\})} \psi(t)\, dy.$$

Since $u'(x) = 0$ in $I \setminus A$, by Corollary 3.14, $\mathcal{L}^1(u(I \setminus A)) = 0$. Hence, the previous equality can be rewritten as

$$\int_I \psi(x) \left|u'(x)\right| dx = \int_{\mathbb{R}\setminus u(I\setminus A)} \sum_{t\in A\cap u^{-1}(\{y\})} \psi(t)\ dy.$$

On the other hand, if $y \in \mathbb{R} \setminus u(I \setminus A)$, then

$$\begin{aligned} u^{-1}(\{y\}) &= \left(A \cap u^{-1}(\{y\})\right) \cup \left((I \setminus A) \cap u^{-1}(\{y\})\right) \\ &= A \cap u^{-1}(\{y\}), \end{aligned}$$

and so

$$\begin{aligned} \int_{\mathbb{R}\setminus u(I\setminus A)} \sum_{t\in A\cap u^{-1}(\{y\})} \psi(t)\ dy &= \int_{\mathbb{R}\setminus u(I\setminus A)} \sum_{t\in u^{-1}(\{y\})} \psi(t)\ dy \\ &= \int_{\mathbb{R}} \sum_{t\in u^{-1}(\{y\})} \psi(t)\ dy, \end{aligned}$$

where in the last equality we have used the fact that $\mathcal{L}^1(u(I \setminus A)) = 0$ once more.

This shows that (3.20) holds.

Step 2: Assume next that $I = (a, b)$, that there exists a compact set $K \subset I$ such that $\psi = 0$ on $I \setminus K$, that u is differentiable for all x in K, and that

$$\lim_{y\in K,\, y\to x} \frac{u(y) - u(x)}{y - x} = u'(x) \quad \text{uniformly for } x \in K.$$

Then by Exercise 3.66 below there exists a function $v \in C_c^1(I)$ such that $v = u$ and $v' = u'$ on K. Applying the previous step to v and with ψ replaced by $\chi_K\psi$, we obtain

$$\begin{aligned} \int_K \psi(x) \left|u'(x)\right| dx &= \int_I \chi_K(x)\, \psi(x) \left|v'(x)\right| dx \\ &= \int_{\mathbb{R}} \sum_{t\in v^{-1}(\{y\})} \chi_K(t)\, \psi(t)\ dy \\ &= \int_{\mathbb{R}} \sum_{t\in K\cap v^{-1}(\{y\})} \psi(t)\ dy = \int_{\mathbb{R}} \sum_{t\in K\cap u^{-1}(\{y\})} \psi(t)\ dy. \end{aligned}$$

Since $\psi = 0$ on $I \setminus K$, we have that (3.20) holds.

Step 3: Assume that $I = (a, b)$. Since u is differentiable $\mathcal{L}^1$-a.e. in I, the sequence of functions

$$u_n(x) := \sup_{y\in\left(\left(x-\frac{1}{n},x+\frac{1}{n}\right)\cap I\right)\setminus\{x\}} \left|\frac{u(y) - u(x)}{y - x} - u'(x)\right|, \quad x \in I,$$

converges to 0 for $\mathcal{L}^1$-a.e. $x \in I$. It follows by Egoroff's theorem that there exists an increasing sequence of compact sets $\{K_j\} \subset I$ such that

$$\mathcal{L}^1\left(I \setminus \bigcup_{j=1}^{\infty} K_j\right) = 0$$

and such that $\{u_n\}$ converges to zero uniformly in K_j for every $j \in \mathbb{N}$. In particular,

$$\lim_{y \in K_j,\, y \to x} \frac{u(y) - u(x)}{y - x} = u'(x) \quad \text{uniformly as } x \in K_j$$

for all $j \in \mathbb{N}$. By the previous step with ψ replaced by $\psi \chi_{K_j}$,

$$\int_{K_j} \psi(x) \left|u'(x)\right| dx = \int_{\mathbb{R}} \sum_{t \in K_j \cap u^{-1}(\{y\})} \psi(t)\, dy.$$

Letting $j \to \infty$, it follows by the Lebesgue monotone convergence theorem that

$$\int_{\bigcup_{j=1}^{\infty} K_j} \psi(x) \left|u'(x)\right| dx = \int_{\mathbb{R}} \sum_{t \in \bigcup_{j=1}^{\infty} K_j \cap u^{-1}(\{y\})} \psi(t)\, dy.$$

Since $\mathcal{L}^1\left(I \setminus \bigcup_{j=1}^{\infty} K_j\right) = 0$, by hypothesis we have that

$$\mathcal{L}^1\left(u\left(I \setminus \bigcup_{j=1}^{\infty} K_j\right)\right) = 0,$$

and so we obtain (3.20).

