Wild and Flat Embeddings

To truly appreciate results about flatness, one must be keenly aware of the existence of wildness. In this chapter we set forth a wealth of examples of wild embeddings, beginning with two classics discovered by Antoine and Alexander in the 1920s. Then we describe a technique involving decomposition spaces by which wild arcs in S^n are transmuted into wild arcs in S^{n+1}. Toward the end of the chapter we introduce additional examples of wildly embedded 1-, 2- and 3-cells in S^3; some of them were discovered in the 1940s by R. H. Fox and E. Artin while others were discovered in the 1960s by R. H. Bing and his followers. The net effect is to make available examples of wild embeddings in all possible dimensions and codimensions.

Partly for contrast, we also present several results about flat embeddings. All are derived by elementary methods, independent of the engulfing techniques to be developed in subsequent chapters. The results include the Generalized Schönflies Theorem of M. Brown confirming the flatness of any codimension-one sphere in S^n that is locally flat, a version of work by J. C. Cantrell assuring the flatness of a codimension-one sphere in $\mathbb{R}^n$, $n > 3$, that is locally flat everywhere except possibly one point and a result of V. Klee attesting to the flatness of an arc in $\mathbb{R}^n$ that lies in a hyperplane.

2.1. Antoine's necklace and Alexander's horned sphere

Here we reproduce two fundamental, historically important examples of wild embeddings in $\mathbb{R}^3$. The first is a wild embedding of the Cantor set and the second is a wild embedding of S^2. For each of them the invariant used to detect wildness is the fundamental group of the complement. Related high-dimensional examples are constructed by suspension.

Example 2.1.1. *There exists a wild Cantor set in* $\mathbb{R}^3$.

A *Cantor set* in a manifold is an embedded copy of the familiar middle-thirds Cantor set. Cantor sets are characterized as the compact, totally disconnected metric spaces that are perfect, meaning that they have no isolated points. The example we construct is known as *Antoine's necklace.*

Remark. It is a mild but common and traditional misnomer to speak of a Cantor set as being "wild"; strictly speaking, a Cantor set cannot be wild because it is not an embedded polyhedron. There is a standard copy of the Cantor set in $[0,1] \subset \mathbb{R}^1 \subset \mathbb{R}^n$, so what we really mean when we call a Cantor set in $\mathbb{R}^n$ wild is that it is not flat. In the same spirit, when we speak of "tame" Cantor sets in $\mathbb{R}^n$, what we really mean is that they are flat.

A *solid torus* is a ∂-manifold homeomorphic to $S^1 \times B^2$. Let T be a solid torus standardly positioned in $\mathbb{R}^3$, and let T_1, T_2, T_3, and T_4 be solid tori embedded in $\operatorname{Int} T$ as shown in Figure 2.1. (Note that $(\mathbb{R}^3, T)$ and $(\mathbb{R}^3, T_i)$ are pairwise homeomorphic.) Set $A_0 = T$ and $A_1 = \cup_{i=1}^4 T_i$. In each T_i let $T_{i1}, T_{i2}, T_{i3}, T_{i4}$ be solid tori embedded there exactly as the T_i are placed in T. Replicate infinitely often, so that at the k-th step we have a ∂-manifold A_k, the union of 4^k (pairwise disjoint) solid tori, where each component τ of A_k contains exactly 4 components of A_{k+1}, and where there exists a homeomorphism of the triples $(\mathbb{R}^3, \tau, \tau \cap A_{k+1})$ and $(\mathbb{R}^3, T, A_1)$. Arrange these pieces so that each component τ of A_k has diameter at most ϵ_k, where $\epsilon_k \to 0$ as $k \to \infty$. (It is permissible to let the number of components of A_k be larger than 4^k in order to achieve small size.)

Set $A = \cap_k A_k$. Then A is a compact, totally disconnected metric space with no isolated points. Hence, A is homeomorphic to the standard middle-thirds Cantor set C in $[0,1] \subset \mathbb{R}^1 \subset \mathbb{R}^3$. (If one wants to secure a homeomorphism between A and C directly, without appeal to the topological characterization of the Cantor set, one easily can show, based on the construction, that $A \cong \Pi_{i=1}^{\infty} X_i$, where $X_i = \{1,2,3,4\}$ is endowed with the discrete topology, and exploit the related, more familiar $C \cong \Pi_{i=1}^{\infty} S_i$, where each S_i is a two-point set with the discrete topology.)

Proposition 2.1.2. $\pi_1(\mathbb{R}^3 \smallsetminus A) \neq \{1\}$.

Since $\pi_1(\mathbb{R}^3 \smallsetminus C) \cong \{1\}$, this proposition will confirm that A is wild. The argument will be based on the following pair of technical facts.

Lemma 2.1.3. *The inclusion-induced* $\phi_\# : \pi_1(\partial T_i) \to \pi_1(\mathbb{R}^3 \smallsetminus \operatorname{Int} A_1)$ *is one-to-one.*

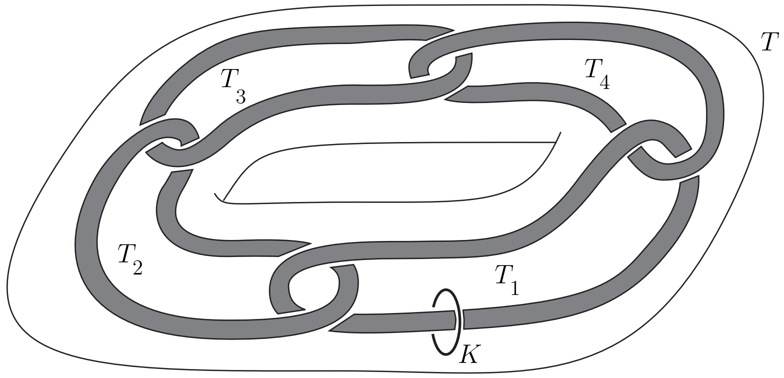

Figure 2.1. The first two stages in the construction of Antoine's necklace

Lemma 2.1.4. *The inclusion-induced* $\phi'_{\#} : \pi_1(\partial T) \to \pi_1(T \smallsetminus \operatorname{Int} A_1)$ *is one-to-one.*

Assuming Lemmas 2.1.3 and 2.1.4 for the moment, we complete the proof of Proposition 2.1.2. Thicken $\mathbb{R}^3 \smallsetminus \operatorname{Int} A_0$ to an open set W_0 that admits a strong deformation retraction to $\mathbb{R}^3 \smallsetminus \operatorname{Int} A_0$. Similarly, for $k \geq 1$ thicken $A_{k-1} \smallsetminus \operatorname{Int} A_k$ to an open set W_k that admits a strong deformation retraction to $A_{k-1} \smallsetminus \operatorname{Int} A_k$. Impose control on these thickenings to ensure that $W_{k-1} \cap W_k$ is naturally homeomorphic to $\partial A_k \times (-1, 1)$. Add the components of W_k to $\cup_{i=0}^{k-1} W_i$ one at a time and apply Lemma 2.1.3 and 2.1.4 in conjunction with Theorem 0.11.5 to establish that $\pi_1(\cup_{i=0}^{k-1} W_i) \to \pi_1(\cup_{i=0}^{k} W_i)$ is 1-1. It follows that $\mathbb{Z} \cong \pi_1(W_0) \to \pi_1(\mathbb{R}^3 \smallsetminus A = \cup_{i=0}^{\infty} W_i)$ is 1-1. Thus, $\pi_1(\mathbb{R}^3 \smallsetminus A) \neq \{1\}$.

We now turn our attention to Lemmas 2.1.3 and 2.1.4. Their proofs are based on the following claims.

Claim 2.1.5. *Let* J *and* C *denote linked circles in* $\mathbb{R}^3$*, as shown in Figure 2.2. Then* J *is a retract of* $\mathbb{R}^3 \smallsetminus C$*.*

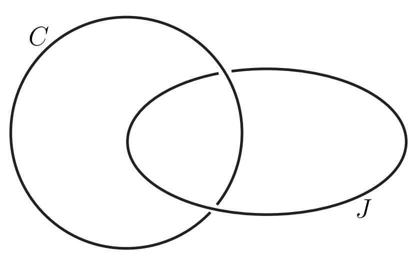

Figure 2.2. Two linked circles

Proof. Build a (round) 3-cell B containing $J \cup C$, and split it with a 2-cell into two hemispherical balls B_L and B_R such that $B_R \supset J$ and B_L intersects C in a standard spanning arc. Find retractions:

(1) of $\mathbb{R}^3 \setminus C$ to $B \setminus C$,

(2) of $B_L \setminus C$ to $\partial B_L \setminus C$ and, by extension, of $B \setminus C$ to $(B_R \cup \partial B_L) \setminus C$,

(3) of the latter to $B_R \setminus C$, and

(4) of $B_R \setminus C$ to J. □

A similar argument yields:

Claim 2.1.6. *Let C and C' denote circles in $\mathbb{R}^3$ and E a planar disk in $\mathbb{R}^3$, as shown in Figure 2.3. Then $E \setminus (C \cup C')$ is a retract of $\mathbb{R}^3 \setminus (C \cup C')$.*

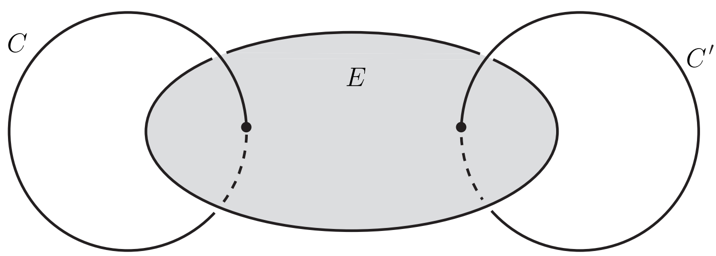

Figure 2.3. Three linked circles

Proof of Lemma 2.1.3. The proof of Lemma 2.1.3 is a relatively straightforward application of Claim 2.1.5 and is left as an exercise. □

Call a simple closed curve C a *center line* of a solid torus T if there exists a homeomorphism of $S^1 \times B^2$ onto T carrying $S^1 \times \{0\}$ onto C.

Proof of Lemma 2.1.4. Let C_1 and C_3 denote center lines of T_1 and T_3, respectively. Find disks E_2 and E_4 in $\operatorname{Int} T$ as shown in Figure 2.4, where $E_j \cap \partial A_1 = \partial E_j \subset \partial T_j$. Since $C_1 \cup E_2 \cup C_3 \cup E_4$ contains a center line of T, there exists a retraction

$$\rho : T \setminus (C_1 \cup E_2 \cup C_3 \cup E_4) \to \partial T.$$

Thicken each E_j $(j = 2, 4)$ to a 3-cell B_j such that, among other things, B_j meets ∂T_j in an annulus and meets each of C_1 and C_3 in a standard arc spanning B_j. Then E_j splits B_j into two 3-cells and the closure of $\partial B_j \setminus T_j$ consists of two parallel copies, E_j^+ and E_j^-, of E_j.

Suppose $f : I^2 \to T \setminus A_1$ with $f(\partial I^2) \subset \partial T$. Find a 2-dimensional PL ∂-manifold M in $\operatorname{Int} I^2$ such that

$$f^{-1}(E_2 \cup E_4) \subset \operatorname{Int} M \subset M \subset f^{-1}(B_2 \cup B_4).$$

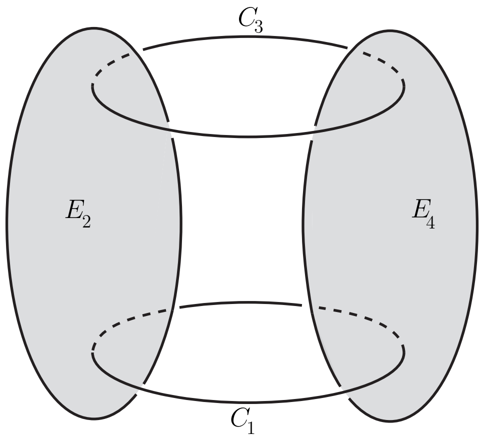

Figure 2.4. Four linked circles

Identify the component P of $I^2 \smallsetminus \operatorname{Int} M$ containing ∂I^2, then name the simple closed curves J_i of $P \cap M$ as well as the 2-cells $D_i \subset \operatorname{Int} I^2$ bounded by J_i $(i = 1, \ldots, k)$.

Fix i. There exists a j (either 2 or 4) such that $f(J_i) \subset B_j \smallsetminus E_j$, so $f(J_i)$ may be homotoped to lie entirely in either E_j^+ or E_j^-. This homotopy takes place inside of $B_j \smallsetminus E_j$, so it may be extended to all of I^2 and we may assume that $f(J_i)$ is contained in either E_j^+ or E_j^-; to be specific let us say $f(J_i) \subset E_j^+$. The map $f|D_i$ shows that $f|J_i$ is nullhomotopic in $\mathbb{R}^3 \smallsetminus (T_1 \cup T_3)$, and Claim 2.1.6 implies that f can be redefined on D_i so that its image lies in $E_j^+ \smallsetminus (C_1 \cup C_3)$.

If this process is carried out for each i, then f will be replaced by a map $F : I^2 \to T \smallsetminus (C_1 \cup E_2 \cup C_3 \cup E_4)$ with $F|\partial I^2 = f|\partial I^2$. Now ρF reveals that $f|\partial I^2$ is nullhomotopic in ∂T. $\square$

This completes the construction of Antoine's necklace. As mentioned earlier, some variation is allowed in the number of solid tori used at each stage of the construction: each solid torus at one stage may be replaced by more than four solid tori at the next stage, and it is even permissible for the number of solid tori to vary from one stage to another and from one link to another. For this reason Antoine's necklace can be regarded as one specific member of a whole class of Antoine Cantor sets. There are two conditions that must be satisfied by the construction of the objects in this class. First, each component τ of the k-th stage ∂-manifold A_k must be an unknotted solid torus and all the next-stage solid tori in τ must be simply linked in a chain that winds exactly once around τ. This will ensure that $\pi_1(\mathbb{R}^3 \smallsetminus A) \neq \{1\}$. The second condition is that each solid torus at one stage must be replaced by enough solid tori at the next stage so that the diameters of the components of A_k approach 0 as $k \to \infty$. This second condition ensures that $\cap_{k=1}^{\infty} A_k$ is totally disconnected and therefore a Cantor set. Figure 2.5 shows three consecutive stages in a typical construction.

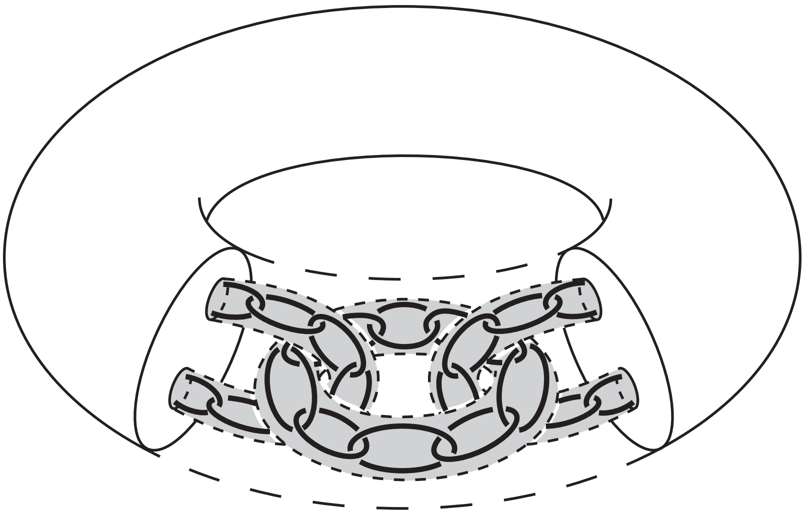

Figure 2.5. An Antoine Cantor set

Different Antoine Cantor sets may be inequivalently embedded. In fact, varying the number of links in the Antoine construction results in an uncountable number of different equivalence classes of embeddings of the Cantor set in $\mathbb{R}^3$ (Sher, 1968).

We now turn our attention to the construction of wild spheres.

Example 2.1.7. *There exist wild 2-spheres in* $\mathbb{R}^3$.

We will describe two different examples. The first is based on Antoine's necklace. Start with a round 3-cell F_0 in $\mathbb{R}^3$ that is disjoint from A. Add a tube F_1 that connects the first 3-cell to the solid torus T. The tube is solid, so $F_0 \cup F_1$ is another 3-cell. Add thin tubes in T to connect $F_1 \cap T$ to the four components of A_1; then do the same at later stages, adding 4^k tubes at stage $k + 1$. There should be four disjoint tubes in each component of A_k as indicated in Figure 2.6. Define the *Antoine 3-cell* to be F, the union of all the tubes together with Antoine's necklace A, and define the *Antoine sphere* to be ∂F.

It is not difficult to construct a homeomorphism from a 3-cell to F. We will not explicitly describe that construction, although we will give some indication of how a similar homeomorphism is constructed when we describe Alexander's horned sphere, below. The Antoine sphere bounds a topological

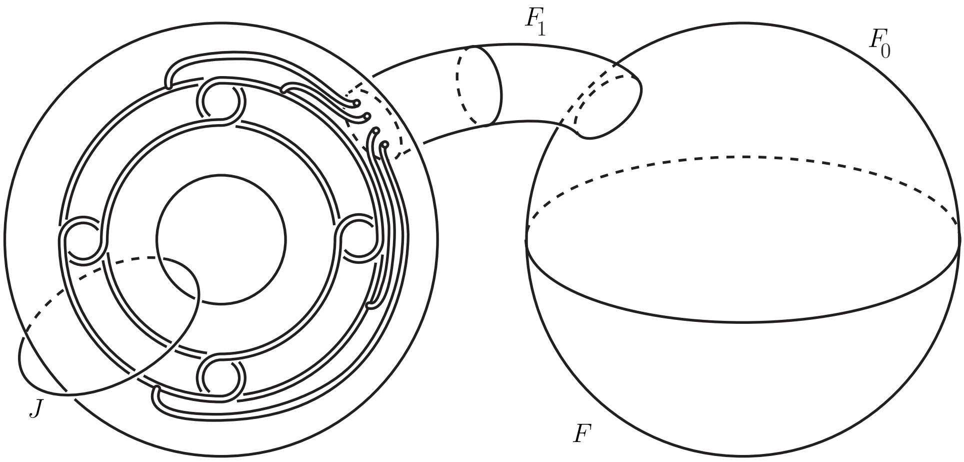

Figure 2.6. The Antoine sphere

3-cell on the inside, but the exterior is not simply connected. In order to see that the exterior is not simply connected, observe that F can be constructed so that it does not intersect the loop $J \subset \mathbb{R}^3 \setminus F$ shown in Figure 2.6. Since J is homotopically essential in the complement of Antoine's necklace, it is homotopically essential in $\mathbb{R}^3 \setminus \partial F$ as well.

Note that we have not only constructed a wild 2-sphere, but have also constructed a wild 3-cell F. Any arc in F that contains Antoine's necklace must be wild since the loop J represents a nontrivial loop in the complement. Similarly any 2-cell in F that contains A must be wild. Hence we have the following.

Example 2.1.8. *There exist wild cells of dimension 1, 2, and 3 in* $\mathbb{R}^3$.

We now construct a second wild 2-sphere in $\mathbb{R}^3$, the famous *Alexander horned sphere.* The construction relies on a certain pillbox replacement procedure.

Definition. A *pillbox* is a cylindrical 3-cell C with top disk τ and bottom disk β containing simply linked solid tori T_1 and T_2, with $T_1 \cap \partial C = \tau$ and $T_2 \cap \partial C = \beta$. (See Figure 2.7.)

Lemma 2.1.9. *Let C be a pillbox, let X be a closed subset of $\mathbb{R}^3$ such that $X \cap C = \tau \cup \beta$, and let J be a 1-sphere in $\mathbb{R}^3 \setminus (X \cup C)$ as shown in Figure 2.8. If $\pi_1(J) \to \pi_1(\mathbb{R}^3 \setminus (X \cup C))$ is one-to-one, then $\pi_1(\mathbb{R}^3 \setminus (X \cup C)) \to \pi_1(\mathbb{R}^3 \setminus (X \cup T_1 \cup T_2))$ is also one-to-one.*

Proof. Use Claim 2.1.5 and the technique of proof of Lemma 2.1.4 to show that if J is null-homotopic in $\mathbb{R}^3 \setminus (X \cup T_1 \cup T_2)$, then J is null-homotopic in $\mathbb{R}^3 \setminus (X \cup C)$. $\square$

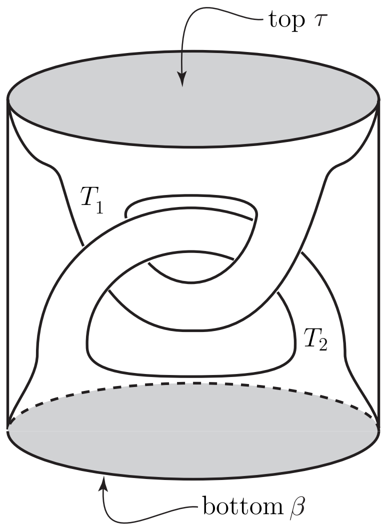

Figure 2.7. A pillbox

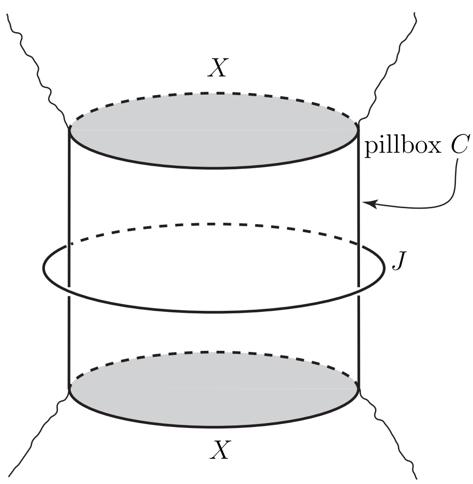

Figure 2.8. Lemma 2.1.9

To begin the construction of the horned sphere, let S_1 be an unknotted solid torus in $\mathbb{R}^3$. We will refer to this solid torus as the first stage in the construction. It is obvious that $\pi_1(\mathbb{R}^3 \setminus S_1) \neq \{1\}$; in fact, the loop J shown in Figure 2.9 represents a nontrivial element of $\pi_1(\mathbb{R}^3 \setminus S_1)$. Inside S_1 identify a pillbox C_1 as indicated in Figure 2.9. Let D_1 denote the complementary 3-cell in S_1. Then $S_1 = C_1 \cup D_1$ and $C_1 \cap D_1 = \tau_1 \cup \beta_1$, the top and bottom of the pillbox.

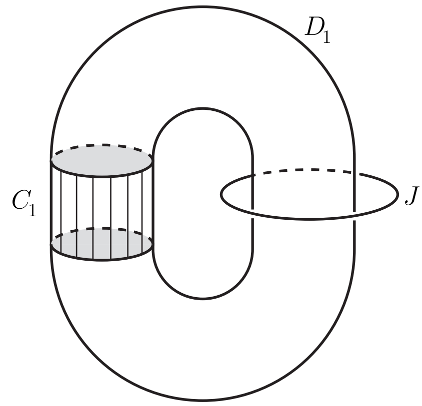

Figure 2.9. The first stage in the construction

Let T_{11} and T_{12} be the two distinguished solid tori in the pillbox C_1. Define the second stage of the construction by $S_2 = (S_1 \smallsetminus C_1) \cup (T_{11} \cup T_{12})$. It follows from Lemma 2.1.9 that the loop J represents a nontrivial element of $\pi_1(\mathbb{R}^3 \smallsetminus S_2)$.

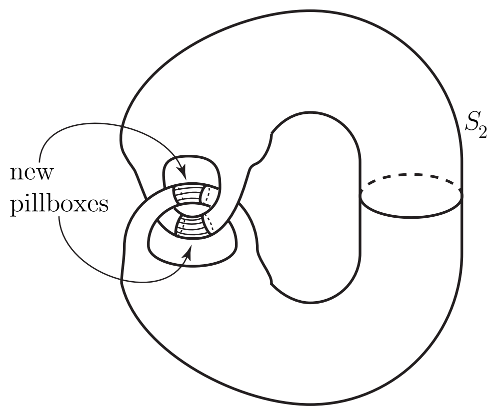

Figure 2.10. The second stage in the construction

Inside T_{11} and T_{12} identify two new pillboxes C_{11} and C_{12} as indicated in Figure 2.10. Inside each of those two pillboxes we can identify two distinguished solid tori. Define the third stage S_3 to be the solid object obtained from S_2 by removing the two new pillboxes and replacing them with the four solid tori just described. This construction is continued inductively, with arrangements to ensure that the diameters of pillboxes at stage $k \geq 2$ is bounded by 2^{-k}. The process results in a nested sequence of compact 3-dimensional solids $S_1 \supset S_2 \supset S_3 \supset \ldots$. Define the *Alexander 3-cell* to be the compact set $B = \cap_{i=1}^{\infty} S_i$ and define the *Alexander horned sphere* to be the boundary of B. Figure 2.11 shows a drawing of Alexander's horned sphere. Color Plates 2–4 display photographs of physical models of the first few stages in the construction.

