
Introduction: The Main Problem

Prerequisites. What background is needed for reading this text? Chiefly, a knowledge of piecewise linear topology. For many years the standard reference in that area has been the text *Introduction to Piecewise-Linear Topology*, by C. P. Rourke and B. J. Sanderson (1972), and we assume familiarity with much of their book. To be honest, that book presumes extensive understanding of both general and algebraic topology; as a consequence we implicitly are assuming those subjects as well. In an attempt to limit our presumptions, we specifically shall take as granted the results from two fairly standard texts on general and algebraic topology, both by J. R. Munkres—namely, his *Topology: Second Edition* (2000) and *Elements of Algebraic Topology* (1984), each of which can be treated quite effectively in a year-long graduate course.

Unfortunately, even those three texts turn out to be insufficient for all our needs. The purpose of the initial Chapter 0, the Prequel, is to correct that deficiency.

Basic Terminology. The notation laid out in this subsection should be familiar to those who have read Rourke and Sanderson's text. Nevertheless, we spell out the essentials needed to fully understand the forthcoming discussion of the primary issues addressed in this book.

Here \mathbb{R} denotes the set of real numbers and \mathbb{R}^n denotes n -dimensional Euclidean space, the Cartesian product of n copies of \mathbb{R} . For $1 \leq k < n$ we regard \mathbb{R}^k as included in \mathbb{R}^n in the obvious way, as the subset containing all points whose final $(n - k)$ -coordinates are all equal to zero.

We use B^n to denote the standard n -ball (or n -cell) in \mathbb{R}^n , $\text{Int } B^n$ to denote its interior, and S^{n-1} to denote the standard $(n-1)$ -sphere, the boundary, ∂B^n , of B^n . Specifically,

$$\begin{aligned} B^n &= \{\langle x_1, x_2, \dots, x_n \rangle \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 \leq 1\}, \\ \text{Int } B^n &= \{\langle x_1, x_2, \dots, x_n \rangle \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 < 1\}, \text{ and} \\ S^{n-1} = \partial B^n &= \{\langle x_1, x_2, \dots, x_n \rangle \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 = 1\}. \end{aligned}$$

We call any space homeomorphic to B^n or S^{n-1} an n -cell or an $(n-1)$ -sphere, respectively. The k -ball B^k is defined as a subset of \mathbb{R}^k , but for each $k < n$ the inclusion $\mathbb{R}^k \subset \mathbb{R}^n$ determines a standard k -ball B^k and a standard $(k-1)$ -sphere S^{k-1} in \mathbb{R}^n as well.

All simplicial complexes and CW complexes are assumed to be locally finite. A *polyhedron* is the underlying space of a simplicial complex. While a simplicial complex K and the underlying polyhedron $|K|$ are two different things, we will not always maintain this distinction in our terminology. Piecewise linear is abbreviated PL.

An n -dimensional (topological) manifold is a separable metric space in which each point has a neighborhood that is homeomorphic to \mathbb{R}^n . Such a neighborhood is called a *coordinate neighborhood* of the point.

The Main Problem. The central topic in this text is topological embeddings. Formally, an *embedding* of one topological space X in another space Y is nothing more than a homeomorphism of X onto a subspace of Y . The domain X is called the *embedded space* and the target Y is called the *ambient space*. Two embeddings $\lambda, \lambda' : X \rightarrow Y$ are *equivalent* if there exists a (topological) homeomorphism Θ of Y onto itself such that $\Theta \circ \lambda = \lambda'$. The main problem in the study of topological embeddings is:

Main Problem. *Which embeddings of X in Y are equivalent?*

In extremely rare circumstances all pairs of embeddings are equivalent. For instance, if X is just a point, the equivalence question for an arbitrary pair of embeddings of X in a given space Y amounts to the question of homogeneity of Y , which has an affirmative answer whenever, for example, Y is a connected manifold.

Ordinarily, then, our interest will turn to conditions under which embeddings are equivalent, and we will limit attention to reasonably well-behaved spaces X and Y . Specifically, in this book the embedded space X will ordinarily be a compact polyhedron¹ and the ambient space Y will always be a manifold, usually a piecewise linear (abbreviated PL) manifold. If there are embeddings of the polyhedron X in the PL manifold Y that are homotopic

¹A major exception is the study of embeddings of the Cantor set.

but not equivalent, then X is said to *knot* in Y . For given polyhedra X and Y , it is often possible to identify a distinguished class of PL embeddings of X in Y that are considered to be *unknotted*; any PL embedding that is not equivalent to an unknotted embedding is then said to be *knotted*.

While we do place limitations on the spaces considered, we intentionally include the most general kinds of topological embeddings in the discussion. Let X be a polyhedron and let Y be a PL manifold. An embedding $X \rightarrow Y$ is said to be a *tame embedding* if it is equivalent to a PL embedding; the others are called *wild*. For embeddings of polyhedra the Main Problem splits off two fundamental special cases, one called the Taming Problem and the other the (PL) Unknotting Problem.