Step 4: If I is an arbitrary interval, let $(a_n, b_n) \subset I$ be such that $a_n \to (\inf I)^+$, $b_n \to (\sup I)^-$. By the previous step, (3.20) holds in each (a_n, b_n). Formula (3.20) now follows in I° from the Lebesgue monotone convergence theorem. If one or both endpoints of I belong to I, we can proceed as in last part of Step 1 to show that (3.20) holds in I. □

Choosing $\psi(x) := g(u(x))$ in (3.20), where $g : \mathbb{R} \to [0, \infty]$ is a Borel function, yields

$$\int_{\mathbb{R}} g(y)\, N_u(y; I)\, dy = \int_I g(u(x)) \left|u'(x)\right| dx,$$

where, we recall, $N_u(\cdot; I)$ is the Banach indicatrix of u. In particular, for $g = 1$, we get the analog of Banach's theorem (see (2.48)), that is,

$$\int_{\mathbb{R}} N_u(y; I)\, dy = \int_I \left|u'(x)\right| dx.$$

Exercise 3.66. Let $K \subset (a,b)$ be a compact set and let $u : K \to \mathbb{R}$ be such that u is differentiable on K and

$$\lim_{y \in K,\, y \to x} \frac{u(y) - u(x)}{y - x} = u'(x) \quad \text{uniformly for } x \in K.$$

Prove that there exists a function $v : (a,b) \to \mathbb{R}$, with $v \in C_c^1((a,b))$, such that $v = u$ and $v' = u'$ on K. Hint: On each connected component (a_k, b_k) of $(a,b) \setminus K$ define v to be a suitable third-order polynomial.

We conclude this section by discussing the analog of Theorem 2.31. The following exercise (see Exercise 2.30) shows that the composition of absolutely continuous functions is not absolutely continuous (see however Exercise 3.51).

Exercise 3.67. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(z) := \begin{cases} 1 & \text{if } z \le -1, \\ \sqrt{|z|} & \text{if } -1 < z < 1, \\ 1 & \text{if } z \ge 1, \end{cases}$$

and let $u : [-1,1] \to \mathbb{R}$ be the function

$$u(x) := \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f and u are absolutely continuous but their composition $f \circ u$ is not.

The next result gives necessary and sufficient conditions on $f : \mathbb{R} \to \mathbb{R}$ for $f \circ u$ to be absolutely continuous for *all* absolutely continuous functions $u : [a,b] \to \mathbb{R}$.

Theorem 3.68 (Superposition)**.** *Let $I \subset \mathbb{R}$ be an interval and let $f : \mathbb{R} \to \mathbb{R}$. Then $f \circ u \in AC_{\mathrm{loc}}(I)$ for* all *functions $u \in AC_{\mathrm{loc}}(I)$ if and only if f is locally Lipschitz. In particular, if f is locally Lipschitz and $u \in AC_{\mathrm{loc}}(I)$, then the chain rule (3.14) holds.*

Proof. Step 1: Assume that f is locally Lipschitz and let $u \in AC_{\mathrm{loc}}(I)$. Fix an interval $[a,b]$. In particular, $|u|$ is bounded in $[a,b]$ by some constant ℓ, and so there exists $L > 0$ such that

$$|f(z_1) - f(z_2)| \le L\,|z_1 - z_2| \tag{3.23}$$

for all $z_1, z_2 \in [-\ell, \ell]$. We claim that $f \circ u$ is absolutely continuous in $[a,b]$. Indeed, since $u \in AC([a,b])$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_k |u(b_k) - u(a_k)| \le \frac{\varepsilon}{L}$$

for every finite number of nonoverlapping intervals $(a_k, b_k) \subset [a, b]$, with

$$\sum_k (b_k - a_k) \le \delta.$$

Hence, by (3.23),

$$\sum_k |(f \circ u)(b_k) - (f \circ u)(a_k)| \le L \sum_k |u(b_k) - u(a_k)| \le \varepsilon,$$

which proves the claim.

The validity of the chain rule follows from Corollary 3.52.