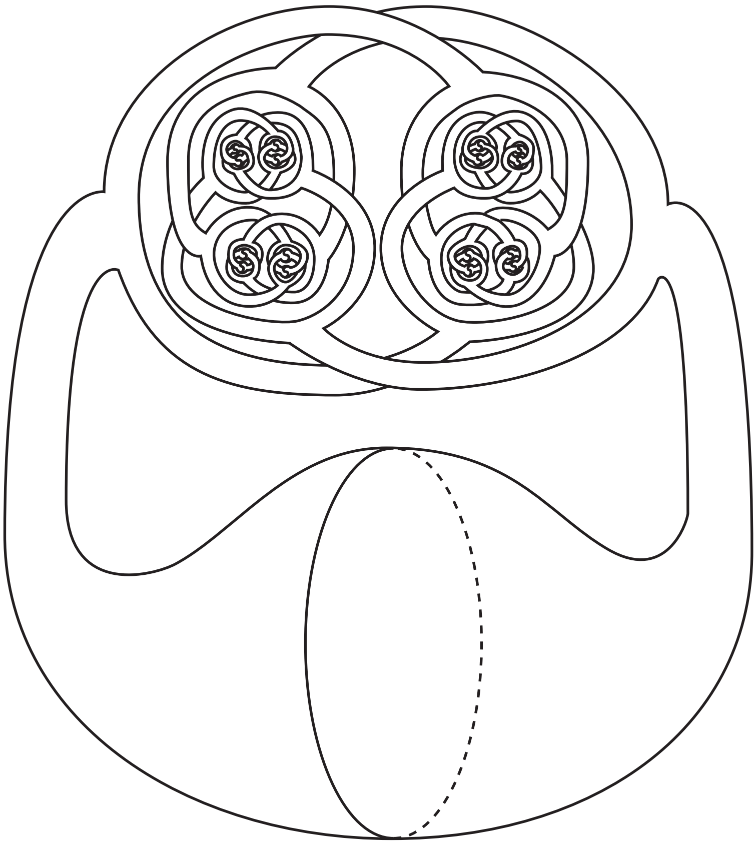

Figure 2.11. The Alexander horned sphere

In order to complete the proof that the horned sphere has the stated properties, we must show two things: first, B is a topological 3-cell and second, $\pi_1(\mathbb{R}^3 \setminus B) \neq \{1\}$. If the loop J shown in Figure 2.9 were inessential in $\mathbb{R}^3 \setminus B$, then compactness of the track of the shrinking homotopy would provide an n such that J is inessential in $\mathbb{R}^3 \setminus S_n$. But induction and Lemma 2.1.9 show that J is essential in $\mathbb{R}^3 \setminus S_n$ for every n. Hence J is essential in $\mathbb{R}^3 \setminus B$.

To see that B is a 3-cell, it helps to think of it as a union rather than an intersection. At the nth stage of the construction we have 2^{n-1} pillboxes, each of which contains two distinguished solid tori. Each of these 2^n solid tori is then divided into a pillbox and a complementary 3-cell (the 3-cell D_1 at the first stage). Let D_n denote the union of the 2^{n-1} complementary

3-cells at the nth stage. Inductively define $B_1 = D_1$ and $B_n = B_{n-1} \cup D_n$. Observe that

$$B = \overline{\cup_{n=1}^{\infty} B_n}.$$

It is relatively simple to use the B_n to construct a homeomorphism from a 3-cell to B. The construction is indicated in Figure 2.12, which shows the domain of the homeomorphism. Map the large region at the bottom to D_1, map the union of the next two regions to D_2, map the union of the next four regions to D_3, etc. Note that $B \setminus \cup_{n=1}^{\infty} B_n$ is a Cantor set. We will call this Cantor set the *Alexander Cantor set*. This completes the construction of the Alexander horned sphere.

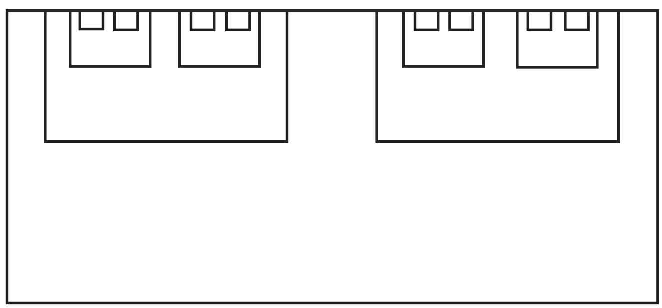

Figure 2.12. Construction of a homeomorphism from a 3-cell to B

Remark. It is interesting to compare the wildness of the two embeddings of S^2 that were constructed in this section. The Antoine sphere and the Alexander horned sphere are alike in that each has one complementary domain whose closure is a 3-cell while the other complementary domain fails to be simply connected. The two embeddings are also alike in that each of them is locally flat except at the points of a Cantor set. There is, however, a significant qualitative difference in the wildness exhibited by the two embeddings. In each case the Cantor set of wild points can be considered either as a subset of the 2-sphere itself or as a subset of $\mathbb{R}^3$. Antoine's necklace is flat when considered as a subset of the Antoine sphere, but it is wild when considered as a subset of $\mathbb{R}^3$. By contrast, the Alexander Cantor set is twice flat in the sense that it is flat both as a subset of the Alexander 2-sphere and as a subset of $\mathbb{R}^3$ (Exercise 2.1.3).

It is clear from the definition that local flatness is an open condition, so the set of points at which an embedding is wild is always a closed set. We will refer to this set as the *wild set* of the embedding and a point in this set is called a *wild point* of the embedding. The wild set of each of the spheres constructed in this section is a Cantor set. Later in the chapter we will construct wild embeddings of the 2-sphere in $\mathbb{R}^3$ whose wild sets are as small as a single point or as large as the entire sphere.

High-dimensional examples are constructed by suspension.

Example 2.1.10. *There exist wild cells and spheres in S^n for all $n \geq 3$.*

Proof. For $n > 3$, iteration of the suspension operator applied to the examples constructed earlier in the section produces examples of nonflat codimension-two and codimension-one spheres in S^n, as well as nonflat cells in codimensions 0, 1, and 2. By Lemma 1.4.1, cells whose complements have nontrivial fundamental groups suspend to cells with the same property and the same codimension. As a result, the existence of wild cells and spheres in all dimensions $n \geq 3$ follows immediately from the 3-dimensional examples. □

The wild cells constructed in Example 2.1.10 have dimensions n, $n-1$, and $n-2$ and are locally flat except at the points of the iterated suspension of a Cantor set. Later in the chapter we will use other methods to construct everywhere wild cells in $\mathbb{R}^n$ of all codimensions.

Historical Notes. Antoine's necklace and the Alexander horned sphere are named for their inventors, L. Antoine and J. W. Alexander, respectively. The discovery of these two examples dates back to the 1920s; see (Antoine, 1921) and (Alexander, 1924b). Alexander pointed out (1924c) that Antoine's construction of a wild Cantor set could also be used to construct a wild 2-sphere, as detailed in this section.

Exercises

2.1.1. Prove Lemma 2.1.3.

2.1.2. Let H be a compact, 2-dimensional ∂-manifold in $\mathbb{R}^2$. Show that for each map $f : H \to T \smallsetminus A_1$ with $f(\partial H) \subset \partial T$ there exists a map $F : H \to \partial T$ with $F|\partial H = f|\partial H$. [Hints: Show that every loop in ∂T is null-homotopic in $\mathbb{R}^3 \smallsetminus (T_i \cup T_{i+1} \cup T_{i+2})$; then show that there exists a map $f' : H \to T \smallsetminus A_1$ such that $f'|\partial H = f|\partial H$ and $f'(H) \cap (E_2 \cup E_4) = \emptyset$.]

2.1.3. The Alexander Cantor set $\mathcal{A}$ is tame in $\mathbb{R}^3$. [Hint: Alter the embedding of $\mathcal{A}$ so linear projection to the axis perpendicular to the plane of the page in Figure 2.11 restricts to an embedding on $\mathcal{A}$.]

2.1.4. Let C be a Cantor set in a connected n-manifold M. Construct an arc α satisfying $C \subset \alpha \subset M$.

2.1.5. Let C be a Cantor set in a connected n-manifold M, $n > 2$. Construct an n-cell B satisfying $C \subset \partial B \subset B \subset M$, with C flat as a subset of ∂B.

2.1.6. Construct a 2-sphere in $\mathbb{R}^3$ such that neither complementary domain is simply connected.

2.1.7. Every compact, totally disconnected subset C of an n-manifold M has a neighborhood $U \supset C$ that can be embedded in $\mathbb{R}^n$.

2.2. Function spaces

Several subsets of the function space $C(X, Y)$—the space of all continuous functions of X to Y—will prove useful. For this discussion, one should assume that Y admits a complete and bounded[1] metric d and that $C(X, Y)$ is endowed with the complete metric ρ defined by

$$\rho(f, g) = \text{lub}\{d(f(x), g(x)) \mid x \in X\}.$$

We will be interested in the following subsets of $C(X, Y)$:

$\text{Surj}(X, Y)$ = the set of all mappings of X onto Y (the surjections);

$\text{Emb}(X, Y)$ = the set of all embeddings of X in Y; and

$\text{Homeo}(X, Y)$ = the set of all homeomorphisms of X onto Y.

In case X and Y both are simplicial complexes, we will use $C_{\text{PL}}(X, Y)$, $\text{Emb}_{\text{PL}}(X, Y)$, $\text{Homeo}_{\text{PL}}(X, Y)$ and $\text{Surj}_{\text{PL}}(X, Y)$ to denote the collection of all PL mappings of the specified type. For the Main Problem to be non-vacuous, in this notation we must have that $\text{Emb}(X, Y)$ is nonempty. Correspondingly, to solve the Taming Problem, we must determine which elements of $\text{Emb}(X, Y)$ are equivalent to elements of $\text{Emb}_{\text{PL}}(X, Y)$, whereas to answer the PL Unknotting Problem, we must decide which elements of $\text{Emb}_{\text{PL}}(X, Y)$ are equivalent.

Lemma 2.2.1. *Let (X, d_X) be a compact metric space, (Y, d_Y) a complete metric space, and $C(X, Y)$ the space of all continuous functions of X to Y with metric ρ, as above. Then* $\text{Surj}(X, Y)$ *is a closed subset of $C(X, Y)$. Moreover,* $\text{Emb}(X, Y)$ *and* $\text{Homeo}(X, Y)$ *are G_δ-subsets of $C(X, Y)$.*

Proof. Showing that the complement of $\text{Surj}(X, Y)$ is open in $C(X, Y)$ is straightforward (even when X is non-metrizable). In order to confirm that $\text{Emb}(X, Y)$ is a G_δ-subset, consider the set of $(1/k)$-mappings

$$A_k = \{f \in C(X, Y) \mid \text{diam}\, f^{-1}f(x) < 1/k \text{ for each } x \in X\}.$$

One can prove that A_k is open in the function space $C(X, Y)$ by producing, for any $f \in A_k$, a corresponding $\eta = \eta(f) > 0$ such that $d_Y(f(x_1), f(x_2)) > \eta$ whenever $x_1, x_2 \in X$ satisfy $d_X(x_1, x_2) \geq 1/k$. Each $g \in C(X, Y)$ with $\rho(g, f) < \eta/2$ belongs to A_k, since then $d_Y(g(x_1), g(x_2)) > 0$ whenever

[1] Recall that if d' is an arbitrary complete metric on Y, then the rule $d(y, y') = \min\{1, d'(y, y')\}$ defines a complete and bounded metric on Y that is equivalent to the original in the sense that they induce the same topology.

$d_X(x_1, x_2) \geq 1/k$. It follows that $\cap_{k=1}^{\infty} A_k = \mathrm{Emb}(X, Y)$ is a G_δ-set in $C(X, Y)$. The G_δ-property also holds for $\mathrm{Homeo}(X, Y)$ because

$$\mathrm{Homeo}(X, Y) = \mathrm{Surj}(X, Y) \cap \mathrm{Emb}(X, Y). \qquad \square$$

The point of Lemma 2.2.1, of course, is that these subsets all admit complete metrics and, therefore, have the Baire property.

One should observe that ordinarily $\mathrm{Homeo}(X, Y)$ fails to be closed in $C(X, Y)$. Consequently, an arbitrary Cauchy sequence $\{h_k\}$ in $\mathrm{Homeo}(X, Y)$ need not converge to a homeomorphism, although it will always converge in $C(X, Y)$ to a surjection. Later we will want to know conditions under which a Cauchy sequence of homeomorphisms *does* converge to a homeomorphism, and, conveniently, one can recover appropriate conditions from the proof of Lemma 2.2.1. Here is an all-important philosophical perspective that should be extracted: when constructing a sequence of homeomorphisms $h_k : X \to Y$ recursively, if for each k one can impose control limiting $\rho(h_{k+1}, h_k)$ that is specified after $h_1, \ldots, h_{k-1}$ and h_k have all been determined, then one can construct the entire sequence $\{h_i\}$ so that it converges to a homeomorphism. The principle is embodied in the next Proposition.

Proposition 2.2.2. *Let (X, d_X) be a compact metric space and (Y, d_Y) a complete metric space. Suppose $\{h_k \mid k = 1, 2, \ldots\}$ is a sequence of embeddings of X in Y and $\{\epsilon_k \mid k = 0, 1, 2, \ldots\}$ is a sequence of positive numbers such that for $k > 0$*

(a) $\epsilon_k < \epsilon_{k-1}/2$;

(b) $d_Y(h_k(x_1), h_k(x_2)) \geq 4\epsilon_k$ *for all* $x_1, x_2 \in X$ *with* $d_X(x_1, x_2) \geq 1/k$, *and*

(c) $\rho(h_{k+1}, h_k) < \epsilon_k$.

Then $\{h_k\}$ converges in $C(X, Y)$ to an embedding $h_\infty : X \to Y$. Moreover, if each h_k is a homeomorphism, then so is h_∞.

Proof. Exercise 2.2.1. $\square$

Remark. Conditions (a) and (c) assure that $\{h_k\}$ forms a Cauchy sequence in $C(X, Y)$. When (b) is added to the mix, the Conditions mimic similar ones appearing in the proof of Lemma 2.2.1 that force all successive h_{k+i} to belong to the open subset A_k of $C(X, Y)$.

Theorem 2.2.3. *If X is a compact metric space of dimension at most k, then $\mathrm{Emb}(X, \mathbb{R}^{2k+1})$ is a dense G_δ-subset of $C(X, \mathbb{R}^{2k+1})$.*

Proof. The fact that $\mathrm{Emb}(X, \mathbb{R}^{2k+1})$ is a G_δ-set follows from Lemma 2.2.1. See (Munkres, 1975, Theorem 7.9.6) for a proof of density. The full theorem may also be found on page 56 of (Hurewicz and Wallman, 1948). $\square$

Theorem 2.2.4. *If K is a finite k-complex and M is a PL m-manifold, $2k < m$, then* $\mathrm{Emb}_{PL}(K, M)$ *is dense in* $C(K, M)$.

Proof. Munkres's argument, which establishes density of $\mathrm{Emb}_{\mathrm{PL}}(K, \mathbb{R}^m)$ in $C(K, \mathbb{R}^m)$, can be applied in one chart at a time to give the result—see (Rourke and Sanderson, 1972, Theorem 5.4). □

Remark. Theorem 2.2.3 is sharp. For $k = 1$ there are famous examples, reproduced in Munkres, of finite 1-complexes that do not embed in $\mathbb{R}^2$. In Chapter 5 we will prove the more general result that the k-skeleton of a $(2k+2)$-simplex cannot be embedded in $\mathbb{R}^{2k}$.

Exercise

2.2.1. Prove Proposition 2.2.2.

2.3. Shrinkable decompositions and the Bing shrinking criterion

Many wild embeddings arise from decompositions: a tame embedding into a manifold is followed by a quotient of the ambient manifold. It becomes important then to have tools available for detecting when the quotient space is a manifold. In this section we develop tools for that purpose.

We begin with a quick review of some basic definitions. A *decomposition* G of a space X is simply a partition of X (ordinarily into closed sets). The *decomposition space* (= quotient space) is the space X/G whose points are the elements of G. There is a natural quotient map $\pi : X \to X/G$ and X/G is assigned the quotient topology. (A subset U of X/G is defined to be open if $\pi^{-1}(U)$ is open in X.) An open set $V \subset X$ is said to be *G-saturated* if it is the union of elements of G; thus, $U \subset X/G$ is open if and only if it is the image of a G-saturated open subset of X.

During the 1950s R. H. Bing introduced and exploited several forms of a remarkable condition now called the *Bing shrinkability criterion* or *Bing shrinking criterion.* It prompted a major change in decomposition theory, shifting the focus from the decomposition space back to the source. The need for a fresh point of view arose from the study of decomposition maps $q : S^3 \to Q$ because, even when it appeared certain that Q had to be homeomorphic to S^3, one then had no effective characterization of S^3 to exploit for establishing the topological equivalence. The shrinkability criterion aimed at realizing Q as the homeomorphic image of the known source space, a realization achieved as the end result of manipulations in the source on the decomposition elements.

In its most general form, the criterion is expressed as follows: a partition G of a space X is *shrinkable* if and only if the following condition is satisfied.

Shrinkability criterion. For each G-saturated open cover $\mathcal{U}$ of X and each arbitrary open cover $\mathcal{V}$ of X there is a homeomorphism h of X onto itself satisfying

(a) for each $x \in X$ there exists $U \in \mathcal{U}$ such that $x, h(x) \in U$, and

(b) for each $g \in G$ there exists $V \in \mathcal{V}$ such that $h(g) \subset V$.

In other words, the homeomorphism h must shrink elements of G to small size, where "small" is determined by $\mathcal{V}$, under an action that is limited by $\mathcal{U}$.

Experience suggests that the decomposition space associated with a shrinkable decomposition is often homeomorphic to the source space S. To guarantee that this is true, additional restrictions, like local compactness or complete metrizability, must be imposed on S. This section explores some relatively coarse aspects of those restrictions. A good starting point is the compact metric case.

Definition. Let ρ denote a complete metric on $C(X, Y)$, where X and Y are compact metric spaces. A surjection $f : X \to Y$ is a *near-homeomorphism* if for each $\epsilon > 0$ there exists $h \in \operatorname{Homeo}(X, Y)$ such that $\rho(h, f) < \epsilon$.

Lemma 2.3.1. *Let X and Y be compact metric spaces. If $f \in \operatorname{Surj}(X, Y)$ and $h \in \operatorname{Homeo}(X, X)$, then $\rho(f, fh) = \rho(f, fh^{-1})$*

Proof. $\rho(f, fh^{-1}) = \rho(fhh^{-1}, fh^{-1}) = \rho(fh, f)$. □

Theorem 2.3.2 (Shrinkability criterion in the compact metric case)**.** *Let X, Y be compact metric spaces and ρ a metric on $C(X, Y)$. Then $f \in \operatorname{Surj}(X, Y)$ is a near-homeomorphism if and only if for each $\epsilon > 0$ there exists $h \in \operatorname{Homeo}(X, X)$ satisfying:*

(a) $\rho(f, fh) < \epsilon$, *and*

(b) $\operatorname{diam} h(f^{-1}(y)) < \epsilon$ *for each $y \in Y$.*

Proof. The forward implication is the easier. Fix a near-homeomorphism f and $\epsilon > 0$. By hypothesis there exists $F \in \operatorname{Homeo}(X, Y)$ with $\rho(F, f) < \epsilon/2$. Uniform continuity of F^{-1} provides $\delta > 0$ such that the image under F^{-1} of each δ-subset of Y has diameter less than ϵ. Again, there exists $F^* \in \operatorname{Homeo}(X, Y)$ with $\rho(F^*, f) < \min\{\epsilon/2, \delta/2\}$. For each $y \in Y$, $F^*(f^{-1}(y))$ lies in the $(\delta/2)$-neighborhood of y, implying that $\operatorname{diam} F^*(f^{-1}(y)) < \delta$. Define $h \in \operatorname{Homeo}(X, X)$ as $F^{-1}F^*$. The choice of δ guarantees that h satisfies condition (b). To see that h satisfies condition (a) as well, note that

$$\begin{aligned}\rho(f, fh) &\leq \rho(f, F^*) + \rho(F^*, fF^{-1}F^*) \\ &< \epsilon/2 + \rho(F(F^{-1}F^*), f(F^{-1}F^*)) \\ &= \epsilon/2 + \rho(F, f) \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

To prove the reverse implication, fix $f \in \mathrm{Surj}(X, Y)$ satisfying shrinkability conditions (a) and (b) and let A denote the closure in $C(X, Y)$ of the subset consisting of all maps fh^{-1}, where $h \in \mathrm{Homeo}(X, X)$. For $n = 1, 2, \ldots$ define

$$A_n = \{\varphi \in A \mid \operatorname{diam} \varphi^{-1}(y) < 1/n \text{ for each } y \in Y\}.$$

The claim is that each A_n is open and dense in A. Openness follows exactly as in the proof of Lemma 2.2.1. To prove denseness, we start with $\varphi \in A$ and $\eta > 0$ and produce $\varphi^* \in A_n$ such that $\rho(\varphi, \varphi^*) < \eta$. To do so, first obtain fh^{-1}, $h \in \mathrm{Homeo}(X, X)$, such that $\rho(\varphi, fh^{-1}) < \eta/2$ and then apply uniform continuity of h and the shrinkability criterion to obtain another $H \in \mathrm{Homeo}(X, X)$ for which $\rho(f, fH) < \eta/2$ and $\operatorname{diam} Hf^{-1}(y)$ is so small that $\operatorname{diam} hHf^{-1}(y) < 1/n$. Clearly the map $\varphi^* = fH^{-1}h^{-1}$ satisfies $\operatorname{diam}(\varphi^*)^{-1}(y) < 1/n$ for each $y \in Y$. Moreover,

$$\begin{aligned}\rho(\varphi, \varphi^*) &= \rho(\varphi, fH^{-1}h^{-1}) \\ &\leq \rho(\varphi, fh^{-1}) + \rho(fh^{-1}, fH^{-1}h^{-1}) \\ &\leq \rho(\varphi, fh^{-1}) + \rho(f, fH^{-1}) \\ &= \rho(\varphi, fh^{-1}) + \rho(f, fH) \qquad \text{(by Lemma 2.3.1)} \\ &< \eta/2 + \eta/2 = \eta.\end{aligned}$$

To conclude the argument, observe that A itself is complete, being a closed subset of the complete metric space $\mathrm{Surj}(X, Y)$. By the Baire Category Theorem, $\cap_n A_n$ is dense in A, and $\cap_n A_n \subset \mathrm{Homeo}(X, Y)$, as before. Thus, $f \in A$ can be approximated by homeomorphisms $F \in \cap_n A_n$. $\square$

Theorem 2.3.3. *Let X be a compact metric space and $f \in \mathrm{Surj}(X, Y)$. Then f is a near-homeomorphism if and only if, for each $\epsilon > 0$, there exists $\mu \in \mathrm{Surj}(X, X)$ such that $\{f^{-1}(y) \mid y \in Y\} = \{\mu^{-1}(x) \mid x \in X\}$ and $\rho(f, f\mu) < \epsilon$.*

Proof. First assume $\mu \in \mathrm{Surj}(X, X)$ satisfies $\{f^{-1}(y) \mid y \in Y\} = \{\mu^{-1}(x) \mid x \in X\}$ and $\rho(f, f\mu) < \epsilon < 1$. Then $F = f\mu^{-1}$ defines a homeomorphism of

X onto Y. Moreover, for each $x \in X$ there exists $x^* \in \mu^{-1}(x)$ and

$$\begin{aligned}\rho(f(x), F(x)) &= \rho(f(x), f\mu^{-1}(x)) \\ &= \rho(f(x), f(x^*)) \\ &= \rho(f\mu(x^*), f(x^*)) \\ &\le \rho(f\mu, f) < \epsilon.\end{aligned}$$

Thus, $\rho(f, F) < \epsilon$ and f is a near-homeomorphism.

Conversely, assume f is a near-homeomorphism. Given ϵ, $0 < \epsilon < 1$, identify $F \in \mathrm{Homeo}(X, Y)$ satisfying $\rho(f, F) < \epsilon$, and define μ as $\mu = F^{-1}f$. Clearly then $\{f^{-1}(y) \mid y \in Y\} = \{\mu^{-1}(x) \mid x \in X\}$, and

$$\rho(f, f\mu) = \rho(f, fF^{-1}f) = \rho(FF^{-1}f, fF^{-1}f) = \rho(F, f) < \epsilon,$$

as required. □

Technical needs make it advantageous to impose further controls on the shrinking process. To that end, given $f \in \mathrm{Surj}(X, Y)$ let N_f denote the *nondegeneracy set* of f, defined by

$$N_f = \left\{x \in X \mid f^{-1}f(x) \neq \{x\}\right\}.$$

Furthermore, given a closed subset C of X missing N_f, say that the induced partition $G_f = \{f^{-1}(y) \mid y \in Y\}$ of X is *shrinkable fixing* C if shrinking homeomorphisms $h : X \to X$ fulfilling the shrinkability criterion can be obtained that keep each point of C fixed, and say that G_f is *strongly shrinkable* if, for every closed set $C \subset X$ with $C \cap N_f = \emptyset$, G_f is shrinkable fixing C.

By restricting the action on C, one can readily adapt the proof given for Theorem 2.3.2 to establish the following, which lends itself to quick application of shrinkability in the locally compact metric case.