Taming Problem. *Which topological embeddings of X in Y are equivalent to PL embeddings?*

Unknotting Problem. *Which PL embeddings of X in Y are equivalent?*

The point is, for tame embeddings the Main Problem reduces to the Unknotting Problem, and PL methods provide effective – occasionally complete – answers to the latter. As we shall see, local homotopy properties give very precise answers to the Taming Problem. This also means that local homotopy properties make detection of wildness quite easy. There are related crude measures that adequately differentiate certain types of wildness, but the category of wild embeddings is highly chaotic. In fact, at the time of this writing very little effort had been devoted to classifying in any systematic way the wild embeddings of polyhedra in manifolds.

A closed subset X of a PL manifold N is said to be *tame* (or, *tame as a subspace*) if there exists a homeomorphism h of N onto itself such that $h(X)$ is a subpolyhedron; X itself is *wild* if it is homeomorphic to a simplicial complex but is not tame. Here the focus is more on the subspace X than on a particular embedding. One can provide a direct connection, of course: a closed subset X of a PL manifold N is tame as a subspace if and only if there exist a polyhedron K and a homeomorphism $g : K \rightarrow X$ such that $\lambda = \text{inclusion} \circ g : K \rightarrow N$ is a tame embedding.

We say that a k -cell or $(k - 1)$ -sphere X in \mathbb{R}^n is *flat* if there exists a homeomorphism h of \mathbb{R}^n such that $h(X)$ is the standard object of its type. Generally, whenever we have some standard object $S \subset \mathbb{R}^n$ and a subset X of \mathbb{R}^n homeomorphic to S , we will say that X is *flat* if there is a homeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $h(X) = S$. In other words, S represents the preferred copy in \mathbb{R}^n , and another copy in \mathbb{R}^n is flat if it is ambiently equivalent (setwise) to S .

Flatness Problem. *Under what conditions is a cell or sphere in \mathbb{R}^n flat?*

The problems listed above are the main ones that will occupy attention in this text. They all can be viewed as uniqueness questions in the sense that they ask whether given embeddings are equivalent. There are also existence questions for embeddings, which will be studied alongside the uniqueness questions. We identify two such: one global, the other local.

Existence Problem. *Given a map $f : X \rightarrow Y$, is f homotopic to a topological embedding or a PL embedding?*

Approximation Problem. *Which topological embeddings of X in Y can be approximated by PL embeddings?*

The flatness concept has a local version. A topological embedding $e : M \rightarrow N$ of a k -dimensional manifold M into an n -dimensional manifold N is *locally flat at $x \in M$* if there exists a neighborhood U of $e(x)$ in N such that $(U, U \cap e(M)) \cong (\mathbb{R}^n, \mathbb{R}^k)$. An embedding is said to be *locally flat* if it is locally flat at each point x of its domain. The last two problems have local variations: for example, one can ask whether a map of manifolds is homotopic to a locally flat embedding or whether a topological embedding of manifolds can be approximated by locally flat embeddings.

When considering an embedding $e : X \rightarrow Y$, the dimension of Y is called the *ambient dimension*. Almost all of the examples and theorems in this book involve embeddings in manifolds of ambient dimension three or more. We skip dimension two because classical results like the famous Schönflies theorem (Theorem 0.11.1) imply that no nonstandard local phenomena arise in conjunction with embeddings into manifolds of that dimension.

While isolated examples of wild embeddings were discovered earlier, the work of R. H. Bing in the 1950s and 1960s revealed the pervasiveness of wildness in dimensions three and higher. His pioneering work led to a proliferation of embedding results, first concentrating on dimension three, but soon expanding to include higher dimensions as well. The subject of topological embeddings is now a mature branch of geometric topology, and this book is meant to be a summary and exposition of the fundamental results in the area.

Organization. As mentioned earlier, the initial Chapter 0 addresses background matters. The real beginning, Chapter 1, treats knottedness, tameness and local flatness; it provides examples of knotted, PL codimension-two sphere pairs in all sufficiently large dimensions, and it delves into the local homotopy properties of nicely embedded objects. Chapter 2 presents the basic examples that motivate the study and offers context for theorems to come later; it also includes several flatness theorems that can be proved without the use of engulfing. Engulfing – the fundamental technical tool for

the subject – is introduced and carefully examined in Chapter 3. The remaining chapters strive to systematically investigate the central embedding problems. That investigation is organized by codimension. The *codimension* of an embedding $e : X \rightarrow Y$ is defined by $\text{codim}(e) = \dim Y - \dim X$, the difference between the ambient dimension and the dimension of the embedded space. Generally speaking, the greater the codimension the easier it is to prove positive theorems about embeddings. Chapter 4 treats the *trivial range*, the range in which the codimension of the embedded space exceeds its dimension, where the most general theorems hold. Next, Chapter 5 moves on to codimension three, to which many trivial-range theorems extend with appropriate modifications. However, very few of the codimension-three theorems extend to codimension two, so Chapter 6 is largely devoted to the construction of codimension-two counterexamples. In codimension one the situation changes once more, and again there are many positive results, which form the subject of Chapter 7. The book concludes in Chapter 8 with a quick description of some codimension-zero results.