Step 2: Assume that $f \circ u \in AC_{\text{loc}}(I)$ for all functions $u \in AC_{\text{loc}}(I)$. We claim that f is locally Lipschitz. The proof follows closely that of Theorem 2.31, with the only difference that instead of discontinuous functions u (see (2.30), (2.35)) we will use piecewise affine functions. Fix $[a, b] \subset I$. We begin by showing that f is locally bounded. Consider an interval $[-r, r]$, where $r > 0$. We claim that f is bounded in $[-r, r]$. Indeed, for every $z_0 \in [-r, r]$ consider the function

$$u(x) := \begin{cases} z_0 + x - \frac{a+b}{2} & \text{if } x \in [a, b], \\ z_0 - \frac{b-a}{2} & \text{if } x < a, \\ z_0 + \frac{b-a}{2} & \text{if } x > b. \end{cases}$$

Since $u \in AC_{\text{loc}}(I)$, by hypothesis $(f \circ u) \in AC([a, b])$. In particular, it is bounded in $[a, b]$. Thus, there exists a constant $M_{z_0} = M_{z_0}(a, b) > 0$ such that

$$\left| f\left(z_0 + x - \frac{a+b}{2}\right) \right| \le M_{t_0}$$

for all $x \in [a, b]$, which implies that

$$|f(z)| \le M_{y_0}$$

for all $z \in \left(z_0 - \frac{b-a}{2}, z_0 + \frac{b-a}{2}\right)$. A compactness argument shows that f is bounded in $[-r, r]$ by some constant $M_r > 0$.

Next we claim that f is Lipschitz in $[-r, r]$. Indeed, assume by contradiction that this is not the case. Then we may find two sequences $\{s_n\}, \{t_n\} \subset [-r, r]$ such that $s_n \ne t_n$ and

$$\frac{|f(s_n) - f(t_n)|}{|s_n - t_n|} > 2\left(n^2 + n\right) \tag{3.24}$$

for all $n \in \mathbb{N}$. Since $\{s_n\}$ is bounded, we may extract a subsequence (not relabeled) such that $s_n \to s_\infty$. Take a further subsequence (not relabeled) such that (3.24) continues to hold and

$$|s_n - s_\infty| < \frac{1}{(n+1)^2}. \tag{3.25}$$

Since f is bounded in $[-r,r]$ by M_r, by (3.24) for all $n\in\mathbb{N}$ we have

$$2M_r \geq |f(s_n)-f(t_n)| > 2\left(n^2+n\right)|s_n-t_n|\,. \tag{3.26}$$

Hence,

$$0<\delta_n := \frac{|s_n-t_n|\,(b-a)}{2M_r} < \frac{(b-a)}{2\,(n^2+n)} \to 0.$$

For every $n\in\mathbb{N}$, we divide the interval $\left[a+\frac{b-a}{n+1}, a+\frac{b-a}{n}\right]$ into subintervals of length δ_n. Thus, let

$$\ell_n := \frac{\operatorname{diam} I_n}{\delta_n} = \frac{b-a}{\delta_n\,(n^2+n)} = \frac{2M_r}{|s_n-t_n|\,(n^2+n)} > 2 \tag{3.27}$$

and set $m_n := \max\{j\in\mathbb{N}_0 : j<\ell_n\}$. Since $\ell_n>2$, we have

$$\frac{\ell_n}{2} \leq m_n < \ell_n. \tag{3.28}$$

Consider the partition P_n of $\left[a+\frac{b-a}{n+1}, a+\frac{b-a}{n}\right]$ given by

$$\begin{aligned} P_n &:= \left\{a+\frac{b-a}{n+1}+\frac{1}{2}j\delta_n : j=0,\dots,2m_n\right\}\cup\left\{a+\frac{b-a}{n}\right\} \\ &=: \left\{x_0^{(n)},\dots,x_{2m_n+1}^{(n)}\right\}. \end{aligned}$$

We now define the piecewise affine function $u: I\to\mathbb{R}$ in the following way. Set $s_0 := s_1$. Define $u(x) := s_\infty$ if $x\leq a$, $u(x) := s_0$ if $x\geq b$, while in each interval $\left[a+\frac{b-a}{n+1}, a+\frac{b-a}{n}\right)$, $n\in\mathbb{N}$, set

(3.29)

$$u(x) := \begin{cases} \frac{2(s_n-t_n)}{\delta_n}\left(x-x_{2i-1}^{(n)}\right)+t_n & \text{if } x_{2i-1}^{(n)}\leq x\leq x_{2i}^{(n)}, \\ & 1\leq i\leq m_n-1, \\ \frac{2(t_n-s_n)}{\delta_n}\left(x-x_{2i}^{(n)}\right)+s_n & \text{if } x_{2i}^{(n)}\leq x\leq x_{2i+1}^{(n)}, \\ & 0\leq i\leq m_n-1, \\ \frac{s_{n-1}-t_n}{x_{2m_n+1}^{(n)}-x_{2m_n-1}^{(n)}}\left(x-x_{2m_n-1}^{(n)}\right)+t_n & \text{if } x_{2m_n-1}^{(n)}\leq x\leq x_{2m_n+1}^{(n)}. \end{cases}$$