Theorem 2.3.4. *Suppose X is a compact metric space, $f \in \mathrm{Surj}(X, Y)$, and C is a closed subset of X with $C \cap N_f = \emptyset$. Then f can be approximated by homeomorphisms agreeing with f on C if and only if G_f is shrinkable fixing C.*

A mapping $f : X \to Y$ is *proper* if, for each compact subset C of Y, $f^{-1}(C)$ is compact. Several key results concerning near-homeomorphisms between compact metric spaces have analogs pertaining to proper mappings between locally compact metric spaces.

Theorem 2.3.5. *Suppose (X, d_X) and (Y, d_Y) are locally compact metric spaces. Then a proper, surjective mapping $f : X \to Y$ can be approximated (in the space of maps $X \to Y$ endowed with the compact-open topology) by homeomorphisms if for each compact subset C of Y and each $\epsilon > 0$ there exists a homeomorphism $h : X \to X$ satisfying*

(a) $d_Y(f(x), fh(x)) < \epsilon$ *for each* $x \in f^{-1}(C) \cup h^{-1}f^{-1}(C)$, *and*
(b) $\operatorname{diam} hf^{-1}(c) < \epsilon$ *for each* $c \in C$.

Proof. Let X^* and Y^* denote the one-point compactifications of X and Y, respectively, and $f^* : X^* \to Y^*$ the obvious extension of f. Properness of f is equivalent to continuity of f^*. Since X and Y are locally compact and second countable, X^* and Y^* are compact metric spaces. The point is that f can be approximated (in the compact-open topology) by homeomorphisms if f^* can be approximated in $C(X^*, Y^*)$ by homeomorphisms preserving the points at infinity, which reduces Theorem 2.3.5 to Theorem 2.3.4. □

Historical Notes. The shrinking criterion is a profound insight of R. H. Bing. It appeared implicitly in (Bing, 1952) and developed over time into a general method; see (Bing, 1957a), for example, or (van Mill, 1989, §6.1).

Exercises

2.3.1. Every proper continuous mapping $f : X \to Y$ between metric spaces is a closed mapping.

2.3.2. Let X, Y be locally compact metric (or even Hausdorff) spaces and $X^* = X \cup \{\infty\}$, $Y^* = Y \cup \{\infty'\}$ their one-point compactifications. Then $f \in C(X, Y)$ is proper if and only if the obvious extension $f^* : X^* \to Y^*$ (where $f^*(\infty) = \infty'$) is continuous.

2.3.3. Let C denote the Cantor set. Show that each $f \in \operatorname{Surj}(C, C)$ is a near-homeomorphism. [Hint: any subset $X \subset C$ that is both open and closed in C is homeomorphic to C.]

2.4. Cellular sets and the Generalized Schönflies Theorem

Next we identify a crucial property possessed by the point preimages of a near-homeomorphism of manifolds. The first application, later in the section, will be the proof of a Generalized Schönflies Theorem. Historically this argument was an early signal of the crucial relationship between topological embeddings and decompositions of manifolds.

Definition. A subset X of $\mathbb{R}^n$ (or, more generally, of an n-manifold) is said to be *cellular* if there exists a sequence $\{B_i\}$ of n-cells in $\mathbb{R}^n$ such that $B_{i+1} \subset \operatorname{Int} B_i$ and $X = \cap B_i$. Alternatively, a compact $X \subset \mathbb{R}^n$ is cellular if each neighborhood U of X contains an n-cell B such that $X \subset \operatorname{Int} B \subset B \subset U$. As yet another possibility, a compact $X \subset \mathbb{R}^n$ is cellular if and only if it has arbitrarily small neighborhoods homeomorphic to $\mathbb{R}^n$.

Cellular sets are compact and connected, but they need not be locally connected. (Consider the $\sin(1/x)$-continuum in $\mathbb{R}^2$, Figure 2.13.)

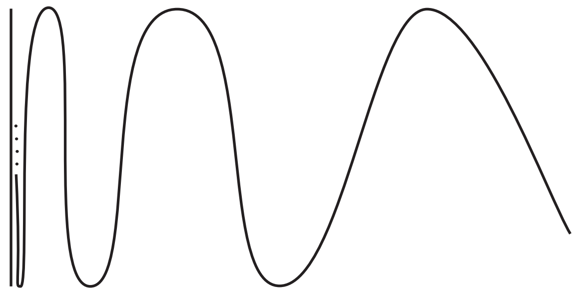

Figure 2.13. The $\sin(1/x)$-continuum

Definition. A map $f : M \to Y$ defined on an n-manifold M is said to be a *cellular map* if $f^{-1}(y)$ is a nonempty cellular set in M for every $y \in Y$.

We use $\mathrm{Cell}(M, Y)$ to denote the set of all cellular maps. Note that

$$\mathrm{Cell}(M, Y) \subset \mathrm{Surj}(M, Y) \subset C(M, Y).$$

Cellular maps defined on manifolds and near-homeomorphisms are closely linked; in fact, we will see that under various special conditions the two kinds of maps are the same. The next theorem asserts that cellularity of point preimages is a necessary condition for a map defined on an n-manifold to be a near-homeomorphism. It is not, however, a sufficient condition in general: the quotient map defining the famous dogbone space (Bing, 1957b) is a counterexample, but that example is too specialized for treatment here.

Proposition 2.4.1. *If $f \in \mathrm{Surj}(X, M^n)$ is a near-homeomorphism, M^n is a compact n-manifold, and $z \in M^n$, then $f^{-1}(z)$ is cellular.*

Proof. Let U be any neighborhood in X of $f^{-1}(z)$, find an n-cell B in M^n satisfying $z \in \mathrm{Int}\, B \subset B \subset M^n \smallsetminus f(X \smallsetminus U)$, and choose $\epsilon > 0$ smaller than both $d(x, M^n \smallsetminus B)$ and $d(B, M^n \smallsetminus f(X \smallsetminus U))$. By hypothesis there exists $F \in \mathrm{Homeo}(X, M^n)$ with $\rho(F, f) < \epsilon/2$. Then $F^{-1}(B)$ is an n-cell in X, and a routine check indicates that $f^{-1}(z) \subset \mathrm{Int}\, F^{-1}(B) \subset F^{-1}(B) \subset U$, so $f^{-1}(z)$ is cellular. $\square$

Corollary 2.4.2. *If $f \in \mathrm{Surj}(M^n, Y)$ is a near-homeomorphism and $y \in Y$, then $f^{-1}(y)$ is cellular.*

A closed subset X of a space M determines a decomposition whose only nondegenerate element is X. We use M/X to denote the associated decomposition space. In this special case cellularity is sufficient to imply that the quotient map is a near-homeomorphism.

Proposition 2.4.3. *If X is a cellular subset of an n-manifold M and Q is the quotient space M/X, then the quotient map $q : M \to Q$ is a near-homeomorphism.*

Proof. Given $\epsilon > 0$, let U denote the $(\epsilon/2)$-neighborhood of $q(X)$ in Q. Apply cellularity of X to obtain an n-cell B such that $X \subset \operatorname{Int} B \subset B \subset q^{-1}(U)$. Equate B with B^n, the standard n-cell; interior to $B = B^n$ construct another round n-cell $B' \supset X$ centered at the origin of $B = B^n$; radially compress B' very near the origin, keeping ∂B pointwise fixed, via a homeomorphism $h^* : B \to B$ such that $\operatorname{diam} h^*(X) < \epsilon$. By Theorem 2.3.2 or 2.3.5 the extension of h^* across $M \smallsetminus B$ via the identity to $h \in \operatorname{Homeo}(M, M)$ shows that q is a near-homeomorphism. $\square$

Definition. An *inverse set* of a map $f : X \to Y$ is a nondegenerate point preimage of f; i.e., an inverse set is a set of the form $f^{-1}(y)$ that contains more than one point.

Corollary 2.4.4. *If U is an open subset of an n-manifold and f is a closed map of $\overline{U}$ onto an n-cell B for which the only inverse set under f is a cellular subset X of U, then $\overline{U}$ is an n-cell.*

The topic of cellularity leads to one of the major themes of this book: the intimate connections between decomposition theory and taming theory. J. W. Cannon probably was the first to stress this theme explicitly, but the connections themselves have been, or should have been, visible from the outset, in the work dating back to the 1950s of R. H. Bing, E. E. Moise, and M. Brown. Brown's important Generalized Schönflies Theorem (1960), one of the first and perhaps the most elegant flatness theorem, displays an aspect of that connection through its dependence on decomposition methods. As noted in §1.1, an $(n-1)$-sphere Σ in S^n is flat if and only if it bounds two n-cells. It is this observation that allows us to make the connection between flatness of $(n-1)$-spheres in S^n and certain decompositions of S^n.

Proposition 2.4.5. *Let Q be an n-cell and let X be a compact subset of* $\operatorname{Int} Q$. *If $f \in C(Q, S^n)$ has X as its only inverse set and $f(\operatorname{Int} Q)$ is open, then X is cellular in Q.*

Proof. Since f is one-to-one on ∂Q, $f(\partial Q)$ is an $(n-1)$-sphere. The inverse set does not touch ∂Q, so the connected set $f(\operatorname{Int} Q)$ must be contained in one of the two complementary domains of $f(\partial Q)$; in particular, f is not onto. Choose a point $z \in S^n \smallsetminus f(Q)$. Then $S^n \smallsetminus \{z\} \cong \mathbb{R}^n$, so $S^n \smallsetminus \{z\}$ has a radial structure centered at the point $f(X)$.

Let U denote an open subset of $\operatorname{Int} Q$ containing X. Then $f(U) = f(\operatorname{Int} Q) \smallsetminus f(Q \smallsetminus U)$ is an open subset of S^n. Use the radial structure of $S^n \smallsetminus \{z\}$ to construct a homeomorphism $\theta : S^n \to S^n$, fixed on some neighborhood V of $f(X)$ and a neighborhood of z, such that $\theta(f(Q)) \subset f(U)$. Define $F : Q \to U$ as the identity on $f^{-1}(V)$ and as $f^{-1}\theta f$ on $Q \smallsetminus X$. Note that F is well defined, continuous, and one-to-one. Thus F is an embedding and $F(Q)$ is an n-cell in U that contains X in its interior. □

Proposition 2.4.6. *If $\psi \in \mathrm{Surj}(S^n, S^n)$ has exactly two inverse sets, then each of them is cellular.*

Proof. Let A and B denote the inverse sets of ψ. We will show that B is cellular. Let Q be an n-cell in S^n containing $A \cup B$ in its interior. Then $\psi(\operatorname{Int} Q)$ is open and contains an open set U for which $\psi(A) \in U$ but $\psi(B) \notin U$. Use the structure of S^n as the union of two n-cells to find $\theta \in \mathrm{Homeo}(S^n, S^n)$ such that $\theta(\psi(Q)) \subset U$ and θ fixes some neighborhood V of $\psi(A)$. Define $f \in C(Q, S^n)$ as the identity on $\psi^{-1}(V)$ and as $\psi^{-1}\theta\psi$ on $Q \smallsetminus A$. Then $f(\operatorname{Int} Q)$ is open and B is the only inverse set of f. By Proposition 2.4.5, B is cellular. □

A similar proof establishes the following generalization.

Proposition 2.4.7. *If $\psi \in \mathrm{Surj}(S^n, S^n)$ has only a finite number of inverse sets, then $f \in \mathrm{Cell}(S^n, S^n)$.*

The next theorem is the main theorem of the section.

Theorem 2.4.8 (Generalized Schönflies)**.** *If h is an embedding of $S^{n-1} \times [-1, 1]$ in S^n, then $h(S^{n-1} \times \{0\})$ is flat. In particular, the closure of each component of $S^n \smallsetminus h(S^{n-1} \times \{0\})$ is an n-cell.*

Proof. Let A denote the closure of the component of $S^n \smallsetminus h(S^{n-1} \times \{1\})$ that does not contain $\Sigma = h(S^{n-1} \times \{0\})$ and B the closure of the component of $S^n \smallsetminus h(S^{n-1} \times \{-1\})$ that does not contain Σ (see Figure 2.14). Furthermore, let D_A (respectively D_B) denote the closure of that component of $S^n \smallsetminus \Sigma$ containing A (respectively B).

Let $q : S^{n-1} \times [-1, 1] \to Q$ denote the quotient mapping to the quotient space obtained from $S^{n-1} \times [-1, 1]$ by identifying the spheres $S^{n-1} \times \{\pm 1\}$ to (separate) points. As Q is the suspension of S^{n-1}, there exists $\lambda \in \mathrm{Homeo}(Q, S^n)$ sending the image of $S^{n-1} \times \{0\}$ to the standard $S^{n-1} \subset S^n$. Extend the map $\lambda q h^{-1}$ from $h(S^{n-1} \times [-1, 1])$ onto S^n to $f \in \mathrm{Surj}(S^n, S^n)$ by defining $f(A) = \lambda q h^{-1}(h(S^{n-1} \times \{1\}))$ and $f(B) = \lambda q h^{-1}(h(S^{n-1} \times \{1\}))$. Each of A and B is cellular (Proposition 2.4.6) and, therefore, D_A and D_B are n-cells by Corollary 2.4.4. □

An $(n-1)$-manifold Σ contained in an n-manifold M is said to be *bicollared* if there exists an embedding $h : \Sigma \times [-1, 1] \to M$ such that

PLATE 1. Tame sphere, Inner Mongolian black granite, 16" diameter, by Helaman Ferguson

Photograph by Claire Ferguson

PLATE 2. Alexander horned wild sphere, bronze, by Helaman Ferguson

Photograph by Claire Ferguson

PLATE 3. Alexander horned wild sphere, patina bronze, 9" diameter, by Helaman Ferguson

Photograph by Claire Ferguson

PLATE 4. Incised torus wild sphere, polished bronze, 9" diameter, by Helaman Ferguson

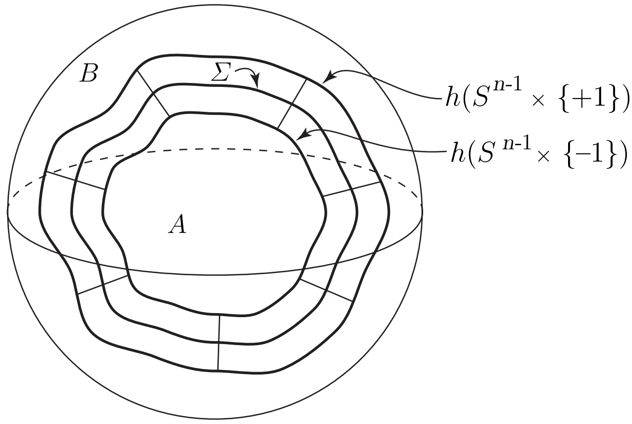

Figure 2.14. Proof of the Generalized Schönflies Theorem

$h(\Sigma \times \{0\}) = \Sigma$. The Generalized Schönflies Theorem can be simply paraphrased using this terminology: every bicollared $(n-1)$-sphere in S^n is flat.

It should be clear from the examples described earlier in the chapter that the bicollar hypothesis is necessary in the Generalized Schönflies Theorem. The complement of any $(n-1)$-sphere embedded in S^n will always have exactly two connected components, but the closure of these complementary domains need not be n-cells. In each of the wild examples constructed earlier, one of their complementary domains was not simply connected. Later in the chapter we will see that the closure of a complementary domain may fail to be an n-cell even if the complementary domain itself is homeomorphic to the interior of an n-cell.

Application of the techniques used in the proof of the Generalized Schönflies Theorem leads to a simple manifold structure theorem.

Proposition 2.4.9. *Any compact n-manifold that can be expressed as the union of two open n-cells is homeomorphic to S^n.*

Proof. Suppose M can be expressed as the union of open sets U and V, each homeomorphic to $\mathbb{R}^n$. Name a homeomorphism $f : V \to \mathbb{R}^n$, and regard the target $\mathbb{R}^n$ as $S^n \smallsetminus \{p\}$. Then f extends to $F \in \mathrm{Surj}(M, S^n)$ by setting $F(M \smallsetminus V) = \{p\}$, and F has $X = M \smallsetminus V$ as its only inverse set. Since X is contained in the interior of some n-cell $Q \subset U$, Proposition 2.4.5 implies that X is cellular. Finally, by Proposition 2.4.3, F is a near-homeomorphism, implying that M is an n-sphere. $\square$

To complete the coverage of the Generalized Schönflies Theorem we show that every locally flat codimension-one sphere is bicollared. It is convenient to work with one-sided collars.

Definition. A subset C of a space X is said to be *collared in* X provided there exists an embedding λ of $C \times [0, 1)$ onto an open subset of X such that $\lambda(c, 0) = c$ for all $c \in C$, and it is said to be *locally collared* if it can be covered by a collection of open sets (relative to C), each of which is collared in X. The image of λ is called a *collar on* C.

Theorem 2.4.10 (Collaring). *The boundary ∂M of a ∂-manifold M is collared in M.*

Proof. Form a new ∂-manifold M' from $M \cup (\partial M \times [-1, 0])$ by identifying each $p \in \partial M$ with $\langle p, 0\rangle \in \partial M \times [-1, 0]$. It has the advantage that $\partial M'$, which corresponds to $\partial M \times \{-1\}$, is clearly collared in M'.

We treat only compact ∂M. Cover ∂M by finitely many open subsets $\{W_i\}$, each collared in M, and let V_i denote a collar on W_i in M. Inductively build collars on $\cup_{i=1}^{k} W_i$; the general case quickly reduces to the case $k = 2$. Find $C_i \subset W_i$, closed in $W_1 \cup W_2$, such that $C_1 \cup C_2 = W_1 \cup W_2$. Name a continuous $\gamma_1 : W_1 \cup W_2 \to [-1, 0]$ with $\gamma_1(C_1) = -1$ and $\gamma_1(W_2 \smallsetminus W_1) = 0$. After parametrizing $V_1 \cup (W_1 \times [-1, 0])$ as $W_1 \times [-1, 1)$ in the natural way, define an embedding $\psi_1 : M \to M'$ by declaring $\psi_1 \mid M \smallsetminus V_1 = \text{incl}$, next specifying (for $w \in W_1$)

$$\langle w, 0\rangle \to \langle w, \gamma_1(w)\rangle \text{ and } \langle w, t\rangle \to \langle w, t\rangle \text{ for } t \in [1/2, 1),$$

and then extending linearly to prescribe correspondences between the various intervals $\{w\} \times [0, 1/2]$ and $\{w\} \times [\gamma_1(w), 1/2]$. A similar construction with the constant function $\gamma_2 : W_1 \cup W_2 \to \{-1\}$ gives an embedding $\psi_2 : \text{image}\, \psi_1 \to M'$ for which the composite $\psi_2 \cdot \psi_1$ sends M homeomorphically onto $M \cup (W_1 \cup W_2) \times [-1, 0]$. The inverse of $\psi_2 \cdot \psi_1$ exposes a collar on $W_1 \cup W_2$. □

A related argument shows that a closed subset C of a metric space X is collared in X if and only if C is locally collared in X.

Corollary 2.4.11. *An $(n-1)$-sphere Σ in S^n is bicollared, and hence flat, if and only if the closure of each component of $S^n \smallsetminus \Sigma$ is a ∂-manifold.*

Corollary 2.4.12. *Every compact ∂-manifold in S^n bounded by an $(n-1)$-sphere is an n-cell.*

The following corollary of Theorems 2.4.8 and 2.4.10 is often called the Generalized Schönflies Theorem.

Corollary 2.4.13. *Every locally flat $(n-1)$-sphere in S^n is flat.*

Corollary 2.4.14. *The boundary of every G-orientable ∂-manifold M is G-orientable.*

Proof. Now we know M contains a copy of $\partial M \times \mathbb{R}$. Corollary 0.3.6 assures that the latter is G-orientable, and Corollary 0.3.8 does the same for ∂M. □

Corollary 2.4.15. *Let M be a ∂-manifold and $\phi_t : \partial M \to \partial M$ an isotopy such that $\phi_0 = \mathrm{Id}_{\partial M}$. Then, for each neighborhood U of ∂M, ϕ_t extends to an ambient isotopy Φ_t of M supported in U such that $\Phi_0 = \mathrm{Id}_M$.*

Proof. Produce a collar $\lambda : \partial M \times [0, 1] \to M$ on ∂M with image in U, where $\lambda_0 = \mathrm{incl}_{\partial M}$. Then define $\Phi_1 : M \to M$ as the identity on $M \smallsetminus \lambda(\partial M \times [0, 1])$ and as $\lambda(\phi_{1-t}(x), t)$ for $\lambda(x, t) \in \lambda(\partial M \times [0, 1])$. Specification of an isotopy Φ_t extending ϕ_t and running from $\Phi_0 = \mathrm{Id}_M$ to Φ_1 is left to the reader. □

Historical Notes. The generalized Schönflies theorem was first proved by M. Brown (1960), who developed the elegant method of shrinking cellular sets used in the proof. Earlier B. Mazur (1959) had proved the theorem with an additional technical hypothesis, and eventually M. Morse (1960) showed how to remove that condition to provide an alternative proof of the theorem.

Cellularity, as an important concept, not the term itself, appeared in the 1920s with the analysis by R. L. Moore (1925) of cellular decompositions of 2-manifolds.

Collaring Theorem 2.4.10 is also due to Brown (1960). The argument here is taken from R. Connelly (1971), who conceived the simplification of appending an abstract collar.

Exercises

2.4.1. The three definitions of cellular set given at the beginning of the section are equivalent.

2.4.2. A compact set X in S^n is cellular if and only if $S^n \smallsetminus X \cong \mathbb{R}^n$.

2.4.3. Every arc $\alpha \subset \mathbb{R}^n$ that is locally polyhedral modulo one point is cellular.

2.4.4. (A one-sided Schönflies theorem.) Let $\Sigma \subset S^n$ be an embedded $(n-1)$-sphere and let U be one of its complementary domains. If $\overline{U}$ is a ∂-manifold, then $\overline{U}$ is an n-cell.

2.4.5. Let Σ_1 and Σ_2 be two disjoint $(n-1)$-spheres in S^n, let U_1 be the complementary domain of Σ_1 that contains Σ_2, and let U_2 be the complementary domain of Σ_2 that contains Σ_1. Define $A = U_1 \cap U_2$. Prove that $\overline{A} \smallsetminus \Sigma_i \cong S^{n-1} \times [0, 1)$.[2]

[2] In a later chapter we will make use of the annulus theorem, which asserts that $\overline{A} \cong S^{n-1} \times [0, 1]$.

2.5. The Klee trick

A simple, elegant application of the Tietze Extension Theorem leads to an unknotting result for embeddings into hyperplanes.

Theorem 2.5.1. *Suppose $\lambda : C \to \mathbb{R}^n$ and $\lambda' : C \to \mathbb{R}^m$ are embeddings of a compact metric space C. Then the associated embeddings $e, e' : C \to \mathbb{R}^n \times \mathbb{R}^m$, where $e(c) = \langle \lambda(c), 0\rangle$ and $e'(c) = \langle 0, \lambda'(c)\rangle$, are each equivalent to the diagonal embedding $d = \lambda \times \lambda' : C \to \mathbb{R}^n \times \mathbb{R}^m$.*

Proof. It suffices to show that e is equivalent to d. Since $\mathbb{R}^m$ has the universal extension property (Munkres, 1975, page 216), the map $\lambda'\lambda^{-1} : \lambda(C) \to \mathbb{R}^m$ can be extended to a map $\psi : \mathbb{R}^n \to \mathbb{R}^m$. Define $\Psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ as $\Psi(\langle x, y\rangle) = \langle x, y + \psi(x)\rangle$. Clearly Ψ is continuous; indeed, it is a homeomorphism, for the map $\langle x, y\rangle \to \langle x, y - \psi(x)\rangle$ acts as its inverse. Furthermore,

$$\Psi e(c) = \Psi(\langle \lambda(c), 0\rangle) = \langle \lambda(c), \psi(\lambda(c))\rangle = \langle \lambda(c), \lambda'(c)\rangle = d(c),$$

as required. $\square$

Corollary 2.5.2. *Any two embeddings λ, λ' of a compact metric space into $\mathbb{R}^n$ are equivalent when considered as embeddings to their images in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.*

Corollary 2.5.3. *Every arc in $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ is flat in $\mathbb{R}^{n+1}$.*

Another corollary could be listed—that every k-cell in $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+k}$ is flat in $\mathbb{R}^{n+k}$—but for $k > 1$ this is far from best possible. In later chapters we shall learn that all k-cells in $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ are flat in $\mathbb{R}^{n+1}$.

Historical Notes. Theorem 2.5.1 is due to V. Klee (1955).

Exercises

2.5.1. Show that for every arc $A \subset \mathbb{R}^n$, $A \times [-1, 1]$ is a cellular subset of $\mathbb{R}^n \times \mathbb{R}^1$.

2.5.2. Suppose $X \times \mathbb{R}^1$ is a manifold. Show that each arc of the form $\{x\} \times [-1, 1]$ is cellular in $X \times \mathbb{R}^1$.

2.5.3. Let X be a compact subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}^m$ be continuous. Show that $X \times \{0\}$ and the graph of f are equivalently embedded in $\mathbb{R}^n \times \mathbb{R}^m$.

2.5.4. Any arc $\alpha \subset \mathbb{R}^n$, $n > 3$, that is a countable union of points and segments is flat. [Hint: Find a line L such that any line parallel to L intersects α in at most one point.]

2.6. The product of $\mathbb{R}^1$ with an arc decomposition

Next we turn to the construction of everywhere wild embeddings in all dimensions and all codimensions. The examples of wild embeddings constructed in §1.4 all have relatively low codimension, and a new technique is required to produce examples in codimensions greater than two. The idea is this: start with an arc in S^n, suspend it to produce a 2-cell in S^{n+1}, and then shrink out the arcs in the levels of the suspension to produce a new arc in S^{n+1}. The necessary shrinking theorem is proved in this section and the examples will be constructed in the following section.

Let A be an arc in $\mathbb{R}^n$ and $q : \mathbb{R}^n \to \mathbb{R}^n/A$ the quotient map. Urysohn's Metrization Theorem assures that $\mathbb{R}^n/A$ is a locally compact metric space.

Theorem 2.6.1. *If A is an arc in $\mathbb{R}^n$, then $(\mathbb{R}^n/A) \times \mathbb{R}^1 \approx \mathbb{R}^{n+1}$.*

Note that $(\mathbb{R}^n/A) \times \mathbb{R}^1$ is the same as $(\mathbb{R}^n \times \mathbb{R}^1)/\{A \times \{t\} \mid t \in \mathbb{R}\}$. We intend to prove that the decomposition of $\mathbb{R}^n \times \mathbb{R}^1$ into points and the arcs $A \times \{t\}$, $t \in \mathbb{R}^1$, is shrinkable. To that end, name a homeomorphism $\alpha : [0, 1] \to A$, and fix $\epsilon > 0$. Partition $[0, 1]$ by points $\{t_i\}$ with $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$ such that $\operatorname{diam} \alpha([t_{i-1}, t_{i+3}]) < \epsilon$ for $i \in \{1, 2, \dots, m-2\}$. Expand each $\alpha([t_{i-1}, t_i])$ slightly to an open subset U_i of $\mathbb{R}^n$, where

$$U_i \cap U_j \neq \emptyset \text{ if and only if } |i - j| \leq 1, \text{ and}$$
$$\operatorname{diam}(U_i \cup U_{i+1} \cup U_{i+2} \cup U_{i+3}) < \epsilon \quad (i = 1, 2, \dots, m-2).$$

These U_i's will supply motion controls on $h \in \operatorname{Homeo}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ for the $\mathbb{R}^n$ factor; to maintain control in the $\mathbb{R}^1$ direction, we identify some intervals and related sets: for $i \in \{1, 2, \dots, m-1\}$ let $J_i = [i, 2m - i]$ and then let $L_i = J_i \smallsetminus \operatorname{Int} J_{i+1}$ $(i < m - 1)$.

According to Corollary 2.5.3, each level arc $A \times \{t\}$ is flat, so any one of them can be shrunk to small size. To support our aim of shrinking all level arcs simultaneously, Lemma 2.6.2 shows how to combine a vertical compression with the pinching of one level arc to achieve shrinking of the product of J_{i+1} and a subarc of A. This basic move is applied finitely often in Lemma 2.6.3 to achieve a partial shrinking of certain blocks, and these block moves, carefully arranged, achieve the desired shrinking of the entire family of arcs.

Lemma 2.6.2. *Let V_i be a neighborhood of $\alpha([0, t_i])$ in $\mathbb{R}^n$. Then there exists $h_i \in \operatorname{Homeo}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ satisfying:*

(1) $h_i \mid \mathbb{R}^{n+1} \smallsetminus (V_i \times J_i) = \operatorname{Id}$,
(2) $h_i \mid \alpha([t_i, 1]) \times \mathbb{R}^1 = \operatorname{Id}$, *and*
(3) $h_i(\alpha([0, t_i]) \times J_{i+1}) \subset U_{i+1} \times J_i$.

Proof. By Corollary 2.5.3, $A \times \{i+1\}$ is flat. One can shrink the subarc $\alpha([0,t_i]) \times \{i+1\}$ near the point $\alpha(t_i) \times \{i+1\}$ via $\mu \in \mathrm{Homeo}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ such that

$$\begin{aligned}
&\mu \mid \mathbb{R}^{n+1} \smallsetminus (V_i \times J_i) = \mathrm{Id},\\
&\mu \mid \alpha([t_i,1]) \times \mathbb{R}^1 = \mathrm{Id}, \text{ and}\\
&\mu(\alpha([0,t_i]) \times \{i+1\}) \subset U_{i+1} \times J_i.
\end{aligned}$$

It follows that $\mu^{-1}(U_{i+1} \times \mathrm{Int}\, J_i) \supset (\alpha([0,t_i]) \times \{i+1\}) \cup (\alpha(t_i) \times J_{i+1})$, so certainly there exists $\delta > 0$ such that

$$\mu^{-1}(U_{i+1} \times \mathrm{Int}\, J_i) \supset \alpha([t_i - \delta, t_i]) \times J_{i+1}).$$

Now one can produce $\upsilon \in \mathrm{Homeo}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$, which compresses points of $\alpha([0,t_i]) \times J_{i+1}$ into $\mu^{-1}(U_{i+1} \times \mathrm{Int}\, J_i)$ and changes only the $\mathbb{R}^1$ coordinates, subject to the restrictions

$$\begin{aligned}
&\upsilon \mid \mathbb{R}^{n+1} \smallsetminus (V_i \times J_i) = \mathrm{Id},\\
&\upsilon \mid \alpha([t_i,1]) \times \mathbb{R}^1 = \mathrm{Id}, \text{ and}\\
&\upsilon(\alpha([0,t_i]) \times J_{i+1}) \subset \mu^{-1}(U_{i+1} \times \mathrm{Int}\, J_i).
\end{aligned}$$

To produce υ more explicitly, name $d \in (0,1)$ for which

$$\mu^{-1}(U_{i+1} \times \mathrm{Int}\, J_i) \supset \alpha([0,t_i]) \times [i+1, i+1+d];$$

use Urysohn's Lemma to define a map $s : \mathbb{R}^n \to [i+1+d, 2m-i-1]$ sending $(\mathbb{R}^n \smallsetminus V_i) \cup \alpha([t_i,1])$ to $\{2m-i-1\}$ while sending $\alpha([0, t_i \smallsetminus \delta])$ to $\{i+1+d\}$. Finally, define $\upsilon \in \mathrm{Homeo}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ as the identity above $R^n \times \{2m-i\}$ and below $\mathbb{R}^n \times \{i+1\}$, with $\upsilon(\langle p, 2m-i-1\rangle) = \langle p, s(p)\rangle$ for each $p \in \mathbb{R}^n$, and with υ acting as the obvious linear homeomorphism on all vertical intervals $\{p\} \times J_{i+1}$ and $\{p\} \times [2m-i-1, 2m-i]$. The effect of υ is illustrated in Figure 2.15. Note that υ is the identity in a neighborhood of the shaded region.

Now simply define h_i as $\mu\upsilon$. Then

$$h_i(\alpha([0,t_i]) \times J_{i+1}) = \mu\upsilon(\alpha([0,t_i]) \times J_{i+1}) \subset \mu\mu^{-1}(U_{i+1} \times J_i) = U_{i+1} \times J_i,$$

as desired. The other requirements of Lemma 2.6.2 are easily confirmed. □

Lemma 2.6.3. *There exists* $\lambda \in \mathrm{Homeo}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ *satisfying:*

(1) $\lambda \mid \mathbb{R}^{n+1} \smallsetminus \cup_{i=1}^{m-2}(U_i \times J_i) = \mathrm{Id}$,

(2) $\lambda(\alpha([0,t_{i+1}]) \times L_i) \subset (U_i \cup U_{i+1}) \times J_{i-1}$ *for* $i \in \{1, 2, \ldots, m-2\}$,

(3) $\lambda(\alpha([0,t_m]) \times J_{m-1}) \subset (U_{m-1} \cup U_m) \times J_{m-2} \subset (U_{m-1} \cup U_m) \times J_1$.

Proof. Here λ will arise as a composition $h_1 h_2 \cdots h_{m-2}$ of homeomorphisms from Lemma 2.6.2. To get started, obtain h_1 from 2.6.2 for the neighborhood $V_1 = U_1$ of $\alpha([0,t_1])$.

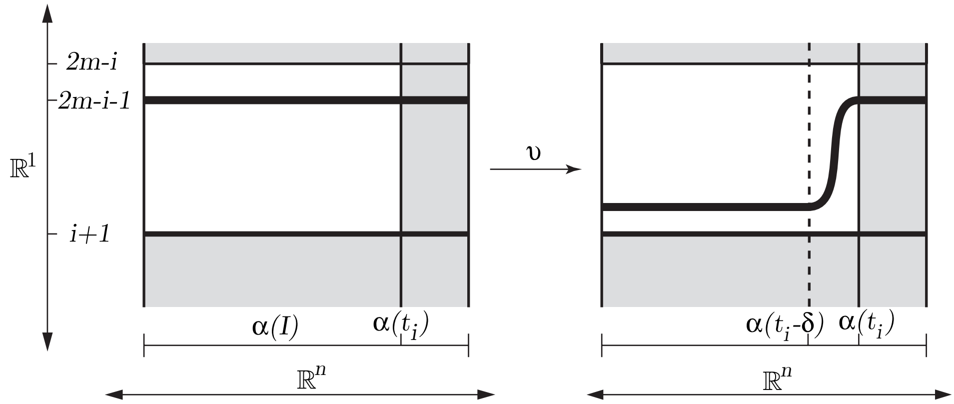

Figure 2.15. The homeomorphism υ

Since h_1 acts as the identity on $\alpha([t_1, t_2]) \times \mathbb{R}^1$ and carries $\alpha([0, t_1]) \times J_2$ into $U_2 \times J_1$, there exists a neighborhood $V_2 \subset U_1 \cup U_2$ of $\alpha([0, t_2])$ such that $h_1(V_2 \times J_2) \subset U_2 \times J_1$. Apply Lemma 2.6.2 again with this neighborhood V_2 to obtain h_2.

The iterative step repeats the pattern of the second step. After h_{i-1} has been obtained, subject to the conditions

$$h_{i-1} \mid \alpha([t_{i-1}, 1]) \times J_{i-1} = \mathrm{Id} \text{ and}$$
$$h_{i-1}(\alpha([0, t_{i-1}]) \times J_i) \subset U_i \times J_{i-1},$$

determine a neighborhood V_i of $\alpha([0, t_i])$ in $U_1 \cup \cdots \cup U_i$ such that $h_{i-1}(V_i \times J_i) \subset U_i \times J_{i-1}$ and then apply Lemma 2.6.2 with this neighborhood V_i to obtain h_i.

The composition $\lambda = h_1 h_2 \cdots h_{m-2}$ is shown in Figure 2.16. In a neighborhood of the shaded region, λ is the identity.

It should be obvious from the choices of $V_i \subset U_1 \cup \cdots \cup U_i$ and conclusion (1) of Lemma 2.6.2 that $\lambda = h_1 h_2 \ldots h_{m-2}$ satisfies conclusion (1) above. In analyzing conclusions (2) and (3), it is useful to keep in mind that $U_1 \cup \cdots \cup U_i$ and $U_{i+2} \cup \cdots \cup U_{m+1}$ are disjoint. Due to the choices of V_i, conclusion (1) of Lemma 2.6.2 then yields, for $j \geq i$,

$$(*) \qquad \begin{aligned} &h_i \mid (U_{j+2} \cup \cdots \cup U_{m+1}) \times \mathbb{R}^1 = \mathrm{Id} \text{ and} \\ &h_i \mid \mathbb{R}^n \times (\mathbb{R}^1 \setminus J_j) = \mathrm{Id}. \end{aligned}$$

Since $L_i \subset \mathbb{R}^1 \setminus \mathrm{Int}\, J_{i+1}$, the latter implies

$$(**) \qquad h_j \mid \mathbb{R}^n \times L_i = \mathrm{Id} \text{ whenever } i < j.$$

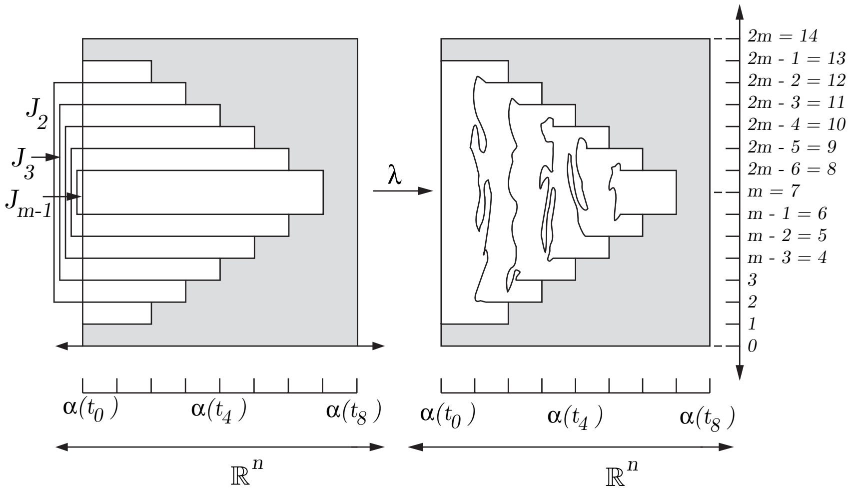

Figure 2.16

To see why conclusion (3) holds, note that

$$\begin{aligned}\lambda(\alpha([0,t_{m-2}])\times J_{m-1}) &= h_1h_2\cdots h_{m-2}(\alpha([0,t_{m-2}])\times J_{m-1})\\ &\subset h_1h_2\cdots h_{m-3}(U_{m-1}\times J_{m-2})\\ &\subset U_{m-1}\times J_{m-2}\end{aligned}$$

by (3) of Lemma 2.6.2 and (*). In addition, by conclusion (2) of the Lemma,

$$\lambda(\alpha([t_{m-2},t_m])\times J_{m-1}) = \alpha([t_{m-2},t_m])\times J_{m-1}\subset (U_{m-1}\cup U_m)\times J_{m-1},$$

and these two inclusions quickly combine to yield (3).

To verify conclusion (2), first observe that $h_i(\alpha([0,t_i])\times J_i)\subset h_i(V_i\times J_i) = V_i\times J_i$ by conclusion (1) of Lemma 2.6.2. Then

$$\begin{aligned}\lambda(\alpha([0,t_{i+1}])\times L_i) &= h_1h_2\cdots h_{m-2}(\alpha([0,t_{i+1}])\times L_i)\\ &= h_1h_2\cdots h_i(\alpha([0,t_{i+1}])\times L_i) \qquad \text{by } (^{**})\\ &\subset h_1h_2\cdots h_i((\alpha([0,t_i])\times L_i)\cup(\alpha([t_i,t_{i+1}])\times J_i))\\ &\subset h_1h_2\cdots h_i(\alpha([0,t_i])\times J_i)\cup(U_{i+1}\times J_i)\\ &\qquad\qquad \text{by (2) of Lemma 2.6.2}\\ &\subset h_1h_2\cdots h_{i-1}(V_i\times J_i)\cup(U_{i+1}\times J_i) \qquad \text{as above}\\ &\subset h_1h_2\cdots h_{i-2}(U_i\times J_{i-1})\cup(U_{i+1}\times J_i) \text{ by choice of } V_i\\ &= (U_i\times J_{i-1})\cup(U_{i+1}\times J_i) \qquad \text{by } (^*)\\ &\subset (U_i\cup U_{i+1})\times J_{i-1}.\end{aligned}$$

Why the conclusion also holds for $i=1$ should be evident to anyone who understands the preceding lines. □

Proof of Theorem 2.6.1. Let $q' : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1 \to (\mathbb{R}^n/A) \times \mathbb{R}^1$ be the map $q \times \mathrm{Id}$. Our intention is to show that q' is a near homeomorphism, which will follow from Theorem 2.3.5 almost instantly, once we construct $h \in \mathrm{Homeo}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ satisfying

$$\begin{aligned} &h \mid (\mathbb{R}^n \smallsetminus N(A;\epsilon)) \times \mathbb{R}^1 = \mathrm{Id}, \\ &h(\mathbb{R}^n \times t) \subset \mathbb{R}^n \times [t-\epsilon, t+\epsilon], \text{ and} \\ &\operatorname{diam} h(A \times t) < 3\epsilon \text{ for each } t \in \mathbb{R}^1. \end{aligned}$$

This will be accomplished by exploiting the structures named for Lemmas 2.6.2 and 2.6.3, carefully pieced together.

Formally, let k range over the integers and set

$$\begin{aligned} D_k &= \cup_{i=1}^{m-2} U_i \times [2mk+i, 2mk+2m-i] \text{ and} \\ D'_k &= \cup_{i=1}^{m-2} U_{m+2-i} \times [2mk+m+i, 2mk+3m-i]. \end{aligned}$$

In addition, let $D = \cup_k D_k$ and $D' = \cup_k D'_k$; D and D' are mirror images of each other, and act as supports for homeomorphisms obtained from Lemma 2.6.3. A key feature is $D \cap D' = \emptyset$. For details in a particular instance, consider $\langle x,t\rangle \in D_0$, where $t \le m$. Choose the least integer i such that $x \in U_i$; then $t \ge i$, by definition of D_0. The only possible D'_k that might contain $\langle x,t\rangle$ is D'_{-1}. If that were the case, note that $x \in U_{m+2-j}$ for $j = m+2-i$, so $x \in U_{m+2-j}$ can hold only for $j \in \{m+2-i, m+1-i\}$. In either situation, the definition of D'_{-1} forces

$$t \le m - (m+1-i) = i-1 < i,$$

a contradiction.

For $k \in Z$ and $i \in \{1, 2, \ldots, m-1\}$, define

$$\begin{aligned} J'_i &= \cup_k [2mk+i, 2mk+2m-i] \\ P'_i &= \cup_k [2mk+m+i, 2mk+3m-i] \end{aligned}$$

and for $i < m-1$ let $L'_i = J'_i \smallsetminus \operatorname{Int} J'_{i+1}$ and $Q'_i = P'_i \smallsetminus \operatorname{Int} P'_{i+1}$. According to Lemma 2.6.3, there exist a homeomorphism λ_R for the J'_i and L'_i and another homeomorphism λ_L for the P'_i and Q'_i, each involving translates of the Lemma 2.6.3 homeomorphism λ to the appropriate levels, satisfying

(1) $\lambda_R \mid \mathbb{R}^{n+1} \smallsetminus D = \mathrm{Id}$ and $\lambda_L \mid \mathbb{R}^{n+1} \smallsetminus D' = \mathrm{Id}$;

(2) $\lambda_R(\alpha([0, t_{i+1}]) \times L'_i) \subset (U_i \cup U_{i+1}) \times J'_i$ and $\lambda_L(\alpha([t_{m-i}, 1]) \times Q'_i) \subset (U_{m-i+1} \cup U_{m-i+2}) \times P'_i$;

(3) $\lambda_R(\alpha([0, t_m]) \times J'_{m-1}) \subset (U_{m-1} \cup U_m) \times J'_1$ and $\lambda_L(\alpha([t_1, 1]) \times P'_{m-1}) \subset (U_2 \cup U_3) \times P'_1$.

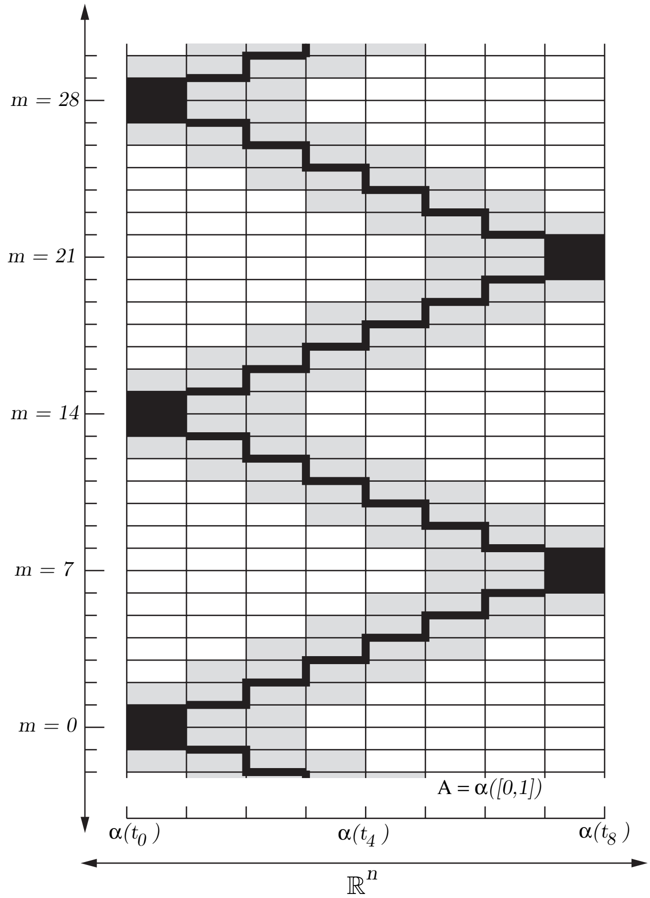

Figure 2.17

Now define h as $\lambda_R\lambda_L$. The resultant shrinking depends on this explicit juxtaposition of left and right stacks. For example, if $t \in \operatorname{Int} L_i'$ then $t \in Q_j'$, where $j = m - 1 - i$. Thus, by (2),

$$\begin{aligned} h(A \times t) &\subset \lambda_R\lambda_L(\alpha([0, t_{i+1}]) \times L_i') \cup \lambda_R\lambda_L(\alpha([t_{i+1}, 1]) \times Q_j') \\ &\subset \lambda_R(\alpha([0, t_{i+1}]) \times L_i') \cup \lambda_R((U_{i+3} \cup U_{m-i+3}) \times P_i') \\ &\subset [(U_i \cup U_{i+1}) \times \mathbb{R}^1] \cup [(U_{i+2} \cup U_{i+3}) \times \mathbb{R}^1]. \end{aligned}$$

Of course, neither λ_R nor λ_L moves points vertically more than $2m$, and due to the disjointness of D and D', this gives both

$$h(A \times t) \subset (U_i \cup U_{i+1} \cup U_{i+2} \cup U_{i+3}) \times [t - 2m, t + 2m] \text{ and}$$
$$h(\mathbb{R}^n \times t) \subset \mathbb{R}^n \times [t - 2m, t + 2m].$$

Recall that initially the U_i's were chosen so that the diameters of any four consecutive ones were small. To complete this proof, rescale the $\mathbb{R}^1$-coordinate with $2m < \epsilon$. $\square$

Corollary 2.6.4. *For each arc A in S^n, the suspension of the quotient space S^n/A is homeomorphic to S^{n+1}.*

Historical Notes. Theorem 2.6.1 is due to J. J. Andrews and M. L. Curtis (1962). The dogbone space constructed by R. H. Bing (1959a) was the first example of a non-manifold decomposition space X such that $X \times \mathbb{R}^1$ is a manifold.

2.7. Everywhere wild cells and spheres

Straightforward application of the product arc-shrinking theorem from §2.6 leads to embeddings that are wild at every point. An embedding of a manifold or ∂-manifold that is not locally flat at any point is called *everywhere wild.*

Example 2.7.1. *For each $n \geq 3$ and $0 < k < n$, S^n contains an everywhere wild k-cell.*

Lemma 2.7.2. *For each $n \geq 3$, S^n contains a wild arc α for which $S^n \smallsetminus \alpha$ fails to be simply connected.*

Proof. Such an arc in S^3 was described in Example 2.1.8. Given an arc A in S^{n-1}, $n > 3$, with nonsimply connected complement, Corollary 2.6.4 allows us to identify S^n with the suspension of S^{n-1}/A. Let α be the arc in S^n that corresponds to the suspension of the special point in the quotient space S^{n-1}/A. Then $S^n \smallsetminus \alpha$ is topologically equivalent to $(S^{n-1} \smallsetminus A) \times (-1, 1)$, which is not simply connected. $\square$

The arcs α provided in the preceding lemma are everywhere wild, starting in dimension four. In order to prove this, we need pinpoint information about the way in which the complement fails to be simply connected: we need to know that the complement of α contains loops that are very close to α but essential in the complement, and the following condition—known as the "cellularity criterion" because it implies cellularity for certain subsets of manifolds, a result to be proved in the next chapter—paves the way. The cellularity criterion is a global version of the 1-LCC condition.

Definition. A compact set X in an m-manifold M is said to satisfy the *cellularity criterion* if for every open neighborhood U of X in M there exists an open set V such that $X \subset V \subset U$ and every map $\partial I^2 \to V \smallsetminus X$ extends to a map $I^2 \to U \smallsetminus X$.

Lemma 2.7.3. *The arc α in Lemma 2.7.2 fails to satisfy the cellularity criterion.*

Proof. Start with $n = 3$; let A be the wild arc in Example 2.1.8, which contains a copy of Antoine's necklace. The simple closed curve J shown in Figure 2.6 is essential in the complement of A and there are related curves just like J that link later stages of the construction. Hence every neighborhood V of A contains a simple closed curve that is essential in $S^3 \smallsetminus A$.

We now show that if A is an arc in S^{n-1} such that $S^{n-1} \smallsetminus A$ is non-simply connected, then the arc α constructed from it (as in the proof of Lemma 2.7.2) fails to satisfy the cellularity criterion. Suppose there exists a neighborhood of α such that every loop in $V \smallsetminus \alpha$ is null-homotopic in $S^n \smallsetminus \alpha$. A loop in $S^n \smallsetminus \alpha$ can be pushed up the product structure on $S^n \smallsetminus \alpha \cong (S^{n-1} \smallsetminus A) \times (-1, 1)$ into V and so the existence of V would mean that $S^n \smallsetminus \alpha$ is simply connected. This contradicts the conclusion of Lemma 2.7.2. $\square$

The proof of the following lemma is left as an exercise.