Note that $\frac{\delta_n}{2}\leq x_{2m_n+1}^{(n)}-x_{2m_n-1}^{(n)}\leq 2\delta_n$. We claim that $u\in AC(I)$. Since $s_n\to s_\infty$ and $t_n\to s_\infty$, we have that u is continuous at $x=a$. Hence, the function u is continuous and differentiable except for a countable number of points, and so, in view of Exercise 3.23, to prove that it is locally absolutely continuous in I, it remains to show that u' is integrable. By (3.25), (3.26), and (3.28) we have

$$\begin{aligned}\int_I |u'|\, dx = \int_a^b |u'|\, dx &\leq \sum_{n=1}^{\infty} (2m_n |s_n - t_n| + |t_n - s_{n-1}|) \\ &\leq \sum_{n=1}^{\infty} (2\ell_n |s_n - t_n| + |t_n - s_n| + |s_n - s_\infty| + |s_\infty - s_{n-1}|) \\ &\leq \sum_{n=1}^{\infty} \frac{M_r}{n^2+n} + \frac{M_r}{n^2+n} + \frac{2}{n^2} \leq \sum_{n=1}^{\infty} \frac{2M_r+2}{n^2} < \infty.\end{aligned}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we have that u' is integrable in I.

On the other hand, by (3.26)–(3.28),

$$\begin{aligned}\sum_{i=1}^{2m_n+1} \left| f\left(u\left(x_i^{(n)}\right)\right) - f\left(u\left(x_{i-1}^{(n)}\right)\right)\right| &\geq 2m_n |f(s_n) - f(t_n)| \\ &> 2\ell_n \left(n^2+n\right) |s_n - t_n| = 4M_r,\end{aligned}$$

and so for every $n \in \mathbb{N}$ by Remark 2.7 we obtain that

$$\operatorname{Var}_{[a,b]}(f \circ u) \geq \sum_{k=1}^{n} \operatorname{Var}_{\overline{I_k}}(f \circ u) \geq 4M_r n \to \infty$$

as $n \to \infty$. Hence, we have obtained a contradiction. □

Remark 3.69. Note that in the necessity part of the theorem we have actually proved a much stronger result, namely that if $f : \mathbb{R} \to \mathbb{R}$ is such that $f \circ u \in BPV_{\text{loc}}(I)$ for all functions $u \in AC(I) \subset AC_{\text{loc}}(I)$, then f is locally Lipschitz.

Remark 3.70. Note that the previous proof continues to hold if $f : \mathbb{R}^d \to \mathbb{R}$; namely $f \circ u$ belongs to $AC_{\text{loc}}(I)$ for all functions $u \in AC_{\text{loc}}\left(I; \mathbb{R}^d\right)$ if and only if f is locally Lipschitz.

3.3. Singular Functions

In this section we prove that every function of bounded pointwise variation may be decomposed into the sum of an absolutely continuous function and a singular function.

Definition 3.71. Let $I \subset \mathbb{R}$ be an interval. A nonconstant function $u : I \to \mathbb{R}$ is said to be *singular* if it is differentiable at $\mathcal{L}^1$-a.e. $x \in I$ with $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in I$.

The jump function u_J of a function $u \in BPV_{\text{loc}}(I)$ is an example of a singular function. Another example is the Cantor function or the function given in Theorem 1.47.

The following theorem provides a characterization of singular functions.

Theorem 3.72 (Singular functions)**.** *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be a nonconstant function such that $u'(x)$ exists (possibly infinite) for $\mathcal{L}^1$-a.e. $x \in I$. Then u is a singular function if and only if there exists a Lebesgue measurable set $E \subset I$ such that $\mathcal{L}^1(I \setminus E) = 0$ and $\mathcal{L}^1(u(E)) = 0$.*

Proof. Assume that u is singular and let $E := \{x \in I : u'(x) = 0\}$. Then $\mathcal{L}^1(I \setminus E) = 0$. By Corollary 3.14 we have that $\mathcal{L}^1(u(E)) = 0$. Conversely, assume that there exists a Lebesgue measurable set $E \subset I$ such that $\mathcal{L}^1(I \setminus E) = 0$ and $\mathcal{L}^1(u(E)) = 0$. Then by Lemma 3.45, $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in E$. Since $\mathcal{L}^1(I \setminus E) = 0$, we have that $u'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in I$. □

As an application of Lemma 3.31 we obtain the standard decomposition of a monotone function into an absolutely continuous monotone function and a singular monotone function.