Lemma 2.7.4. *A compact set $X \subset S^n$ satisfies the cellularity criterion in S^n if and only if the arc corresponding to the suspension of X in* $\mathrm{Susp}(S^n/X)$ *is 1-LCC at each interior point.*

Proof of Example 2.7.1. Consider, first, the case $k = 1$ and $n \geq 4$. By Lemma 2.7.4, the arcs constructed in Lemmas 2.7.2 and 2.7.3 fail to be 1-LCC at all interior points. Proposition 1.3.1 and Exercise 2.7.1 imply that these arcs are everywhere wild.

Now assume that $k > 1$ and $n - k > 2$. By the previous paragraph, there is an arc in S^{n-k+1} that fails to be 1-LCC at each interior point. The $(k-1)$-fold suspension is a k-cell in S^n that is everywhere wild because, by Lemma 1.4.1, it fails to be 1-LCC at each interior point.

Finally, consider the cases $k = n - 2$ and $k = n - 1$. Example 2.1.10 provides wild cells in those codimensions, but they are not everywhere wild, since the basic examples of wild arcs and disks in $\mathbb{R}^3$ on which they are based are not everywhere wild. To address this issue, in the next section we will produce examples of everywhere wild arcs and disks in S^3. Once those examples are in place, multiple suspension to S^n yields everywhere wild cells of dimensions $n - 2$ and $n - 1$. $\square$

Historical Notes. The idea of exploiting the Andrews-Curtis Theorem to produce everywhere wild embeddings is due to Brown (1967). Earlier, W. A. Blankinship (1951) devised wild embeddings in all dimensions and codimensions, based on his construction of wild Cantor sets in $\mathbb{R}^n$, $n > 3$; his Cantor set construction will be set forth in §4.7.

Exercises

2.7.1. Let $C \subset M$ be a k-cell topologically embedded in an n-manifold. If C is nonlocally flat at every interior point, then C is nonlocally flat at every boundary point as well.

2.7.2. Prove Lemma 2.7.4.

2.7.3. Every cellular subset of an n-manifold ($n > 2$) satisfies the cellularity criterion.

2.7.4. If $\alpha \subset S^n$ is an arc that satisfies the cellularity criterion, then $S^n \smallsetminus \alpha$ is contractible.

2.8. Miscellaneous examples of wild embeddings

This section offers more examples of wild embeddings in $\mathbb{R}^3$. These new examples exhibit wildness that is qualitatively different from that of the examples presented earlier in the chapter. The two original examples of wildly embedded 2-spheres in $\mathbb{R}^3$, the Antoine sphere and the Alexander horned sphere, share one property: each of them contains a Cantor set such that the embedding is wild at every point of the Cantor set and is locally flat at all other points. The examples in the section show that a variety of wild sets are possible; the first examples to be presented are wild at just one point while the later examples are wild at positive-dimensional sets. In particular, among the later ones are some everywhere wild codimension-one and -two cells in $\mathbb{R}^3$, which fill a gap in the proof of Example 2.7.1. The section contains an outline of the proofs that the examples have the properties indicated, but many details are left as exercises.

2.8.1. The Fox-Artin arc. The first example is an arc whose wildness is minimal in the sense that the arc is locally flat at every point except one and the complement of the arc is the same as that of a flat arc. The construction begins with the basic building block shown in Figure 2.18. The building block consists of three arcs A, B, and C embedded in a 3-cell as indicted in the figure.

Put an infinite sequence of these building blocks together in such a way that they converge to a point p. Include the point p and delete the first copy of B to form the arc α pictured in Figure 2.19. This arc, known as the *Fox-Artin arc*, is wild because it fails to be locally flat at the endpoint p.

The arc α is obviously locally flat and PL at every point other than p. In order to see that α is not locally flat at p one must prove that α is not

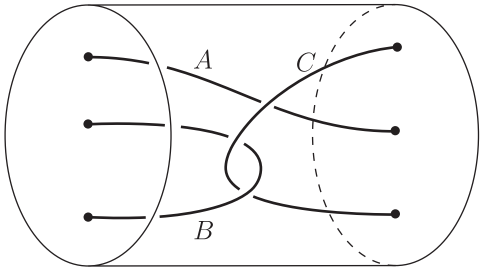

Figure 2.18. The basic Fox-Artin building block

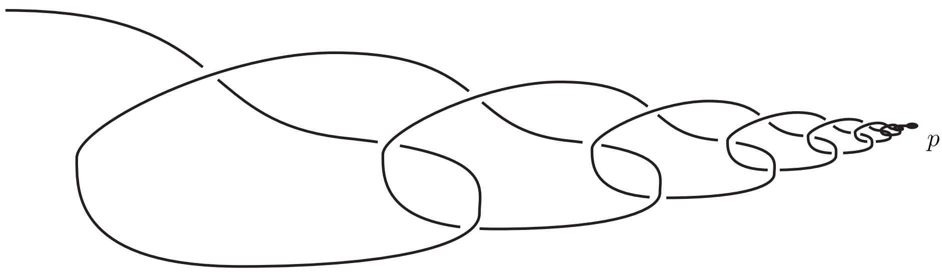

Figure 2.19. The Fox-Artin arc

1-LCC at p. (A tame arc is 1-LCC at each of its endpoints.) The Fox-Artin arc is not 1-LCC at p because for any small neighborhood U of p there exist loops in $U \setminus \alpha$ that cannot be shrunk to a point in a small subset of $S^3 - \alpha$; in fact, they cannot be shrunk to a point without going all the way over the other end of α. A proof is sketched in the exercise below.

Notice that α is cellular (Exercise 2.4.3) and thus the complement of α in S^3 is an open 3-cell. In particular, $S^3 \setminus \alpha$ is simply connected, which means that the wildness of the Fox-Artin arc is more subtle than that of the wild arcs studied earlier, which were known to be wild because their complements were not simply connected. One can obtain wild but cellular embeddings in higher dimensions by suspending the Fox-Artin arc.

This example is unique to S^3 in the sense that three is the only ambient dimension in which an arc can fail to be locally flat at just a single point (Exercise 2.5.4).

Exercise 2.8.1. This exercise contains an outline of the proof that α is not 1-LCC at p. The problem is to fill in the details in the argument. First we need some notation. Choose a sequence $D_1, D_2, D_3, \dots$ of 3-cells such that $D_{i+1} \subset \operatorname{Int} D_i$ for each i, $\cap_{i=1}^{\infty} D_i = \{p\}$, and D_i intersects α as indicated in Figure 2.20. Let A_1, B_1, and C_1 be the arcs in $D_1 \setminus \operatorname{Int} D_2$ that correspond to A, B, and C, respectively, and let E_1 be a flat disk in $D_1 \setminus \operatorname{Int} D_2$ such that $C_1 \subset \partial E_1$ and $\partial E_1 \setminus C_1 \subset \partial D_2$ (Figure 2.20).

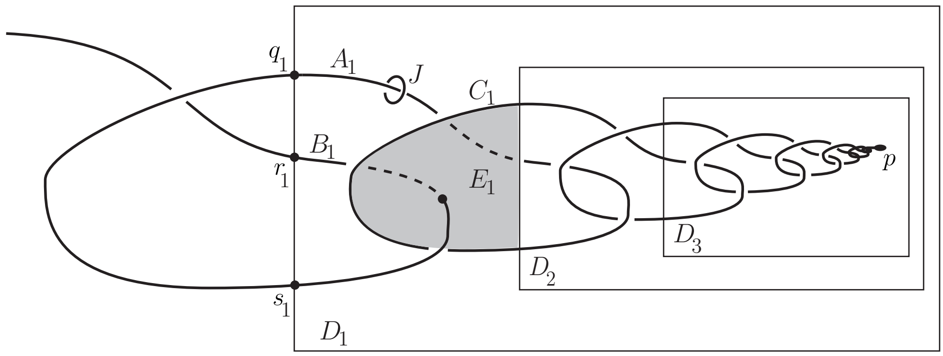

Figure 2.20. A sequence of 3-cells and 2-cells

(a) Let q_1, r_1, and s_1 be the three points at which α intersects ∂D_1 and view D_1 as the cone on its boundary. Show that

$$D_1 \setminus (A_1 \cup D_2 \cup E_1 \cup B_1) \cong D_1 \setminus \operatorname{Cone}(\{q_1, r_1, s_1\})$$

via a homeomorphism that is the identity on the boundary.

(b) Use an argument like that in the proof of Lemma 2.1.4 to prove that the inclusion induced homomorphism $\pi_1(D_1 \setminus (A_1 \cup D_2 \cup E_1 \cup B_1)) \to \pi_1(D_1 \setminus (A_1 \cup D_2 \cup C_1 \cup B_1))$ is one-to-one.

(c) Combine the preceding results to show that $\pi_1(\partial D_1 \setminus \{q_1, r_1, s_1\}) \to \pi_1(D_1 \setminus (\alpha \cup D_2))$ is one-to-one.

(d) Use an argument like that in the proof of Theorem 0.11.5 to show that $\pi_1(\partial D_1 \setminus \{q_1, r_1, s_1\}) \to \pi_1(D_1 \setminus \alpha)$ is one-to-one. Conclude, in particular, that the loop J shown in Figure 2.20 is essential in $D_1 \setminus \alpha$.

(e) Observe that each D_i contains a loop that is homotopic to J in $D_1 \setminus \alpha$ and use this observation to prove that the inclusion induced homomorphism $\pi_1(D_i \setminus \alpha) \to \pi_1(D_1 \setminus \alpha)$ is nontrivial for every i.

(f) Prove that α is not 1-LCC at p.

2.8.2. Double Fox-Artin arcs. Variations on the Fox-Artin arc can have interesting properties. Two that are worthy of mention are the "double Fox-Artin arcs" shown in Figures 2.21 and 2.22.

The double Fox-Artin arc in Figure 2.21 is constructed from a doubly infinite sequence of copies of the basic Fox-Artin building block. It is a wild arc because it fails to be 1-LCC at both endpoints. Its complement is not simply connected.

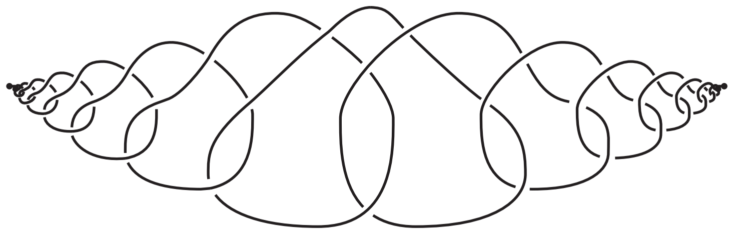

Figure 2.21. A double Fox-Artin arc with nonsimply connected complement

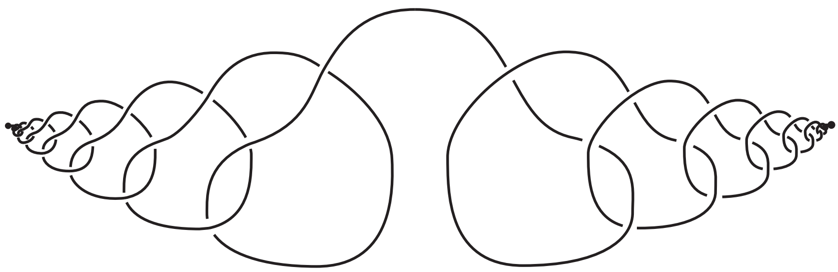

Figure 2.22. A double Fox-Artin arc with simply connected complement

The double Fox-Artin arc shown in Figure 2.22 is the wedge of two copies of α. The unusual feature of this second double Fox-Artin arc is that its complement is simply connected but is not an open 3-cell. In other words, the complement of this arc is simply connected but the arc is not cellular, because it does not satisfy the cellularity criterion. Notice that the only difference between the two arcs is that one of the two crossings in the center of Figure 2.21 has been changed to produce Figure 2.22.

Exercise 2.8.2. Use the techniques of Exercise 2.8.1 to prove the following.

(a) The complement of the arc in Figure 2.21 is not simply connected.

(b) The complement of the arc in Figure 2.22 is simply connected.

(c) The arc in Figure 2.22 does not satisfy the cellularity criterion.

2.8.3. Fox-Artin spheres. The Fox-Artin arc can be used to construct wild embeddings of spheres in S^3. To do so, start with the round 1-sphere or 2-sphere and add a feeler that follows the Fox-Artin arc. This construction is indicated in Figures 2.23 and 2.24. These embeddings are examples of what are called weakly flat spheres. An embedding $e : S^k \to S^n$ is *weakly flat* if $S^n \smallsetminus e(S^k) \cong S^n \smallsetminus S^k$. Neither example is flat since each contains the nonflat arc α. In particular, neither sphere is 1-LCC at the endpoint of the feeler.

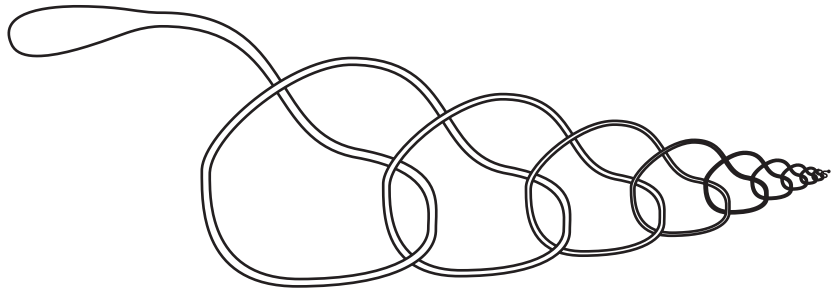

Figure 2.23. The Fox-Artin 1-sphere

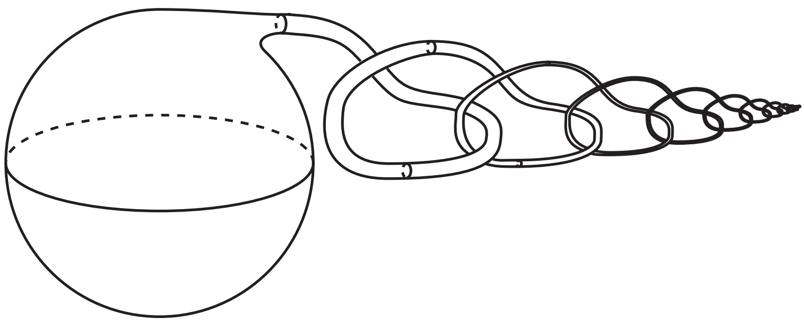

Figure 2.24. The Fox-Artin 2-sphere

Exercise 2.8.3.

(a) The Fox-Artin 1-sphere is not 1-alg at the exceptional point.

(b) The Fox-Artin 2-sphere is not 1-LCC at the exceptional point.

(c) The Fox-Artin 1-sphere is weakly flat.

(d) The Fox-Artin 2-sphere is weakly flat.

2.8.4. Mildly wild arcs. Not every arc that is formed by concatenating an infinite converging sequence of polygonal blocks is wild. In fact, the arc shown in Figure 2.25 is a tame arc.

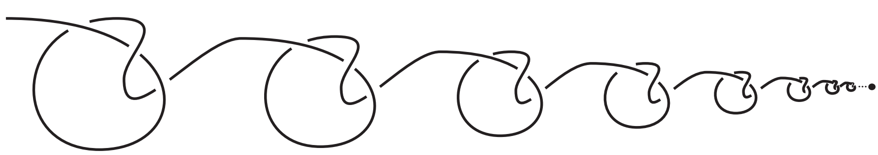

Figure 2.25. A tame arc

Interestingly, if two such arcs are joined end-to-end, the resulting arc is wild. An arc in S^3 is said to be *mildly wild* if it is wild but can be written as the union of two flat arcs. Figure 2.26 shows an example of a mildly wild arc.

Figure 2.26. A mildly wild arc

Exercise 2.8.4.

(a) Prove that the arc in Figure 2.25 is flat. [Hint: Find a nested sequence $B_1 \supset B_2 \supset \ldots$ of 3-cells such that $\cap_{i=1}^{\infty} B_i$ is the endpoint of the arc and each B_i intersects the arc in a single point. For each i there is an ambient homeomorphism that is the identity on $(S^3 \smallsetminus \operatorname{Int} B_i) \cup B_{i+1}$ and that straightens out $\alpha \cap (B_i \smallsetminus \operatorname{Int} B_{i+1})$. The flattening homeomorphism is a limit of a composition of such homeomorphisms.]

(b) Prove that the arc in Figure 2.26 is not flat. [Hint: Use the Seifert-van Kampen Theorem to prove that the arc is not 1-alg at the wedge point.]

2.8.5. The Bing sling. The Bing sling is an example of an everywhere wild 1-sphere $\Sigma \subset \mathbb{R}^3$. Moreover, any arc in Σ is an everywhere wild 1-cell.

The construction begins with the basic building block shown in Figure 2.27. The building block consists of three arcs embedded in a cylindrical 3-cell; it is nearly identical to the one used in the Fox-Artin construction, but for historical accuracy we use this variation.

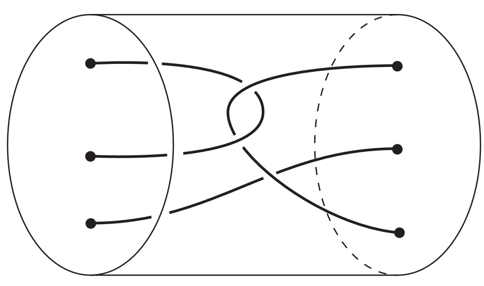

Figure 2.27. The basic building block for the Bing sling

The Bing sling arises as the intersection of a nested sequence of solid tori $T_1 \supset T_2 \supset \cdots$. The first solid torus T_1 is formed from six copies of the basic building block fit together end-to-end in a cycle as shown in Figure 2.28. The core of T_1 is a circle J_1. Inside T_1 there is a distinguished simple closed curve J_2 formed by the union of the subarcs of the six blocks that constitute T_1. This simple closed curve is the center line of a second solid torus, T_2, which is composed of many copies of the basic building block placed end-to-end along J_2. Just a few of those blocks are indicated in Figure 2.28. The subarcs of the blocks that make up T_2 combine to form a simple closed curve J_3, which is the centerline of a third solid torus T_3. The construction is continued recursively and Σ is defined by

$$\Sigma = \cap_{i=1}^{\infty} T_i.$$

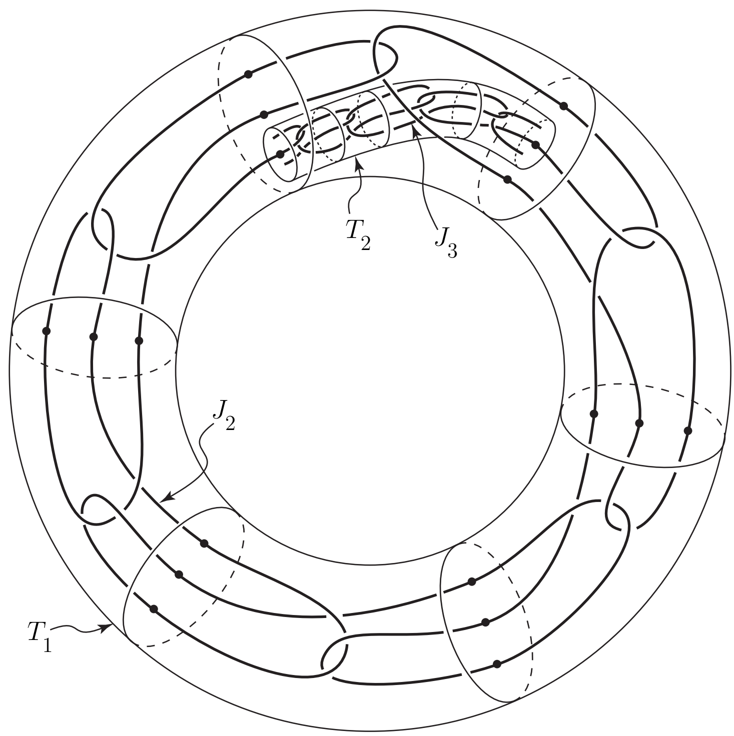

Figure 2.28. The Bing sling

At first glance it might appear that the intersection of the solid tori will be a complicated continuum, but it is, in fact, a simple closed curve. To verify this, observe that there is a homeomorphism h_i from J_i to J_{i+1}, and h_i can be kept as close to the identity as we wish by inserting multiple copies of the basic building block into the ith stage of the construction. Here h_i can be specified so as to send the portion of J_i in a block B from T_i into $J_{i+1} \cap (B \cup B')$, where B' is one of the two blocks from T_i touching B. Thus Proposition 2.2.2 shows that we can perform the construction in such a way that the composition of these homeomorphisms converges to an embedding $e : J_1 \to \mathbb{R}^3$. It is not hard to see that $e(J_1) = \Sigma$ (Exercise 2.8.5(a)).

To prove that Σ is everywhere wild, we show that Σ fails to be 1-alg at every point. Fix a point $x \in \Sigma$ and a neighborhood U of x. Assume that U is contained in the union of two of the building blocks in T_1 so that homotopies in U cannot go all the way around T_1. For any smaller neighborhood V of x there is an index i and one of the building blocks B_0 that make up T_i such that $x \in B_0 \subset V$. Consider the loop K shown in Figure 2.29. It is clear that K is null-homologous in $B_0 \smallsetminus J_{i+1}$, as it bounds an orientable surface there; thus, K represents a commutator in $\pi_1(V \smallsetminus \Sigma)$. If Σ were 1-alg at x, we would be able to choose V small enough so that K is inessential in $U \smallsetminus \Sigma$. In the next two paragraphs we will show that, to the contrary, K is essential in $U \smallsetminus \Sigma$, so no such V exists and we can conclude that Σ is not 1-alg at x.

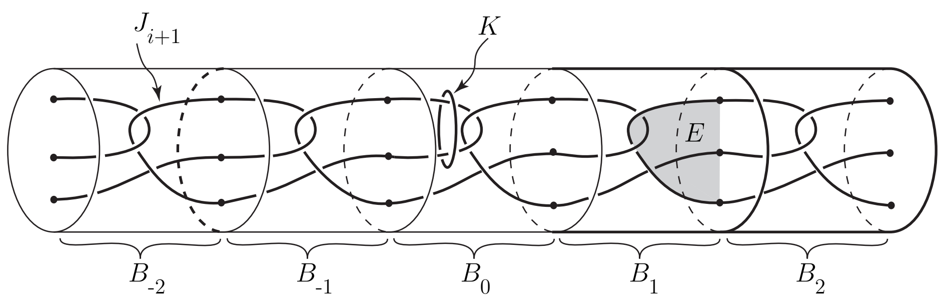

Figure 2.29. The loop K is linked around J_{i+1}

Suppose an embedding $S^1 \to K$ extends to a map $g : B^2 \to U \smallsetminus \Sigma$. The choice of U implies that $g(B^2)$ will miss at least one of the blocks in T_i, so we can find a sequence of consecutive blocks $B_{-n}, \dots, B_n$ such that $g(B^2) \cap T_i$ is contained in the 3-cell $A = B_{-n} \cup \cdots \cup B_n$ and that $g(B^2)$ does not intersect either end of A (see Figure 2.29). Put g in general position relative to ∂T_i. Then $g^{-1}(\partial T_i)$ will consist of a finite number of disjoint simple closed curves. Consider one such simple closed curve C that is innermost in the sense that no other curve is in its interior relative to B^2. The interior of C (in B^2) is mapped by g either to $\mathbb{R}^3 \smallsetminus J_i$ or to $J_i \smallsetminus \Sigma$. In either case, it follows that $g(C)$ does not link J_i homologically and thus is an inessential curve on the annulus $\partial A \cap \partial T_i$. Hence we can modify g so that it maps the interior of C into $\partial A \cap \partial T_i$ and then push the image to one side to eliminate C from $g(B^2) \cap \partial T_i$. This process can be continued inductively and results in a new map g with the property that $g(B^2) \cap \partial T_i = \emptyset$, which means that $g(B^2) \subset A \smallsetminus \Sigma$.

The previous paragraph shows that if K is inessential in $U \smallsetminus \Sigma$, then K is inessential in $A \smallsetminus \Sigma$. In this paragraph we show that K is essential in $A \smallsetminus \Sigma$, which completes the proof that Σ is not 1-alg at x. An argument like that in the construction of Antoine's necklace (see Exercise 2.8.5(b)) establishes that K is essential in $A \smallsetminus J_{i+1}$. Assume that $S^1 \to K$ extends to a map $g : B^2 \to A \smallsetminus \Sigma$. Put g in general position with respect to ∂T_{i+1}. Then $g^{-1}(\partial T_{i+1})$ will consist of a finite number of simple closed curves. Since the images of these curves do not go all the way around T_{i+1}, each of them represents some multiple of the meridian of T_{i+1}. Let C be one of the curves. If $g(C)$ is inessential on ∂T_{i+1}, then g can be modified (as in the previous paragraph) to eliminate that curve of intersection. Thus there must be at least one of these curves C whose image is a nonzero multiple of the meridian of T_{i+1}. But then $g(C)$ homologically links Σ and so $g(B^2) \cap \Sigma \neq \emptyset$.

Exercise 2.8.5.

(a) Prove that the Bing sling Σ is a simple closed curve by verifying that the embedding $e : J_1 \to \mathbb{R}^3$ described above satisfies $e(J_1) = \Sigma$.

(b) Prove that the loop K shown in Figure 2.29 is essential in $A \smallsetminus J_{i+1}$, where $A = B_{-n} \cup \cdots \cup B_n$. [Hint: First observe that K is essential in $B_0 \smallsetminus J_{i+1}$ by results established earlier in the chapter. Then consider the inclusion of K into $(B_0 \cup B_1) \smallsetminus J_{i+1}$. Use an argument like that in the proof of Lemma 2.1.4 to show that if K is inessential in $(B_0 \cup B_1) \smallsetminus J_{i+1}$ then K is inessential in $(B_0 \cup B_1) \smallsetminus (J_{i+1} \cup E)$, where E is the disk shown in Figure 2.29. Check that the embedding of J_{i+1} in $(B_0 \cup B_1) \smallsetminus E$ is the same as the embedding of J_{i+1} in B_0. Next add in B_{-1} and proceed inductively.]

(c) Let $f : S^1 \to \mathbb{R}^3 \smallsetminus \Sigma$ be a map such that $f(S^1)$ homologically links Σ and let $F : B^2 \to \mathbb{R}^3$ be an extension of f. Prove that $F^{-1}(\Sigma)$ contains a Cantor set. Use this fact to give an alternative proof that Σ is everywhere wild.

(d) Prove that Σ is homogeneously embedded; i.e., for every pair of points $x, y \in \Sigma$ there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(\Sigma) = \Sigma$ and $h(x) = y$.

2.8.6. Bing's hooked rug. Bing's hooked rug is an example of an everywhere wild 2-sphere in $\mathbb{R}^3$. The wild sets of the Alexander and Antoine spheres are Cantor sets while the wildness of the Fox-Artin sphere is concentrated at a single point. By contrast, the wildness of the hooked rug is totally diffused: the embedding is wild at every point. Nevertheless, each arc in the 2-sphere is tame. The complement of Bing's hooked rug is not simply connected. To the contrary, near any point of the 2-sphere one can find a loop that is essential in the complement.

Like the previous examples, the construction of Bing's hooked rug is described in two different ways. The hooked rug can be understood as the boundary of the intersection of a nested sequence of compact ∂-manifolds; this view is useful in proving that the example fails to be 1-LCC (and is therefore wild). The example can also be realized as the limit of a sequence of embeddings of the 2-sphere; this view is useful in proving that it is a topologically embedded sphere.

The construction begins with a round 3-cell F_0. Cover the surface of F_0 with a sequence $E_1, E_2, \ldots, E_n$ of disks that have disjoint interiors and such that $E_i \cap E_{i+1}$ is an arc in the boundary of each. (Count cyclically so that $E_n \cap E_1$ is also an edge of each.) Attach to each E_i a tube with a solid torus at the end. The union of the tube and solid torus is called an *eyebolt*—see Figure 2.30.

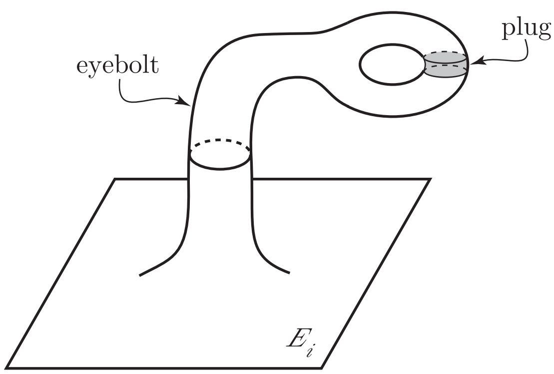

Figure 2.30. A disk E_i with an eyebolt attached

Hook the eyebolt on E_i to the base of the eyebolt on E_{i+1} and the eyebolt on E_n to the base of that on E_1 in a cyclic pattern as indicated in Figure 2.31. The original ball F_0 together with the union of all the eyebolts forms a solid 3-dimensional object H_1. Note that H_1 consists of a 3-cell with eyebolts attached, so H_1 is a cube with handles.[3] Shrink F_0 slightly before attaching the eyebolts so that H_1 is contained in the interior of F_0.

A *plug* for an eyebolt is a copy of $B^2 \times (0, 1)$ that cuts off the eyebolt as shown in Figure 2.30. Remove a plug from each of the eyebolts in H_1; the resulting solid is a 3-cell F_1. The 2-sphere ∂F_1 is the first approximation to Bing's hooked rug. There is an obvious homeomorphism $F_0 \to F_1$. The distance any point is moved by this homeomorphism is at most twice the maximum diameter of the disks $E_1, E_2, \ldots, E_n$, so we can control the size of this homeomorphism by controlling the number and size of the disks E_i.

[3]A *cube with handles* is the regular neighborhood in $\mathbb{R}^3$ of a 1-dimensional polyhedron.

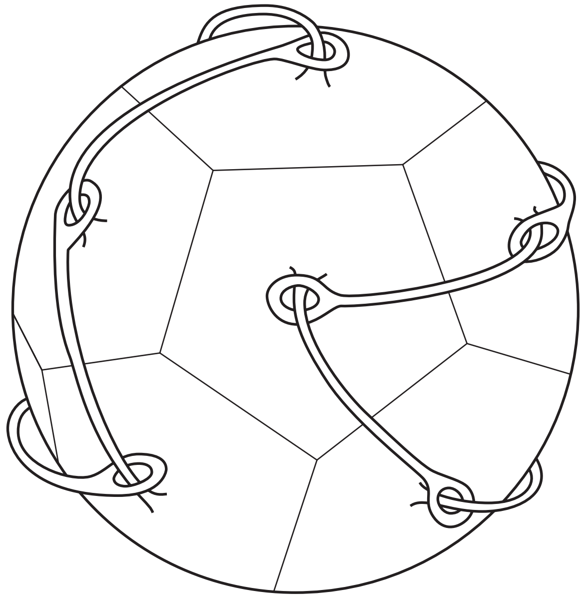

Figure 2.31. H_1, the first stage in the hooked rug construction

The surface of F_1 is covered by disks $E'_1, E'_2, \ldots, E'_n$ that have the same boundaries as the original disks $E_1, E_2, \ldots, E_n$. Cover each E'_i with a sequence of 15 (or more) disks and erect a new, smaller eyebolt on each of the disks. Hook the eyebolts on E'_i together in a circular pattern as indicated in Figure 2.32. Define H_2 to be the union of F_1 and all the second-stage eyebolts. Then H_2 is another cube with handles. We again shrink F_1 slightly before attaching the second-stage eyebolts so that $H_2 \subset \operatorname{Int} H_1$.

Figures 2.31 and 2.32 provide drawings of stage one and stage two, respectively, of the hooked rug construction; Color Plates 5–6 display photographs of physical models of those same stages.

Remove plugs from each of the second-stage eyebolts to form a 3-cell F_2. Note that again there is a homeomorphism $F_1 \to F_2$ and that the distance any point is moved by this homeomorphism is at most twice the maximum diameter of any of the disks used at the second stage. Thus we can make the homeomorphism $F_1 \to F_2$ close to the identity by simply subdividing into more second stage disks and making the corresponding eyebolts small.

The construction is continued inductively to produce a nested sequence $H_1 \supset H_2 \supset H_3 \supset \ldots$ of cubes with handles and a sequence $F_0, F_1, F_2, \ldots$ of 3-cells. Define

$$H = \cap_{i=1}^{\infty} H_i$$

and define Bing's hooked rug to be the boundary of H. We claim that H is a topological 3-cell and that ∂H is an everywhere wild 2-sphere.

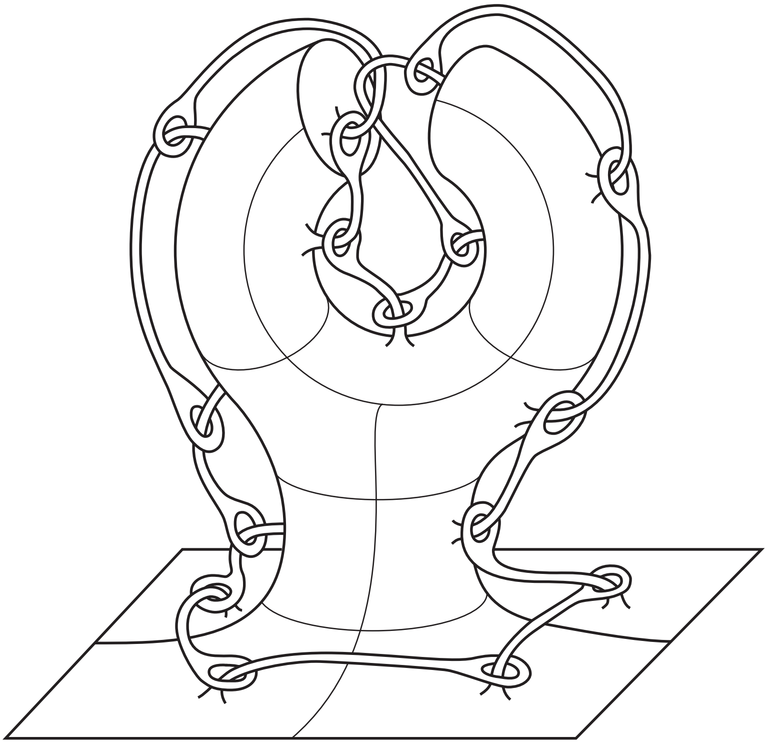

Figure 2.32. The second stage in the hooked rug construction

To prove that H is a 3-cell, we observe that the construction can be done in such a way that the homeomorphism $F_{i-1} \to F_i$ is close to the identity. While Proposition 2.2.2 does not quite apply, the same kind of proof as was used for the Alexander horned sphere shows that the composition of these homeomorphisms converges to an embedding $h : F_0 \to \mathbb{R}^3$. It is not difficult to see that h maps F_0 onto H (Exercise 2.8.6 (a)).

We prove that ∂H is everywhere wild by showing that H fails to be 1-LCC at each point of ∂H. Specifically, we prove that a loop in the complement of H that circles the base of one of the eyebolts at stage i is essential in the complement at each subsequent stage and therefore will be essential in the complement of the intersection H. The small eyebolts at a later stage are spread densely over the sphere, so there is such a loop near every point on the limiting sphere. Thus H fails to be 1-LCC at any point of ∂H.

In order to prove the claims in the preceding paragraph we break down the transition from H_{i-1} to H_i into three steps. Start with H_{i-1}. Remove a plug from each of the eyebolts in H_{i-1} and replace it with a pillbox (see definition on page 47). Call the new ∂-manifold H_i'. Note that H_i' is a 3-cell with two solid handles attached for each eyebolt in H_{i-1}. Now shrink

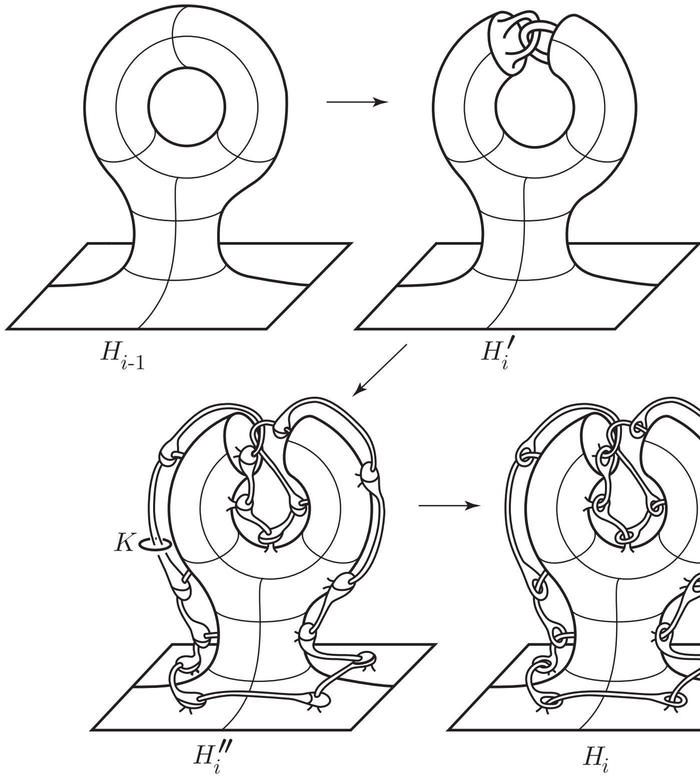

Figure 2.33. The three-step transition from H_{i-1} to H_i

H_i' slightly and add additional simple unlinked handles until there is one handle for each eyebolt in H_i. These handles can be isotoped to resemble the eyebolts that we want to add at this stage. More specifically, each of them can be deformed to look like a tube with a bulb at the end and with the base of one handle going straight through the bulb of the previous handle. The ∂-manifold formed in this way is called H_i'' and it is illustrated in Figure 2.33. Finally, convert the handles to eyebolts by drilling a tunnel through the bulb in each handle to create a hole for the next handle to pass through. The resulting ∂-manifold is H_i.

Repeated application of the following lemma shows that $\pi_1(\mathbb{R}^3 \smallsetminus H_i'') \to \pi_1(\mathbb{R}^3 \smallsetminus H_i)$ is one-to-one.

Lemma 2.8.1. *Let C be a 3-cell in $\mathbb{R}^3$, let B_1, B_2, and B_3 be three disjoint disks on ∂C, let T be a solid torus in C such that $T \cap \partial C = B_1$, and let S be a 3-cell in C such that $S \cap \partial C = B_1 \cup B_2$. Assume T and S are linked as indicated in Figure 2.34. Let X be a closed subset of $\mathbb{R}^3$ such that $X \cap C = B_1 \cup B_2 \cup B_3$. If $\pi_1(\partial C \smallsetminus (B_1 \cup B_2 \cup B_3)) \to \pi_1(\mathbb{R}^3 \smallsetminus (X \cup \operatorname{Int} C))$ is one-to-one, then $\pi_1(\mathbb{R}^3 \smallsetminus (X \cup C)) \to \pi_1(\mathbb{R}^3 \smallsetminus (X \cup S \cup T))$ is one-to-one.*

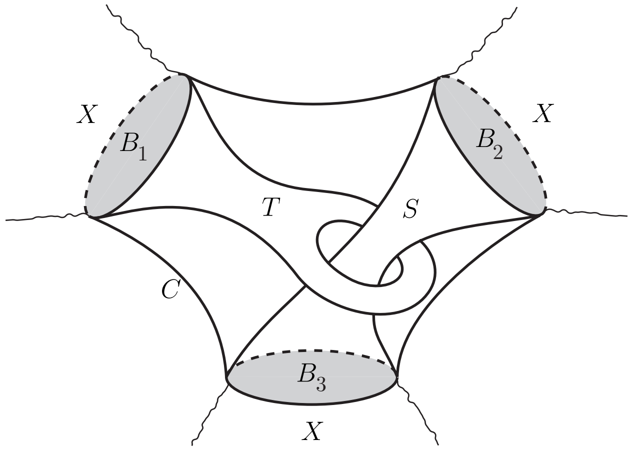

Figure 2.34. Detail of $H_i'' \to H_i$

Proof. Exercise 2.8.6 (b). □

It is clear that a loop K that goes around one of the small handles of H_i'' is essential in $\mathbb{R}^3 \smallsetminus H_i''$ (see Figure 2.33). By Lemma 2.8.1, K is also essential in $\mathbb{R}^3 \smallsetminus H_i$. But then K is essential in $\mathbb{R}^3 \smallsetminus H_{i+1}'$ by Lemma 2.1.9. It is clear that any loop that is essential in $\mathbb{R}^3 \smallsetminus H_{i+1}'$ is also essential in $\mathbb{R}^3 \smallsetminus H_{i+1}''$. Several applications of Lemma 2.8.1 show that K is also essential in $\mathbb{R}^3 \smallsetminus H_{i+1}$. Since this argument can be continued inductively, we see that K is essential in $\mathbb{R}^3 \smallsetminus H$. This completes the proof that H fails to be 1-LCC at every point of ∂H.

As asserted earlier, every arc in ∂H is tame in $\mathbb{R}^3$. We will not prove this, but in a later chapter we will develop tools that could be used to show that every arc in ∂H is 1-alg. A complete proof that arcs in ∂H are tame may be found in (Bing, 1961a).

Exercise 2.8.6.

(a) Prove that Bing's hooked rug is a topological sphere by verifying that there is an embedding $h : F_0 \to \mathbb{R}^3$ such that $h(F_0) = H$.

(b) Prove Lemma 2.8.1.

2.8.7. The Alford sphere. Our final example is a 2-sphere in $\mathbb{R}^3$ whose wild set is an arc. Its construction involves a retooling of the one just given for Bing's hooked rug. Instead of having the eyebolts wander all over the sphere so as to be dense in the limit, we will erect eyebolts just along an arc in the 2-sphere.

Start with a round 3-cell. On its boundary identify a narrow rectangle R with centerline L. Subdivide R into a large number of squares and then attach an eyebolt to each of the squares. Two consecutive eyebolts should be hooked together, but the last one should be left dangling as shown in Figure 2.35.

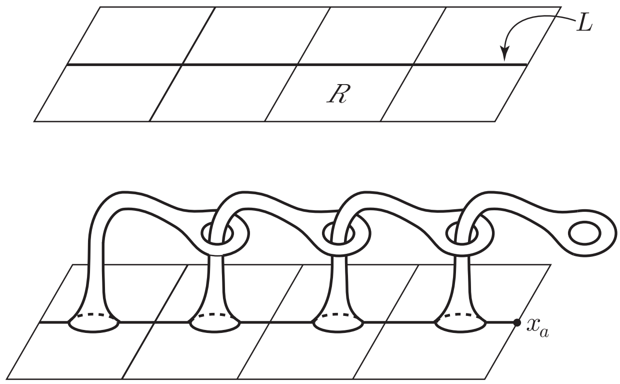

Figure 2.35. The first stage in the construction of the Alford sphere

Now remove a plug from each eyebolt. The resulting solid is a 3-cell and there is a new segment L on its boundary that goes up and over each cut eyebolt. Cover L with a sequence of much smaller squares and erect a new sequence of smaller eyebolts, one in each of the small squares. Again hook consecutive eyebolts together in a linear chain and leave the last one dangling. More specifically, the second-stage eyebolts associated with the ith disk from the first stage should be hooked together as indicated in Figure 2.36; the last second-stage eyebolt for the ith disk should be hooked to the first second-stage eyebolt associated with the $(i+1)$st disk and the very last second-stage eyebolt should be left dangling.

The process is continued inductively and the limit is a 3-cell. The boundary of this 3-cell is the Alford 2-sphere S_A. It is clear that S_A is locally flat at each point not on the limit arc. The limit arc γ_A is called the Alford arc, and x_A is used to denote the endpoint of γ_A near which the eyebolts are left dangling. We prove that S_A fails to be 1-LCC at each point of $\gamma_A \smallsetminus x_A$, so the wild set of S_A is exactly γ_A. In fact, the proof shows that the Alford arc itself fails to be 1-alg at points of $\gamma_A \smallsetminus x_A$, so γ_A is a new example of an everywhere wild arc.

In order to demonstrate that S_A fails to be 1-LCC at points of $\gamma_A \smallsetminus x_A$, we identify a small loop near most points of γ_A that cannot be shrunk to a point in a small set without hitting the Alford sphere. Specifically, let K be a small loop in the complement of S_A that circles the base of one of the ith stage eyebolts as shown in Figure 2.37. Add a short arc α that connects the

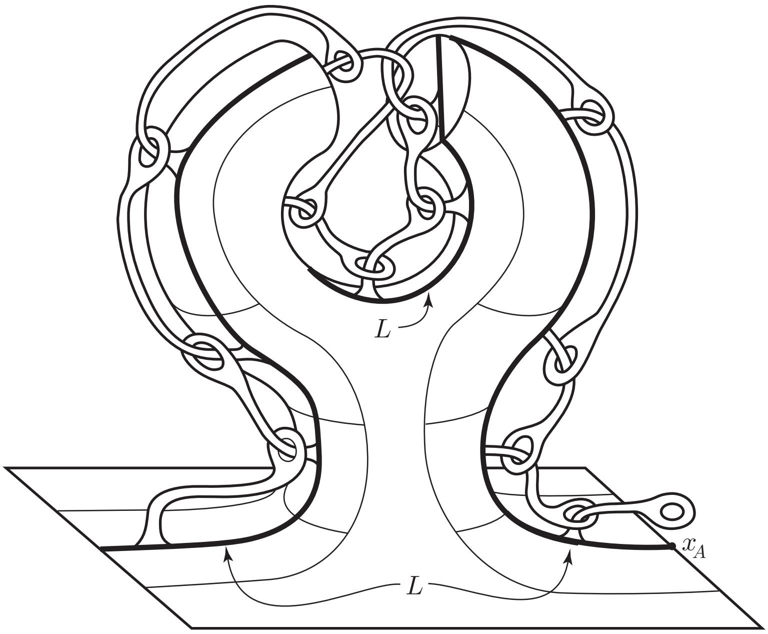

Figure 2.36. The linking of the second stage eyebolts

end of the dangling ith-stage eyebolt to the end of L. By Lemma 2.8.1, K is essential in the complement of the ith stage with α added. Any homotopy that shrinks K to a point in the complement of the ith stage must therefore intersect α and cannot be confined to a small neighborhood of a point on $\gamma_A \smallsetminus x_A$.

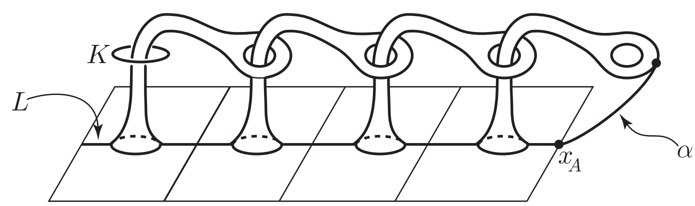

Figure 2.37. The arc α

In fact, K cannot be shrunk to a point in a small subset of the complement of any subsequent stage either. In order to see this, let us say that the construction is done in such a way that α touches the end of the dangling eyebolt at each subsequent stage of the construction. Then the techniques of the preceding section can be used to show that K is essential in the complement of each stage with a subarc of α added. It follows that K cannot be shrunk to a point in a small subset of the complement of S_A.

Exercise 2.8.7.

(a) Prove that the construction described above can be carried out in such a way that the limit is an embedded 3-cell.

(b) Fill in the details of the proof that the Alford sphere is wild at each point of the Alford arc. Prove that the Alford arc itself is an everywhere wild embedding of $[0,1]$ in $\mathbb{R}^3$.

(c) Prove that the Alford arc is 1-LCC at one of its endpoints. [This shows that an embedding can be 1-LCC at a wild point.]

(d) Prove that the Alford arc is cellular.

(e) Prove that the Alford construction can be modified to produce an example of an embedding of S^2 in $\mathbb{R}^3$ whose wild set is homeomorphic to any finite tree. Prove that this construction can be done in such a way that the wild set is cellular.

(f) Prove that there are uncountably many inequivalent embeddings of S^2 in $\mathbb{R}^3$ by producing embeddings whose wild sets are 1-dimensional compacta that are limits of trees.

Historical Notes. The Fox-Artin arc is one of many examples of wild embeddings discovered by R. H. Fox and E. Artin (1948). The mildly wild arc shown in Figure 2.26 was described by R. H. Fox and O. G. Harrold (1962); they named such arcs Wilder arcs after R. L. Wilder, who was the first to consider them. The Bing sling was described in (Bing, 1956). Bing's hooked rug appeared in (Bing, 1961a). D. Gillman (1964) revised the hooked rug technology to produce an everywhere wild 2-sphere in $\mathbb{R}^3$ that bounds a cellular 3-cell. W. R. Alford (1962) capitalized on the work of both Bing and Gillman in developing the Alford sphere.

2.9. Embeddings that are piecewise linear modulo one point

We conclude the chapter with a flattening theorem for codimension-one spheres in S^n. It assures that any $(n-1)$-sphere in S^n, $n \geq 4$, that is piecewise linear modulo one point is flat. This contrasts with the situation in ambient dimension three, where the Fox-Artin sphere is locally PL modulo one point but still wild.

The promised result stands among many flattening theorems to be proved in the text. We include it here in this preliminary chapter because its proof stems from a marvelous argument, one that does not rely on the more elaborate techniques to be developed later, and because it serves as an early indication of the fact that high-dimensional embedding phenomena differ from those encountered in dimension three.

To briefly describe the generalizations to be proved later, we need one additional bit of terminology. Let Σ be an $(n-1)$-sphere topologically embedded in S^n. A subset K of Σ that is homeomorphic to either a cell or a sphere is said to be *twice flat* provided it is flat when considered as a subset of the sphere Σ and also flat when considered as a subset of S^n. In Chapter 7 we will generalize Theorem 2.9.3 in two different ways, showing that Σ is flat if it is locally flat modulo a twice flat Cantor set or if it is locally flat modulo a twice flat cell of dimension not equal to $n-3$.

We make no attempt to state the theorems in this section in their ultimate generality since we plan to improve them later. Instead we state them with hypotheses strong enough to minimize proof technicalities, in order to more easily expose the pivotal ideas.