Theorem 3.73. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be an increasing function. Then u may be decomposed as the sum of three increasing functions*

$$u = u_{\mathrm{AC}} + u_{\mathrm{C}} + u_{\mathrm{J}}, \tag{3.30}$$

where $u_{\mathrm{AC}} \in AC_{\mathrm{loc}}(I)$, u_{C} is continuous and singular, and u_{J} is the jump function of u.

Proof. Define $v := u - u_{\mathrm{J}}$. By Exercises 1.5 and 1.50 we have that v is increasing, continuous, and $v'(x) = u'(x)$ for $\mathcal{L}^1$-a.e. $x \in I$. Fix $x_0 \in I$ and for every $x \in I$ define

$$u_{\mathrm{AC}}(x) := \int_{x_0}^{x} v'(t)\, dt = \int_{x_0}^{x} u'(t)\, dt, \quad u_{\mathrm{C}}(x) := v(x) - u_{\mathrm{AC}}(x). \tag{3.31}$$

Then the decomposition (3.30) holds. Moreover by Lemma 3.31 we have $u_{\mathrm{C}}'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in I$. It remains to show that u_{C} is increasing. Let $x, y \in I$, with $x < y$. By Corollary 1.37,

$$u_{\mathrm{AC}}(y) - u_{\mathrm{AC}}(x) = \int_{x}^{y} v'(t)\, dt \le v(y) - v(x),$$

and so $u_{\mathrm{C}}(y) \ge u_{\mathrm{C}}(x)$ by $(3.31)_2$. □

The function u_{C} is called the *Cantor part* of u.

Since every function with bounded pointwise variation may be written as a difference of two increasing functions, an analogous result holds for functions of bounded pointwise variation.

Corollary 3.74. *Let $I \subset \mathbb{R}$ be an interval and let $u \in BPV_{\text{loc}}(I)$. Then u may be decomposed as the sum of three functions in $BPV_{\text{loc}}(I)$, i.e.,*

$$u = u_{\text{AC}} + u_{\text{C}} + u_{\text{J}}, \tag{3.32}$$

where $u_{\text{AC}} \in AC_{\text{loc}}(I)$, u_{C} is continuous and singular, and

$$u_{\text{J}}(x) := \sum_{y \in I,\, y < x} (u_+(y) - u_-(y)) + u(x) - u_-(x). \tag{3.33}$$

Moreover for every interval $[a,b] \subset I$,

$$\operatorname{Var}_{[a,b]} u = \operatorname{Var}_{[a,b]} u_{\text{AC}} + \operatorname{Var}_{[a,b]} u_{\text{C}} + \operatorname{Var}_{[a,b]} u_{\text{J}}, \tag{3.34}$$

where

$$\operatorname{Var}_{[a,b]} u_{\text{AC}} = \int_a^b |u'(x)|\, dx, \tag{3.35}$$

$$\operatorname{Var}_{[a,b]} u_{\text{J}} = \sum_{x \in (a,b)} (|u_+(x) - u(x)| + |u(x) - u_-(x)|) + |u(a) - u_+(a)| + |u(b) - u_-(b)|. \tag{3.36}$$

If, in addition, $u \in BPV(I)$, then

$$\begin{aligned} \operatorname{Var} u &= \operatorname{Var} u_{\text{AC}} + \operatorname{Var} u_{\text{C}} + \operatorname{Var} u_{\text{J}} \\ &= \int_I |u'(x)|\, dx + \operatorname{Var} u_{\text{C}} + \sum_{x \in I} (|u_+(x) - u(x)| + |u(x) - u_-(x)|), \end{aligned}$$

where $u_-(\inf I) := u(\inf I)$ if $\inf I \in I$ and $u_+(\sup I) := u(\sup I)$ if $\sup I \in I$.

Proof. The decomposition (3.32) follows either by modifying the proof of the previous theorem or by writing u as a difference of two increasing functions (see Theorem 2.18) and applying the previous theorem to each increasing function. We leave the details as an exercise.