Definition. Let K be a finite simplicial complex with $p \in |K|$. A map $f : |K| \to M$ into a piecewise linear manifold M is said to be *piecewise linear modulo* p if there exists a locally finite triangulation K' of the noncompact polyhedron $|K| \smallsetminus \{p\}$ such that each simplex of K' is contained in a simplex of K and f is linear on each simplex of K'.

We treat a special case first. The hypothesis $n \geq 4$ is already needed in this special case.

Proposition 2.9.1. *Let C^n be an n-simplex. If $e : C^n \to S^n$, $n \geq 4$, is an embedding that is piecewise linear modulo one vertex, then $S^n \smallsetminus \operatorname{Int} e(C^n)$ is a topological n-cell.*

Definition. Let (A, B) be a pair of closed subsets of the space X. Define $G_{A,B}$ to be the decomposition of $A \times [0,1]$ whose nondegenerate elements are the arcs $\{x\} \times [0,1]$ with $x \in B$. A *collar of A pinched at B* is an embedding $h : A \times [0,1]/G_{A,B} \to X$ such that $h(x,0) = x$ for every $x \in A$ and $h(A \times [0,1]/G)$ is a neighborhood of $A \smallsetminus B$. In case the subset B is clear from the context, we will simply refer to h as a *pinched collar*.

Lemma 2.9.2. *Let (A, B) be a pair of closed subsets of the space X. If A is locally collared in X at each point of $A \smallsetminus B$, then there is a collar of A pinched at B.*

Proof. Exercise 2.9.1. □

Proof of Proposition 2.9.1. Let C^n be an n-simplex and let $e : C^n \to S^n$ be an embedding that is piecewise linear modulo the vertex $v \in C^n$. Let D^n be a second n-simplex such that $C^n \subset D^n$ and $C^n \cap \partial D^n = \{v\}$. Let E^n be a third n-simplex such that $E^n \subset \operatorname{Int} C^n$. Pick a vertex w of E^n and let α be the straight line segment from w to v. We may assume that $\alpha \cap E^n = \{w\}$. (See Figure 2.38.)

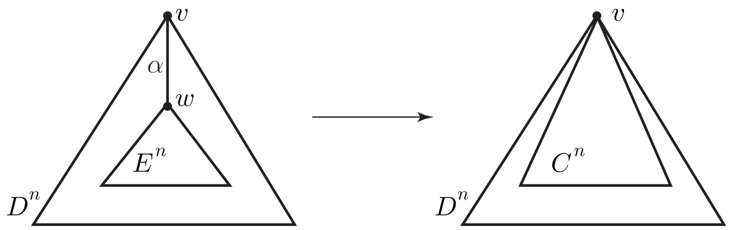

Figure 2.38. Shrink α to a point to map (D^n, E^n) to (D^n, C^n)

By Lemma 2.9.2, e can be extended to a topological embedding $h : D^n \to S^n$. Define

$$G = \overline{S^n \setminus h(C^n)} \quad \text{and} \quad F = \overline{S^n \setminus h(E^n)}.$$

Generalized Schönflies Theorem 2.4.8 guarantees that F is a topological n-cell. We will prove that G is also an n-cell by proving that G is homeomorphic to F.

Note that there is a map $(D^n, E^n) \to (D^n, C^n)$ that is the identity on ∂D^n and whose only nondegenerate inverse set is α. This map of D^n induces a map from $F \cap h(D^n)$ to $G \cap h(D^n)$ that is the identity on $h(\partial D^n)$. Extending via the identity produces a continuous map $g : F \to G$ whose only nondegenerate inverse set is $h(\alpha)$.

Now $h(\alpha)$ is a locally flat arc by Exercise 2.5.4. Hence there is a continuous function $f : F \to F$ whose only nondegenerate inverse set is $h(\alpha)$. It is easy to check that $f \circ g^{-1}$ is a well-defined homeomorphism from G to F. □

Definition. Let $\Sigma \subset S^n$ be a topologically embedded $(n-1)$-sphere and let $p \in \Sigma$. A *bicollar of* Σ *pinched at* p is an embedding

$$c : S^{n-1} \times [-1, 1]/\{v\} \times [-1, 1] \to S^n$$

such that $c(S^{n-1} \times \{0\}) = \Sigma$ and $c(v, 0) = p$. (Here v is a point in S^{n-1} and $S^{n-1} \times [-1, 1]/\{v\} \times [-1, 1]$ is the quotient space of $S^{n-1} \times [-1, 1]$ formed by shrinking $\{v\} \times [-1, 1]$ to a point.)

The following result is the main theorem in the section.

Theorem 2.9.3. *If $\Sigma \subset S^n$, $n \geq 4$, is an embedded $(n-1)$-sphere and Σ has a bicollar pinched at $p \in \Sigma$, where the bicollar is piecewise linear modulo the preimage of p, then Σ is flat.*

Proof. To simplify the notation, we denote $S^{n-1} \times [-1, 1]/\{v\} \times [-1, 1]$ by Q and use v^* to denote the point in Q corresponding to $\{v\} \times [-1, 1]$. By hypothesis, there exists an embedding $c : Q \to S^n$ such that $c(v^*) = p$

and c is piecewise linear modulo v^*. Define H to be the closure of the complementary domain of $c(S^{n-1}\times\{-1\})$ that does not contain Σ and define K to be the closure of the complementary domain of $c(S^{n-1}\times\{1\})$ that does not contain Σ. We will prove that $c(S^{n-1}\times[-1,0]/\{v\}\times[-1,0])\cup H$ and $c(S^{n-1}\times[0,1]/\{v\}\times[0,1])\cup K$ are both n-cells.

Let C^n be an n-simplex in $\operatorname{Int} Q\cup\{v^*\}$ such that v^* is a vertex of C^n and $C^n\cap(S^{n-1}\times\{0\})$ is a flat disk; define $Q'=Q\smallsetminus\operatorname{Int} C^n$. Figure 2.39 shows two views of the pinched bicollar with Q' shaded in each. In the first view, all of Q is shown with $S^{n-1}\times\{0\}$ as its core. In the second view, ∂C^n has been turned inside out so that only Q' is visible. We can choose C^n so that Q' is PL homeomorphic to an n-simplex Δ with the interiors of two PL n-cells removed. Those two n-cells meet at v^* and are otherwise in the interior of Δ. Their boundaries are $S^{n-1}\times\{-1\}$ and $S^{n-1}\times\{1\}$. In addition, $S^{n-1}\times\{0\}$ separates Q' into two pinched collars, Q_- and Q_+, so that Q_- contains $S^{n-1}\times\{-1\}$ and Q_+ contains $S^{n-1}\times\{1\}$. The left half of Figure 2.39 shows that we can choose C^n so that Q_- is naturally homeomorphic to $S^{n-1}\times[-1,0]/\{v\}\times[-1,0]$ and Q_+ is naturally homeomorphic to $S^{n-1}\times[0,1]/\{v\}\times[0,1]$. Thus we can complete the proof by showing that $c(Q_-)\cup H$ and $c(Q_+)\cup K$ are both n-cells.

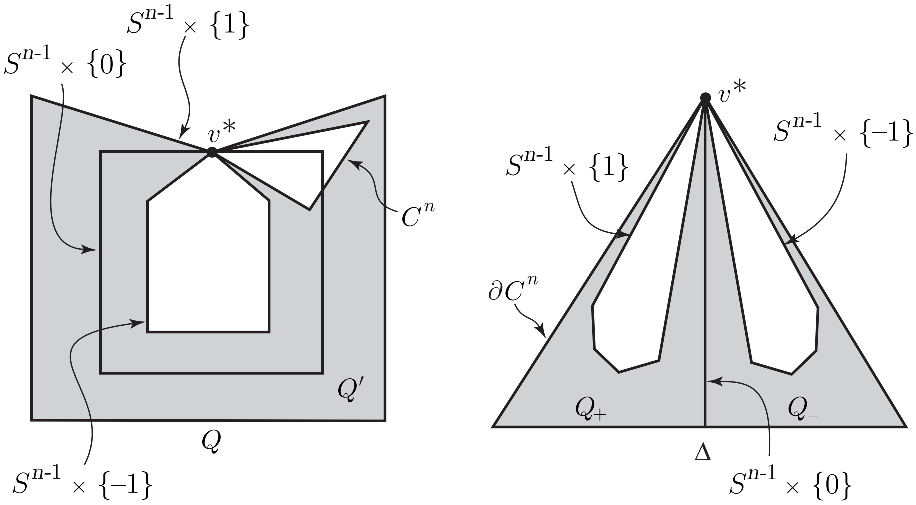

Figure 2.39. Two different views of the bicollar

Proposition 2.9.1 implies that the closure of the complement of $c(C^n)$ is an n-cell. But $S^n\smallsetminus\operatorname{Int} c(C^n)=c(Q')\cup H\cup K$, so $c(Q')\cup H\cup K$ is an n-cell. That is to say, sewing H and K to Q' along $S^{n-1}\times\{-1\}$ and $S^{n-1}\times\{1\}$, respectively, results in an n-cell; more specifically, if h and k are the maps defined by $h=c^{-1}|\operatorname{Fr} H$ and $k=c^{-1}|\operatorname{Fr} K$, then $Q'\cup_h H\cup_k K$ is an n-cell. We will use an infinite construction to show that $Q_-\cup_h H$ and $Q_+\cup_k K$ are also n-cells.


Photograph by Claire Ferguson

PLATE 5. First stage Bing's hooked rug construction, solid bronze, by Helaman Ferguson

Photograph by Claire Ferguson

PLATE 6. Second stage Bing's hooked rug construction, solid bronze, by Helaman Ferguson

Let $B_1, B_2, B_3, \ldots$ be a sequence of n-cells such that, for each i, $B_i \cap B_{i+1}$ is a flat $(n-1)$-cell standardly embedded in both ∂B_i and ∂B_{i+1}, $\cup_{i=1}^{\infty} B_i$ is an n-cell, and there is a point q such that $B_i \cap B_j = \{q\}$ for $|i-j| > 1$. Figure 2.40 shows one way to construct such a sequence by starting with a round n-cell B^n and subdividing it via hyperplanes, any two of which intersect B at a point $q \in \partial B^n$.

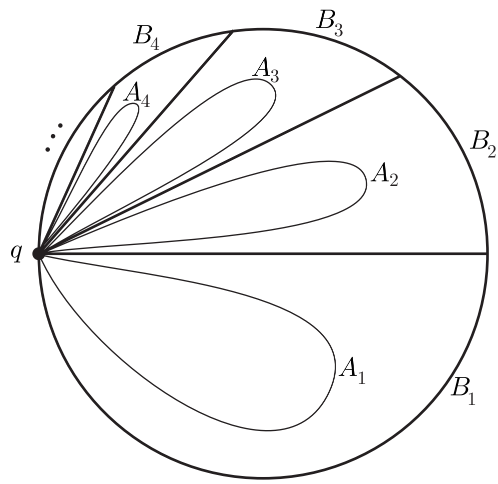

Figure 2.40. A sequence of cells whose union is a cell

For each i, choose a flat n-cell $A_i \subset B_i$ such that $A_i \cap \partial B_i = \{q\}$. If i is odd, define B_i' to be $(B_i \smallsetminus \operatorname{Int} A_i) \cup_{h'} H$; if i is even, define B_i' to be $(B_i \smallsetminus \operatorname{Int} A_i) \cup_{k'} K$. The maps h' and k' are appropriate modifications of h and k, respectively. As demonstrated above, $B_i' \cup B_{i+1}'$ is an n-cell for every odd integer i. Thus $\cup_{i=1}^{\infty} B_i'$ is an n-cell. On the other hand, there is a homeomorphism of Q' to itself that interchanges $S^{n-1} \times \{-1\}$ and $S^{n-1} \times \{1\}$ and is the identity on the other component of $\partial Q'$. [This does not look right in the 2-dimensional figure, but such a self homeomorphism exists as long as $n \geq 3$.] Therefore $B_i' \cup B_{i+1}'$ is also an n-cell for i even and so $\cup_{i=2}^{\infty} B_i'$ is an n-cell. It follows that B_1' is an n-cell. A similar argument shows that B_2' is an n-cell, so the proof is complete. $\square$

Historical Notes. Theorem 2.9.3 is due to J. C. Cantrell. The statement appeared in (Cantrell, 1963a) and the proof is contained in (Cantrell, 1963b) and (Cantrell and Edwards, 1963). The technique of pairing off the infinite sequence of cells in two different ways is often called the "Mazur swindle." Mazur (1959) (1961b) first used the technique to prove the special case of the Generalized Schönflies Theorem in Exercise 2.9.5. Other applications of the technique are described in (Mazur, 1964b) and (Mazur, 1966). The result in Exercise 2.9.4 was first proved by P. H. Doyle and J. G. Hocking (1960).

Exercises

2.9.1. Prove Lemma 2.9.2. [Hint: The open subset $A \setminus B$ has an ordinary collar by Theorem 2.4.10. Carefully trim this collar back to a pinched collar.]

2.9.2. For each $n \geq 3$ there exists a wild $(n-1)$-sphere $\Sigma \subset S^n$ whose wild set is a twice flat $(n-3)$-cell.

2.9.3. For each $n \geq 3$ there exists a wild $(n-1)$-sphere $\Sigma \subset S^n$ whose wild set is an $(n-2)$-cell that is tame in Σ.

2.9.4. If $\Sigma \subset S^3$ is a 2-sphere that is locally flat modulo a point p and there is an arc $A \subset \Sigma$ passing through p that is flat in S^3, then Σ is flat.

2.9.5. Use the technique of proof of Theorem 2.9.3 to give a new proof of the following special case of the Generalized Schönflies Theorem: If $e : S^{n-1} \to S^n$ is a locally flat topological embedding such that $e|U$ is PL for some open subset U of S^{n-1}, then e is flat.

Chapter 7

Codimension-one Embeddings

The optimal codimension-one results arise in the topological category and, for the most part, involve embeddings of codimension-one manifolds, not complexes, in manifolds. Among the positive aspects, which revolve more around local flatness than around PL approximation or ϵ-tameness, there are three prominent results. The first, developed in §7.3, is a local unknottedness theorem for locally flat approximations to a given embedding: any two sufficiently close, locally flat approximations to a given topological embedding of a compact codimension-one manifold are ambient isotopic, with suitable controls on the isotopy. The second, treated in §7.5 and §7.6, is the characterization of locally flat embeddings of codimension-one manifolds in terms of the 1-LCC condition. The third is the locally flat approximation theorem for manifold embeddings, covered in §7.7.

In addition, §7.1 lays out some elementary separation criteria for codimension-one embedded manifolds. §7.4 presents (a statement of) Edwards's Cell-like Approximation Theorem, and makes preparations for later application of that result. §7.8 touches lightly upon codimension-one analogs of the Kirby-Siebenmann obstruction theory, the codimension-two version of which appears in §6.8. §7.9 presents conditions under which an embedding is 1-LCC. §7.10 treats sewings of crumpled cubes; it gives conditions under which prescribed wildness on two sides of a codimension-one manifold can be welded together in an n-manifold, and along the way it gives some additional examples of wild codimension-one embeddings. §7.11 presents an example of a wildly embedded codimension-one sphere with a manifold mapping cylinder neighborhood, and it establishes that codimension-one embedded manifolds with such mapping cylinder neighborhoods are locally flat if they satisfy an additional freeness condition.

As the chapter progresses, it brings to bear several major theorems whose proofs are beyond the scope of this work. These include: local contractibility of the group of homeomorphisms of a compact manifold in §7.4, the Cell-like Approximation Theorem in §7.4, and the Annulus Theorem in §7.5.

7.1. Codimension-one separation properties

Codimension-one submanifolds locally separate their supermanifolds. Global separation can depend on subtler issues. This section explores how (co)homological data affect global separation.

Proposition 7.1.1. *If M is a connected n-manifold and S is a connected $(n-1)$-manifold embedded in M as a closed subset, then $M \setminus S$ has either one or two components. If S separates M and M' is any connected manifold neighborhood of S in M, then S also separates M'. If, in addition, $H_1(M;\mathbb{Z}_2) \cong 0$, then $M \setminus S$ has two components.*

Proof. The first statement follows from exactness of the sequence

$$H_1(M, M \setminus S; \mathbb{Z}_2) \to \widetilde{H}_0(M \setminus S; \mathbb{Z}_2) \to \widetilde{H}_0(M; \mathbb{Z}_2) \cong 0$$

and the duality-based isomorphism $H_1(M, M \setminus S; \mathbb{Z}_2) \cong H_c^{n-1}(S; \mathbb{Z}_2) \cong \mathbb{Z}_2$. If M' is a connected manifold neighborhood of S in M, then the first vertical arrow in the diagram

$$\begin{array}{ccccc} \mathbb{Z}_2 \cong H_1(M', M' \setminus S; \mathbb{Z}_2) & \longrightarrow & \widetilde{H}_0(M' \setminus S; \mathbb{Z}_2) & \longrightarrow & \widetilde{H}_0(M'; \mathbb{Z}_2) \cong 0 \\ \cong\downarrow & & \downarrow & & \\ \mathbb{Z}_2 \cong H_1(M, M \setminus S; \mathbb{Z}_2) & \longrightarrow & \widetilde{H}_0(M \setminus S; \mathbb{Z}_2) & \longrightarrow & \widetilde{H}_0(M; \mathbb{Z}_2) \cong 0 \end{array}$$

is an isomorphism by excision. When $\widetilde{H}_0(M \setminus S; \mathbb{Z}_2) \cong \mathbb{Z}_2$, commutativity and exactness force $\widetilde{H}_0(M' \setminus S; \mathbb{Z}_2) \cong \mathbb{Z}_2$ as well. In case $H_1(M; \mathbb{Z}_2) \cong 0$, the extended sequence

$$0 \cong H_1(M; \mathbb{Z}_2) \to H_1(M, M \setminus S; \mathbb{Z}_2) \to \widetilde{H}_0(M \setminus S; \mathbb{Z}_2) \to \widetilde{H}_0(M; \mathbb{Z}_2) \cong 0$$

allows us to conclude that $M \setminus S$ has exactly two components. □

Definitions. A connected $(n-1)$-manifold S in an n-manifold M is *two-sided (in M)* if S has a connected neighborhood N_S such that $N_S \setminus S$ is disconnected; otherwise S is *one-sided*. Generally, a disconnected $(n-1)$-manifold in M is *two-sided* there if each of its components is.

Proposition 7.1.1 assures that all compact, codimension-one submanifolds separate S^n and, hence, are two-sided.

Corollary 7.1.2. *Every $(n-1)$-manifold S in an n-manifold M is locally two-sided; that is, each $s \in S$ has arbitrarily small connected neighborhoods N_s such that $N_s \setminus S$ has two components. Hence, if S itself is two-sided and U is one of the sides, then S is 0-LCC in $\overline{U}$.*

Proof. In any coordinate neighborhood W of s, take $N_s \subset W$ as a connected neighborhood of s such that $N_s \cap S$ equals the component of $W \cap S$ containing s. □

Lemma 7.1.3. *Let M denote an orientable n-manifold and let S be a connected $(n-1)$-manifold embedded in M as a closed subset. Then S is two-sided if and only if it is orientable.*

Proof. Consider a connected neighborhood N_S of S, where $N_S \setminus S$ is disconnected if and only if S is two-sided. Produce a smaller neighborhood N' of S that deformation retracts to S in N_S. A look at the Mayer-Vietoris sequence for $N_S = (N_S \setminus S) \cup N'$ reveals that $H_1(N_S \setminus S) \oplus H_1(N') \to H_1(N_S)$ is surjective. Note that the image of $H_1(N')$ in $H_1(N_S)$ coincides with that of $H_1(S)$.

When S is two-sided, $\text{incl}_* : H_1(N_S \setminus S; \mathbb{Z}) \to H_1(N_S; \mathbb{Z})$ is surjective: each loop in S is homotopic in N_S to an approximating loop in $N_S \setminus S$ on either side, by the preceding lemma; when S is one-sided, the square of any loop in S is homotopic to one in $N_S \setminus S$. Thus, the cokernel of incl_* is a torsion group. Read off the (non)separation conclusions from the exact sequence:

$$0 \to \text{Torsion} \to H_1(N_S, N_S \setminus S; \mathbb{Z}) \to \widetilde{H}_0(N_S \setminus S; \mathbb{Z}) \to 0$$

and the duality isomorphism $H_c^{n-1}(S; \mathbb{Z}) \cong H_1(N_S, N_S \setminus S; \mathbb{Z})$. □

Corollary 7.1.4. *Suppose M is a connected, orientable n-manifold and $N \subset M$ is a compact, connected, nonorientable $(n-1)$-manifold. Then $M \setminus N$ is connected.*

Corollary 7.1.5. *Let M be an n-manifold with $H_1(M; \mathbb{Z}_2) \cong 0$. If the $(n-1)$-manifold S embeds in M as a closed subset, then S is orientable.*

Lemma 7.1.6. *Let M be a connected n-manifold and let $S \subset M$ be a connected $(n-1)$-manifold embedded in M as a closed subset. Then S separates M if and only if the inclusion-induced homomorphism $H_c^{n-1}(M; \mathbb{Z}_2) \to H_c^{n-1}(S; \mathbb{Z}_2)$ is trivial.*

Proof. This follows from examination of:

$$\begin{tikzcd} H_c^{n-1}(M;\mathbb{Z}_2) \arrow[r] \arrow[d, "\cong"] & H_c^{n-1}(S;\mathbb{Z}_2) \cong \mathbb{Z}_2 \arrow[d, "\cong"] & \\ H_1(M;\mathbb{Z}_2) \arrow[r] & H_1(M, M \setminus S;\mathbb{Z}_2) \arrow[r] & \widetilde{H}_0(M \setminus S;\mathbb{Z}_2) \to 0. \end{tikzcd}$$

□

Corollary 7.1.7. *If M is an n-manifold and S is an $(n-1)$-manifold embedded in M as a closed subset, where $H_1(S;\mathbb{Z}_2) \cong 0$, then S is two-sided in M.*

Proof. Reduce to the case in which S is connected, and let U be a connected neighborhood of S that strong deformation retracts to S in M. Inspection of the diagram ($\mathbb{Z}_2$ coefficients throughout)

$$\begin{tikzcd} H_1(U) \arrow[r] \arrow[d, "0"] & H_1(U, U \setminus S) \arrow[r, "\cong"] \arrow[d, "\cong"] & H_c^{n-1}(S) \arrow[d, "\mathrm{Id}_S^*"] \\ H_1(M) \arrow[r] & H_1(M, M \setminus S) \arrow[r, "\cong"] & H_c^{n-1}(S) \end{tikzcd}$$

yields that $H_1(U) \to H_1(U, U \setminus S)$ is trivial. By duality, $H_c^{n-1}(U) \to H_c^{n-1}(S)$ is trivial, and 7.1.6 applies. □

Corollary 7.1.8. *Given a compact two-sided $(n-1)$-manifold S in the n-manifold M, there exists $\epsilon > 0$ such that for any embedding $\lambda : S \to M$ within ϵ of incl_S, $\lambda(S)$ is 2-sided in M.*

Proof. Identify a neighborhood U_S of S such that each component of S separates the relevant component of U_S. When $\lambda, \mathrm{incl}_S : S \to U_S$ are homotopic, $H_c^{n-1}(U_S;\mathbb{Z}_2) \to H^{n-1}(\lambda(S);\mathbb{Z}_2)$ can be factored through the trivial homomorphism $H_c^{n-1}(U_S;\mathbb{Z}_2) \to H^{n-1}(S;\mathbb{Z}_2)$. □

Corollary 7.1.9. *Given a compact two-sided $(n-1)$-manifold S in the n-manifold M and a neighborhood U of S, there exists $\epsilon > 0$ such that for any two disjoint embeddings $\lambda_0, \lambda_1 : S \to M$ within ϵ of incl_S, U contains a compact subset C with $\lambda_0(S) \cup \lambda_1(S)$ as its frontier.*

Example 7.1.10. *Let $\theta : S' \to S$ be a 2-1 covering map between compact, connected $(n-1)$-manifolds. Then $W = \mathrm{Map}(\theta)$ is a compact n-dimensional ∂-manifold containing S as a one-sided subset of $\mathrm{Int}\, W$, and every embedding $\lambda : S \to \mathrm{Int}\, W$ homotopic to incl_S satisfies $\lambda(S) \cap S \neq \emptyset$.*

Proof. Consider any $\lambda : S \to W$ homotopic to $\mathrm{incl}_S : S \to W$; clearly λ induces an isomorphism at the π_1-level. By definition of the mapping cylinder, $\partial W \cong S'$ is a strong deformation retract of $W \setminus S$. Accordingly, if $\lambda(S)$ were disjoint from S, $\lambda_\# : \pi_1(S) \to \pi_1(W)$ would factor through

$\pi_1(\partial W = S') \to \pi_1(W) \cong \pi_1(S)$, an impossibility, as the latter homomorphism fails to be surjective. □

Proposition 7.1.11. *Let M denote a connected n-manifold, S a connected $(n-1)$-manifold embedded in M as a closed and separating subset, and U a component of $M \smallsetminus S$. Then for each $s \in S$ and neighborhood N of s, there exists a neighborhood $N' \subset N$ of s such that*

$$\text{incl}_* : H_k(N' \cap U; \mathbb{Z}) \to H_k(N \cap U : \mathbb{Z})$$

is trivial for all $k > 0$. Furthermore, if S is 1-LCC in $\overline{U}$, then S is k-LCC in $\overline{U}$ for all $k \geq 0$.