By (2.7) for every interval $J \subset I$ we have

$$\operatorname{Var}_J u \le \operatorname{Var}_J u_{\text{AC}} + \operatorname{Var}_J u_{\text{C}} + \operatorname{Var}_J u_{\text{J}}.$$

Thus, to prove (3.34), it remains to show

$$\operatorname{Var}_{[a,b]} u \ge \operatorname{Var}_{[a,b]} u_{\text{AC}} + \operatorname{Var}_{[a,b]} u_{\text{C}} + \operatorname{Var}_{[a,b]} u_{\text{J}}. \tag{3.37}$$

We divide the proof of (3.37) into five steps.

Step 1: Assume first that u is continuous, so that $u_{\text{J}} = 0$. We claim that for every interval $[\alpha, \beta] \subset I$,

$$\operatorname{Var}_{[\alpha,\beta]} u \ge \operatorname{Var}_{[\alpha,\beta]} u_{\text{AC}} + |u_{\text{C}}(\beta) - u_{\text{C}}(\alpha)|. \tag{3.38}$$

To see this, let

$$E := \left\{x \in (\alpha, \beta) : u'_{\text{C}}(x) = 0\right\}.$$

Fix $\varepsilon > 0$ and let $\delta > 0$ be as in Definition 3.1 for the absolutely continuous function u_{AC}. Using the definition of differentiability, for every $x \in E$ we may find an interval $(a_x, b_x) \subset [\alpha, \beta]$ such that a_x and b_x are rational numbers, and if $a_x < x_1 < x < x_2 < b_x$, then

$$|u_{\mathrm{C}}(x_2) - u_{\mathrm{C}}(x_1)| \le \frac{\varepsilon}{\beta - \alpha} |x_2 - x_1| . \tag{3.39}$$

Since, $\mathcal{L}^1(E) = \beta - \alpha$, from the countable cover $\{(a_x, b_x)\}_{x \in E}$ we may choose a finite subcollection such that

$$\mathcal{L}^1\left(\sum_{i=1}^{n} (a_{x_i}, b_{x_i})\right) \ge \beta - \alpha - \delta. \tag{3.40}$$

By relabeling the points, if necessary, we may assume that

$$x_1 < x_2 < \cdots < x_n,$$

and, by shortening the intervals where necessary, that $b_{x_{i-1}} \le a_{x_i}$ for all $i = 2, \ldots, n$. For simplicity of notation we write $a_i := a_{x_i}$, $b_i := b_{x_i}$ for all $i = 1, \ldots, n$, $b_0 := \alpha$, and $a_{n+1} := \beta$. Consider now the partition

$$P = \{b_0, a_1, b_1, \ldots, a_n, b_n, a_{n+1}\}$$

of $[\alpha, \beta]$. By Remark 2.7,

$$\mathrm{Var}_{[\alpha,\beta]}\, u = \sum_{i=1}^{n} \mathrm{Var}_{[a_i,b_i]}\, u + \sum_{i=1}^{n+1} \mathrm{Var}_{[b_{i-1},a_i]}\, u. \tag{3.41}$$

By Corollary 2.23, the fact that $(u_{\mathrm{AC}})' = u'$ $\mathcal{L}^1$-a.e. in I, and Theorem 3.39, in this order,

$$\begin{aligned} \sum_{i=1}^{n} \mathrm{Var}_{[a_i,b_i]}\, u &\ge \sum_{i=1}^{n} \int_{a_i}^{b_i} \left|u'\right| dx \\ &= \sum_{i=1}^{n} \int_{a_i}^{b_i} \left|(u_{\mathrm{AC}})'\right| dx = \sum_{i=1}^{n} \mathrm{Var}_{[a_i,b_i]}\, u_{\mathrm{AC}}. \end{aligned} \tag{3.42}$$

Using (2.7), we obtain that

$$\begin{aligned} \mathrm{Var}_{[b_{i-1},a_i]}\, u &\ge \mathrm{Var}_{[b_{i-1},a_i]}\, u_{\mathrm{C}} - \mathrm{Var}_{[b_{i-1},a_i]}\, u_{\mathrm{AC}} \\ &\ge |u_{\mathrm{C}}(a_i) - u_{\mathrm{C}}(b_{i-1})| - \mathrm{Var}_{[b_{i-1},a_i]}\, u_{\mathrm{AC}}, \end{aligned}$$

which, together with (3.41) and (3.42), yields

$$\begin{aligned} \mathrm{Var}_{[\alpha,\beta]}\, u \ge &\sum_{i=1}^{n} \mathrm{Var}_{[a_i,b_i]}\, u_{\mathrm{AC}} + \sum_{i=1}^{n+1} |u_{\mathrm{C}}(a_i) - u_{\mathrm{C}}(b_{i-1})| \\ &- \sum_{i=1}^{n+1} \mathrm{Var}_{[b_{i-1},a_i]}\, u_{\mathrm{AC}}. \end{aligned} \tag{3.43}$$

By (3.40) we have

$$\sum_{i=1}^{n+1} (a_i - b_{i-1}) = \beta - \alpha - \sum_{i=1}^{n} (b_i - a_i) \le \delta.$$

Hence (see the proof of Corollary 3.41),

$$\sum_{i=1}^{n+1} \operatorname{Var}_{[b_{i-1},a_i]} u_{\mathrm{AC}} \le \varepsilon. \tag{3.44}$$

On the other hand, by (3.39),

$$\begin{aligned}\sum_{i=1}^{n+1} |u_{\mathrm{C}}(a_i) - u_{\mathrm{C}}(b_{i-1})| &\ge |u_{\mathrm{C}}(\beta) - u_{\mathrm{C}}(\alpha)| - \sum_{i=1}^{n} |u_{\mathrm{C}}(b_i) - u_{\mathrm{C}}(a_i)| \\ &\ge |u_{\mathrm{C}}(\beta) - u_{\mathrm{C}}(\alpha)| - \frac{\varepsilon}{\beta - \alpha} \sum_{i=1}^{n} (b_i - a_i) \qquad (3.45) \\ &\ge |u_{\mathrm{C}}(\beta) - u_{\mathrm{C}}(\alpha)| - \varepsilon.\end{aligned}$$

Combining (3.43), (3.44), and (3.45) and using Remark 2.7 for u_{AC}, we obtain

$$\operatorname{Var}_{[\alpha,\beta]} u \ge \operatorname{Var}_{[\alpha,\beta]} u_{\mathrm{AC}} + |u_{\mathrm{C}}(\beta) - u_{\mathrm{C}}(\alpha)| - 3\varepsilon.$$

By letting $\varepsilon \to 0^+$, we obtain (3.38).

Step 2: Fix an interval $[a,b] \subset I$ and consider a partition P of $[a,b]$, with

$$a = y_0 < y_1 < \cdots < y_m = b.$$

Applying (3.38) in each interval $[y_{i-1}, y_i]$ and using Proposition 2.6 yields

$$\begin{aligned}\operatorname{Var}_{[a,b]} u &= \sum_{i=1}^{m} \operatorname{Var}_{[y_{i-1},y_i]} u \\ &\ge \sum_{i=1}^{m} \operatorname{Var}_{[y_{i-1},y_i]} u_{\mathrm{AC}} + \sum_{i=1}^{m} |u_{\mathrm{C}}(y_i) - u_{\mathrm{C}}(y_{i-1})| \\ &= \operatorname{Var}_{[a,b]} u_{\mathrm{AC}} + \sum_{i=1}^{m} |u_{\mathrm{C}}(y_i) - u_{\mathrm{C}}(y_{i-1})| .\end{aligned}$$

Taking the supremum over all partitions of $[a,b]$ gives

$$\operatorname{Var}_{[a,b]} u \ge \operatorname{Var}_{[a,b]} u_{\mathrm{AC}} + \operatorname{Var}_{[a,b]} u_{\mathrm{C}}.$$

Thus, we have proved that (3.37) holds if u is continuous.

Step 3: Assume that u has a finite number of discontinuity points in (a,b), say

$$a < t_1 < \cdots < t_\ell < b.$$

Let $t_0 := a$, $t_{\ell+1} := b$ and fix $\delta > 0$ so small that

$$0 < \delta < \min\{t_i - t_{i-1} : i = 1, \dots, \ell + 1\}.$$

Using Remark 2.7, we have

$$\begin{aligned}
\operatorname{Var}_{[a,b]} u &= \sum_{i=1}^{\ell+1} \operatorname{Var}_{[t_{i-1}+\delta, t_i-\delta]} u + \sum_{i=0}^{\ell+1} \operatorname{Var}_{[t_i-\delta, t_i+\delta]\cap[a,b]} u \\
&\geq \sum_{i=1}^{\ell+1} \operatorname{Var}_{[t_{i-1}+\delta, t_i-\delta]} u \\
&\quad + \sum_{i=1}^{\ell} \left(|u(t_i+\delta) - u(t_i)| + |u(t_i) - u(t_i-\delta)|\right) \\
&\quad + |u(a+\delta) - u(a)| + |u(b) - u(b-\delta)| \\
&\geq \sum_{i=1}^{\ell+1} \left(\operatorname{Var}_{[t_{i-1}+\delta, t_i-\delta]} u_{\mathrm{AC}} + \operatorname{Var}_{[t_{i-1}+\delta, t_i-\delta]} u_{\mathrm{C}}\right) \\
&\quad + \sum_{i=1}^{\ell} \left(|u(t_i+\delta) - u(t_i)| + |u(t_i) - u(t_i-\delta)|\right) \\
&\quad + |u(a+\delta) - u(a)| + |u(b) - u(b-\delta)|,
\end{aligned}$$

where we have used the previous step together with the fact that in each interval $[t_{i-1} + \delta, t_i - \delta]$ the jump function u_{J} is constant, and so the variation of u in those intervals reduces to the one of $u_{\mathrm{C}} + u_{\mathrm{AC}}$. Letting $\delta \to 0^+$ in the previous inequality and using Remark 2.7 and Exercise 2.8 yields

$$\begin{aligned}
\operatorname{Var}_{[a,b]} u &\geq \sum_{i=1}^{\ell+1} \left(\operatorname{Var}_{[t_{i-1}, t_i]} u_{\mathrm{AC}} + \operatorname{Var}_{[t_{i-1}, t_i]} u_{\mathrm{C}}\right) \\
&\quad + \sum_{i=1}^{\ell} \left(|u_+(t_i) - u(t_i)| + |u(t_i) - u_-(t_i)|\right) \\
&\quad + |u_+(a) - u(a)| + |u(b) - u_-(b)| \\
&= \operatorname{Var}_{[a,b]} u_{\mathrm{AC}} + \operatorname{Var}_{[a,b]} u_{\mathrm{C}} + \operatorname{Var}_{[a,b]} u_{\mathrm{J}}.
\end{aligned}$$

Step 4: Finally, if u has an infinite number of discontinuity points in (a, b), say $\{t_i\}$, consider the saltus function $u_{\mathrm{J},k}$ corresponding to the points t_i with $i < k$. For each $k \in \mathbb{N}$ and $x \in [a, b]$ define

$$u_k(x) := u_{\mathrm{AC}}(x) + u_{\mathrm{C}}(x) + u_{\mathrm{J},k}.$$

Since the discontinuity points of u_k in (a,b) are $\{t_1,\ldots,t_k\}$, by the previous step

$$\begin{aligned}\operatorname{Var}_{[a,b]} u_k = \operatorname{Var}_{[a,b]} u_{\mathrm{AC}} + \operatorname{Var}_{[a,b]} u_{\mathrm{C}} \\ + \sum_{i=1}^{k} \left(\left|u_+ (t_i) - u(t_i)\right| + \left|u(t_i) - u_-(t_i)\right|\right) \\ + \left|u_+(a) - u(a)\right| + \left|u(b) - u_-(b)\right|.\end{aligned}$$

Since

$$\begin{aligned}\operatorname{Var}_{[a,b]} u_k &= \operatorname{Var}_{[a,b]} (u_k - u + u) \le \operatorname{Var}_{[a,b]} (u_k - u) + \operatorname{Var}_{[a,b]} u \\ &\le \sum_{i=k}^{\infty} \left(\left|u_+ (t_i) - u(t_i)\right| + \left|u(t_i) - u_-(t_i)\right|\right) + \operatorname{Var}_{[a,b]} u,\end{aligned}$$

by the previous equality we have

$$\begin{aligned}\operatorname{Var}_{[a,b]} u_{\mathrm{AC}} + \operatorname{Var}_{[a,b]} u_{\mathrm{C}} + \sum_{i=1}^{k} \left(\left|u_+ (t_i) - u(t_i)\right| + \left|u(t_i) - u_-(t_i)\right|\right) \\ + \left|u_+(a) - u(a)\right| + \left|u(b) - u_-(b)\right| \\ \le \sum_{i=k}^{\infty} \left(\left|u_+ (t_i) - u(t_i)\right| + \left|u(t_i) - u_-(t_i)\right|\right) + \operatorname{Var}_{[a,b]} u.\end{aligned}$$

Letting $k \to \infty$, we obtain (3.37).

Step 5: If, in addition, $u \in BPV(I)$, taking a_n and b_n as in Step 2 of the proof of Theorem 3.39, we apply (3.34) in $[a_n, b_n]$ and use Proposition 2.6 and the Lebesgue monotone convergence theorem. □