Proof. Being an ANR, $\overline{U}$ is locally contractible. Hence, given a neighborhood N of $s \in S$, one can find a smaller neighborhood N' such that $\text{incl}_* : H_k(N' \cap \overline{U}) \to H_k(N \cap \overline{U})$ is trivial for all $k > 0$ ($\mathbb{Z}$ coefficients throughout this argument). We will show that when N is chosen so its intersection with S is contractible, then $\text{incl}_* : H_k(N \cap U) \to H_k(N \cap \overline{U})$ will be an isomorphism ($k > 0$), which will give that $\text{incl}_* : H_k(N' \cap U) \to H_k(N \cap U)$ is trivial. Inspection of the long exact sequence for $(N, N \smallsetminus S)$ and duality yields $H_k(N, N \smallsetminus S) \cong H_c^{n-k}(N \cap S) \cong 0$ for $k > 1$, from which it follows that $\text{incl}_* : H_k(N \smallsetminus S) \to H_k(N)$ is an isomorphism when $k > 1$. Diagram chasing assures the same holds true for $k = 1$, because $H_1(N, N \smallsetminus S) \cong H_c^{n-1}(N \cap S) \cong \mathbb{Z}$, so $H_1(N, N \smallsetminus S) \to \widetilde{H}_0(N \smallsetminus S)$ is an isomorphism. Let V denote the other component of $M \smallsetminus S$. Clearly $H_k(N \smallsetminus S) \cong H_k(N \cap U) \oplus H_k(N \cap V)$. A straightforward Mayer-Vietoris argument gives that

$$H_k(N) \cong H_k(N \cap \overline{U}) \oplus H_k(N \cap \overline{V}) \quad (k > 0).$$

Naturality assures that $\text{incl}_* : H_k(N \cap U) \to H_k(N \cap \overline{U})$ is an isomorphism.

Note that S is 0-LCC in $\overline{U}$ by Corollary 7.1.2. When it is also 1-LCC there, application of the local Hurewicz Theorem 0.8.3 confirms that S is k-LCC in $\overline{U}$ for all $k \geq 2$. □

Exercise

7.1.1. Suppose M is a connected n-manifold and $S \subset M$ is a closed $(n-1)$-manifold such that $M \smallsetminus S$ is connected. For each $\alpha_S \in \pi_1(S)$ there exists $\alpha' \in \pi_1(M \smallsetminus S)$ such that $(\text{incl}_{M \smallsetminus S})_\#(\alpha') = 2 \cdot (\text{incl}_S)_\#(\alpha_S)$.

7.2. The 1-LCC characterization of local flatness for collared embeddings

For a compactum in the trivial dimension range, being 1-LCC embedded implies it admits an ϵ-push into its complement. An analog holds for two-sided

1-LCC embeddings of manifolds in codimension one. This new 1-LCC push-off result warrants close attention, as it presents a pivotal codimension-one technique in relatively simple form. The same technique will reappear with more intricate variations in subsequent sections. As a peripheral benefit, the push-off result quickly leads to the 1-LCC characterization of local flatness for codimension-one submanifolds collared on one side. The full 1-LCC characterization of local flatness is treated in §7.6.

Proposition 7.2.1 (1-LCC push-off)**.** *Suppose M is a connected PL n-manifold, $n \geq 5$, and S is a compact, connected, two-sided $(n-1)$-manifold 1-LCC embedded in M, where $M \smallsetminus S$ has two components, U_+ and U_-. Then for each $\epsilon > 0$ there exists an ϵ-push ψ of (M, S) such that $\psi(S) \subset U_+$.*

Proof. Apply Generalized Controlled Engulfing Theorem 3.3.7 for the given integer n and for $r = n-3$ to obtain $\delta > 0$ corresponding to $\epsilon/3$. After noting that both $\overline{U}_+$ and $\overline{U}_-$ are neighborhood retracts, successively choose open neighborhoods $W^+_{n-2} \supset W^+_{n-3} \supset \cdots \supset W^+_1 \supset W^+_0$ of $\overline{U}_+$ for which there exist strong deformation retractions of W^+_i to $\overline{U}_+$ in W^+_{i+1}, $i < n-2$, that move points less than δ and that never move any point of U_- into U_+. Require, in addition, that $W^+_{n-2} \cap U_- \subset B(S; \epsilon)$. Choose open neighborhoods $W^-_3 \supset W^-_2 \supset W^-_1 \supset W^-_0$ of $\overline{U}_-$ with analogous properties, where $W^-_3 \cap U_+ \subset B(S; \epsilon)$.

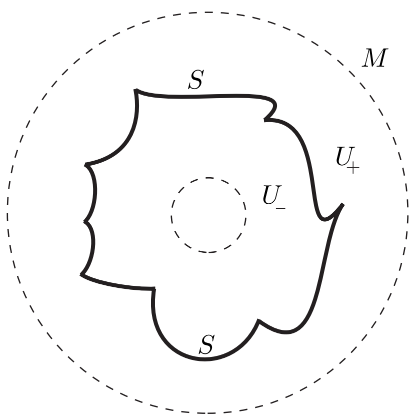

Figure 7.1. The two sides of S in M

Find a compact PL neighborhood P of S in $W^+_0 \cap W^-_0$. Let $C_+ = \mathrm{Cl}(U_+ \smallsetminus P)$ and $C_- = \mathrm{Cl}(U_- \smallsetminus P)$. Subdivide to obtain a triangulation T of P with mesh $T < \epsilon/3$, and let K denote the $(n-3)$-skeleton of T. In T', the first barycentric subdivision of T, let K' denote the simplicial complement of (the subdivided) $K \cup F$, where F denotes the frontier of P. Note that by the special restrictions on the strong deformation retractions of W^+_i to $\overline{U}_+$

spelled out in the preceding paragraph, every relative $(n-3)$-complex in $(W_i^+ \smallsetminus C_+, U_+ \smallsetminus C_+)$ admits a δ-deformation ranging through $W_{i+1}^+ \smallsetminus C_+$, first to $\overline{U}_+ \smallsetminus C_+$, and then into $U_+ \smallsetminus C_+$ by Lemma 3.3.3 (Proposition 7.1.11 assures that S is LCC^n in $\overline{U}_+$).

Now by Theorem 3.3.7 there exists an $(\epsilon/3)$-isotopy ϕ_+ of $M \smallsetminus C_+$ compactly supported in $W_{n-2}^+ \smallsetminus C_+$ such that $\phi_+(U_+ \smallsetminus C_+) \supset K \smallsetminus C_+$; extend via the identity on C_+ to a new push, still denoted as ϕ_+, such that $\phi_+(U_+) \supset K$. Use the same procedure to obtain an $(\epsilon/3)$-isotopy ϕ_- of M supported in $W_3^- \smallsetminus C_-$ such that $\phi_-(U_-) \supset K'$. Stretch across the join structure of T via a third $(\epsilon/3)$-push θ of (M, S) supported in $P \subset W_0^+ \cap W_0^-$ such that

$$\phi_+(U_+) \cup \theta\phi_-(U_-) = C_+ \cup P \cup C_- = M.$$

Apply ϕ_+^{-1} and note that

$$U_+ \cup \phi_+^{-1}\theta\phi_-(U_-) = \phi_+^{-1}(M) = M.$$

As all three pushes are supported in $W_{n-2}^+ \cap W_3^- \subset B(S;\epsilon)$, $\psi = \phi_+^{-1}\theta\phi_-$ is an ϵ-push of (M, S). Most importantly, $\psi(S) \subset U_+$, since obviously $\psi(S)$ is disjoint from $\psi(U_-)$. $\square$

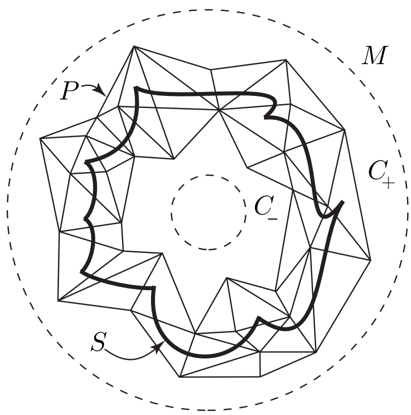

Figure 7.2. The PL neighborhood P of S.

Lemma 7.2.2 (Collar Sliding). *Suppose S is a manifold and $\lambda : S \times [0,1] \to S \times [0,1]$ is an embedding such that $\lambda | S \times 0 = \mathrm{incl}_{S\times 0}$. Then there exists a homeomorphism $h : S \times I \to S \times [0,1] \smallsetminus \lambda(S \times [0, \frac{1}{2}))$ such that $h(s,0) = \lambda(s, \frac{1}{2})$ and $h(s,1) = \langle s, 1\rangle$. Moreover, if each $\lambda(\{s\} \times [0,1])$, $s \in S$, is within $\delta > 0$ of $s \times [0,1]$, then $h(S \times I) \subset B(s \times I; 2\delta)$ for all s.*

Proof. Extend λ to an embedding of $S \times [-1,1]$ in $S \times [-1,1]$ via the Identity on $S \times [-1,0]$. Let ϕ denote the piecewise linear self-homeomorphism of $[-1,1]$ fixing the endpoints, sending 0 to $\frac{1}{2}$ and acting linearly on the complementary subintervals. The self-homeomorphism $\lambda(\mathrm{Id}_S \times \phi)\lambda^{-1}$ defined on $\lambda(S\times[-1,1])$ extends to a self-homeomorphism Φ of $S\times[-1,1]$ via the identity on the complement of $\lambda(S\times[-1,1])$, and Φ restricts to give a homeomorphism $h : S\times I \to S\times[-1,1] \smallsetminus \lambda(S\times[-1,\frac{1}{2}))$. When δ bounds the motion in the S direction under λ, then for any point $\langle s,t\rangle \in S\times I$ moved by h, $\langle s,t\rangle = \lambda(s',t')$ where $d_S(s,s') < \delta$, and

$$h(s,t) \in \lambda(s'\times I) \subset B(s'\times I;\delta) \subset B(s\times I; 2\delta).$$

□

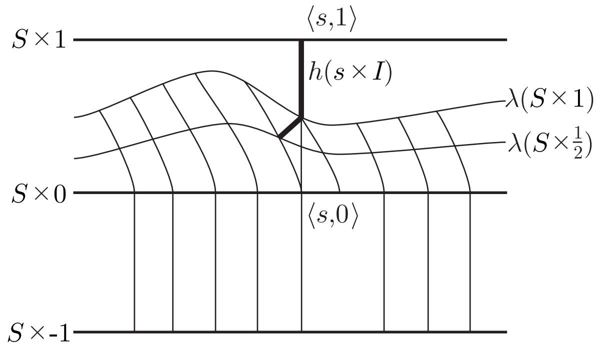

Figure 7.3. A special $S\times I$ product structure

Remark. If $S\times I$ is a PL manifold and λ is a PL collar, then h is a PL embedding.

Theorem 7.2.3. *Suppose M is a PL n-manifold, $n\geq 5$, and S is a compact $(n-1)$-manifold in M such that S is two-sided and 1-LCC embedded. If S has a collar on one side, then S is bicollared.*

Proof. Name a collar $c : S\times I \to M$ on one side of S. Assume both S and M to be connected and M a small enough neighborhood of S that $M \smallsetminus S$ has two components, with U denoting the one missing the image of c. For $i = 1,2,\ldots$ use Proposition 7.2.1 to obtain a $(1/6i)$-push ψ_i of (M,S) such that $\psi_i(S)\subset U$ and ψ_i fixes $c(S\times[u_i,1])$, where $\{u_i\in(0,1]\}_{i=1}^{\infty}$ is a sequence decreasing to 0 and $\operatorname{diam} c(s\times[0,u_i]) < 1/6i$ for all $s\in S$. Do this so $\psi_{i+1}c(S\times I)\subset \psi_i c(S\times(0,1])$ for all i. Then choose $t_i\in(0,u_i)$ such that $\psi_i c(S\times(t_i,1]) \supset \psi_{i+1}c(S\times I)$, and declare $e_i : S\to U$ to be the embedding sending $s\in S$ to $\psi_i c(s\times t_i)$. For notational simplicity, require $\{t_i\}_{i\geq 1}$ to be a strictly decreasing sequence. According to Lemma 7.2.2, the region R_i bounded by e_i and e_{i+1} is a product $S\times[t_{i+1},t_i]$, the arc fibers of which have diameter less than $1/i$, since (by the proof of the Lemma) these arc fibers live in some $\psi_i c(s \times [t_i, u_i])$ plus the union of intersecting arcs $\psi_{i+1} c(s' \times [t_{i+1}, u_i])$. Thus, $S \cup (\cup_i R_i)$ is a collar on S in $\overline{U}$. $\square$

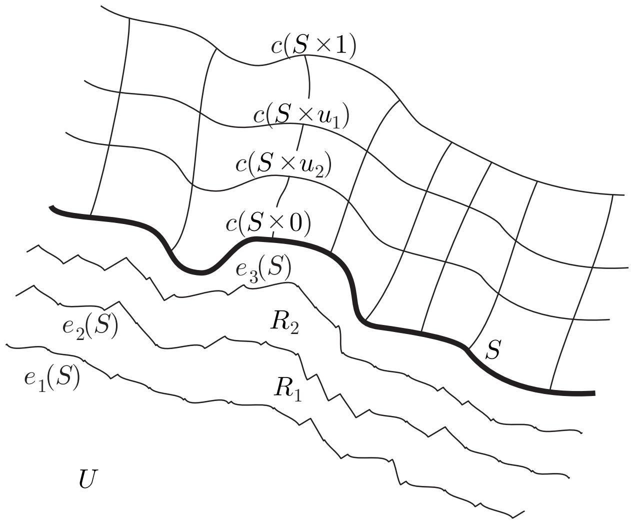

Figure 7.4. The collar on S in $\overline{U}$

Historical Notes. Proposition 7.2.1 and a strengthened Theorem 7.2.3—namely, a 1-LCC local flatness theorem for codimension-one manifolds that can be approximated by locally flat embeddings—were developed in (Seebeck, 1970).

Exercises

7.2.1. Suppose M is a connected PL n-manifold, $n \geq 5$, S is a compact, connected $(n-1)$-manifold that separates M, and U is a component of $M \setminus S$. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that, for any $(n-3)$-complex pair $(K, L) \subset (U \cup B(S; \delta), U)$ and any neighborhood O of S, there is a compactly supported ϵ-push ψ of (M, S) such that $\psi(U \cup O) \supset K$ and $\psi|(U \setminus O) \cup L = \mathrm{Id}|(U \setminus O) \cup L$.

7.2.2. Let $c_0, c_1 : \partial W \times I \to W$ be collars on the boundary of a ∂-manifold W, with $c_0(\partial W \times I) \subset c_1(\partial W \times I)$ and $\operatorname{diam} c_i(w \times I) < \epsilon$ for all $w \in \partial W$, $i = 0, 1$. Then there exists a homeomorphism

$$h : \partial W \times I \to c_1(\partial W \times I) \setminus c_0(\partial W \times [0, \frac{1}{2}))$$

such that $\operatorname{diam} h(w \times I) < 2\epsilon$ for all $w \in \partial W$.

7.2.3. Let W be a ∂-manifold such that ∂W is compact and for every compact subset C of W there is a collar $c : \partial W \times I \to W$ such that

$C \subset c(\partial W \times I)$. Then $W \cong \partial W \times [0, \infty)$. Moreover, if W and the collars are PL, then W is PL homeomorphic to $\partial W \times [0, \infty)$.

7.3. Unknotting close 1-LCC embeddings of manifolds

Throughout §7.3, S will denote a PL $(n-1)$-manifold topologically embedded as a two-sided subset of the PL n-manifold M. The main result, Theorem 7.3.1, assures that any two locally flat approximations to S in M are ambient isotopic under a controlled push of (M, S). It is an exact analog for manifolds of Codimension-three Unknotting Theorem 5.4.2; the codimension-three result cannot be extended to a local unknotting theorem for $(n-1)$-complexes in M, however, since such an extension is known to fail for codimension-two manifolds. En route to establishing 7.3.1, we will show in Theorem 7.3.11 that any two disjoint locally flat approximations cobound an embedded product $S \times I$ with short I-factor.

Theorem 7.3.1 (Local Unknotting for Embeddings of Manifolds)**.** *Let S denote a compact PL $(n-1)$-manifold topologically embedded in a PL n-manifold M^n, $n \geq 5$, as a two-sided subset. Given $\epsilon > 0$ there exists $\delta > 0$ such that, for any two locally flat embeddings λ_0, λ_1 of S in M^n within δ of the inclusion, there exists an ϵ-push θ_t of (M^n, S) such that $\theta_1 \lambda_0 = \lambda_1$.*

The proof, which occupies the rest of this section, also depends heavily upon the following result of Edwards and Kirby (1971).

Theorem 7.3.2 (Local Contractibility)**.** *Given a compact manifold S and $\epsilon > 0$, there exists $\delta > 0$ such that if $\Lambda : S \times [-1, 1] \to S \times [-2, 2]$ is an embedding within δ of the inclusion, then there is an isotopy $\Phi_t : S \times [-1, 1] \to S \times [-2, 2]$ such that $\Phi_0 = \Lambda$, $\Phi_1 | S \times \{0\} = \mathrm{incl}_{S \times \{0\}}$, $\rho(\Phi_t, \mathrm{incl}_{S \times [-1,1]}) < \epsilon$ and $\Phi_t | S \times \{\pm 1\} = \Lambda | S \times \{\pm 1\}$ for each $t \in I$.*

As an immediate consequence of 7.3.2, one can define an ambient isotopy Φ'_t on $S \times [-2, 2]$ as $\Phi'_t = \Phi_t \Lambda^{-1}$ on $\Lambda(S \times [-1, 1])$ and as the identity elsewhere. Clearly $\Phi'_0 = \mathrm{Id}$, $\Phi'_1 \Lambda_0 = \mathrm{incl}_{S \times \{0\}}$ and $\rho(\Phi'_t, \mathrm{Id}) < 2\epsilon$. Application of $(\Phi'_1)^{-1}$ to $S \times I \to S \times [-2, 2]$ implies:

Corollary 7.3.3. *Given a compact manifold S and $\epsilon > 0$, there exists $\delta > 0$ such that if $\Lambda : S \times [-1, 1] \to S \times [-2, 2]$ is an embedding within δ of the inclusion, then there is an embedding $\lambda : S \times I \to S \times [-2, 2]$ such that $\lambda(s, 0) = \langle s, -2 \rangle$, $\lambda(s, 1) = \Lambda_0(s)$, and $\lambda(s \times I)$ is within ϵ of $\{s\} \times [-2, 0]$ for each $s \in S$.*

Corollary 7.3.4. *Given a compact manifold S and $\epsilon > 0$, there exists $\delta > 0$ such that if $\Lambda : S \times [-1, 1] \to S \times [-2, 2]$ is an embedding within δ of the inclusion and $\Lambda_0(S) \cap (S \times [0, 2]) = \emptyset$, then there is an embedding*

$h : S \times I \to S \times [-2, 2]$ such that $h(s, 0) = \Lambda_0(s)$, $h(s, 1) = \langle s, 0\rangle$, and $h(s \times I)$ is within ϵ of $\langle s, 0\rangle$ for each $s \in S$.

Proof. This follows from Corollary 7.3.3 and Corollary 7.2.2, and from the consequence of the latter that

$$h(s \times I) \subset B(s \times [-2, 2]; 2\delta) \cap (S \times [-\delta, 0]). \qquad \square$$

Lemma 7.3.5. *Let Y be a locally compact ANR, $C \subset Y$ a compact ANR, and $W \subset Y$ a neighborhood of C. For each $\delta > 0$ there exist $\eta > 0$ and a neighborhood W' of C such that, given any embedding $e : C \to W'$ within η of incl_C, W' admits a strong deformation retraction $\mu_t : W' \to W$ to $e(C)$ that moves points less than δ.*

Proof. Determine a compact neighborhood $W' \subset W$ of C that admits a $(\delta/2)$-retraction to C, and find $\eta > 0$ such that any embedding $e : C \to Y$ within η of incl_C is $(\delta/2)$-homotopic to incl_C in W'. An application of the Estimated Homotopy Extension Theorem (Corollary 0.6.5) secures a δ-retraction $R_e : W' \to e(C)$. Properties of ANRs allow prearrangements yielding that $R_e, \mathrm{incl}_{W'} : W' \to W$ are δ-homotopic. $\square$

For the next several lemmas, assume that S and M are manifolds satisfying the hypotheses of Theorem 7.3.1.

Lemma 7.3.6. *Given $\epsilon > 0$, there exists $\eta > 0$ such that for every pair $\lambda_0, \lambda_1 : S \to M$ of locally flat embeddings within η of incl_S, there is an ϵ-push ψ of (M, S) such that $\psi(\lambda_0(S)) \cap \lambda_1(S) = \emptyset$.*

Proof. Assume S to be connected and W_S to be a connected neighborhood of S such that $W_S \smallsetminus S$ has two components, U_+ and U_-. Just as in the proof of Proposition 7.2.1, apply Generalized Controlled Engulfing Theorem 3.3.7 for the given integer n and $r = n - 3$ to obtain $\delta > 0$ corresponding to $\epsilon/3$ there. As before, choose open neighborhoods $W^+_{n-2} \supset W^+_{n-3} \supset \cdots \supset W^+_1 \supset W^+_0$ of $\overline{U}_+$ such that not only do there exist strong deformation retractions of W^+_i to $\overline{U}_+$ in W^+_{i+1} $(i = 0, 1, \ldots, n-3)$ moving points less than δ and never moving any point of U_- into U_+, but also (by Lemma 7.3.5) that there exists $\eta_i > 0$ such that for any embedding $\lambda : S \to W^+_i$ within η_i of $\mathrm{incl} : S \to W^+_i$ there is a strong deformation retraction of W^+_i to $\lambda(S)$ in W^+_{i+1} moving points less than δ. Require, in addition, that $W^+_{n-2} \cap U_- \subset B(S; \epsilon) \subset W_S$. Determine open neighborhoods $W^-_3 \supset W^-_2 \supset W^-_1 \supset W^-_0$ of $\overline{U}_-$ and positive numbers $\eta^*_2, \eta^*_1, \eta^*_0$ with analogous properties, where $W^-_3 \cap U_+ \subset B(S; \epsilon)$.

Choose a compact PL neighborhood P of S in $W^+_0 \cap W^-_0$, and impose on P a small mesh triangulation T with specified $(n-3)$-skeleton K and simplicial complement K' of $K \cup F$ (F denoting the frontier of P) in the dual 2-skeleton, as in 7.2.1. Set $\eta = \min\{\eta_i, \eta^*_j, d(S, M \smallsetminus P)\}$. Let $C_+ =$

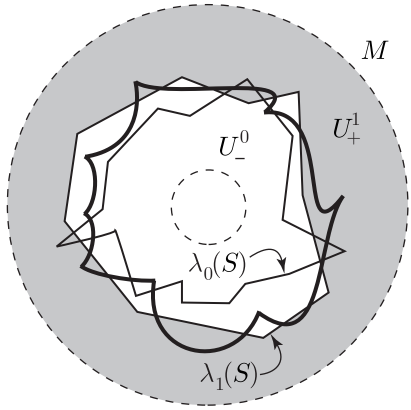

Figure 7.5. The setup for obtaining disjoint approximations.

$\mathrm{Cl}(U_+ \setminus P)$ and $C_- = \mathrm{Cl}(U_- \setminus P)$, as before. Given locally flat embeddings $\lambda_0, \lambda_1 : S \to M$ within η of incl_S, let U_+^1 denote the component of $W_S \setminus \lambda_1(S)$ containing C_+ and U_-^0 the component of $W_S \setminus \lambda_0(S)$ containing C_-. (Use of U_+^1 and U_-^0 represents the most significant change from the proof of Theorem 7.2.1.) Observe that any relative $(n-3)$-complex in $(W_i^+ \setminus C_+, U_+^1 \setminus C_+)$ is δ-homotopic in $(W_{i+1}^+ \setminus C_+, U_+^1 \setminus C_+)$ to a complex mapped into $U_+^1 \setminus C_+$. Similarly, any relative 2-complex in $(W_i^- \setminus C_+, U_-^0 \setminus C_-)$ is δ-homotopic in $(W_{i+1}^- \setminus C_-, U_-^0 \setminus C_-)$ to a complex mapped into U_-^0. Hence, Theorem 3.3.7 promises an $(e/3)$-push ϕ_+ of (M, S) compactly supported in $W_{n-2}^+ \setminus C_+$ such that $\phi_+(U_+^1) \supset K$, and it promises a second $(e/3)$-push ϕ_- of (M, S) compactly supported in $W_3^- \setminus C_-$ such that $\phi_-(U_-^0) \supset K'$. Then there is also a third $(\epsilon/3)$-push θ supported in P—the stretch across the join structure of P—such that

$$\phi_+(U_+^1) \cup \theta\phi_-(U_-^0) \supset C_+ \cup P \cup C_- = W_S.$$

Letting $\psi = \phi_+^{-1}\theta\phi_-$, one sees that ψ is an ϵ-push of (M, S) and $\psi(\lambda_0(S)) \subset U_+^1$, so $\psi(\lambda_0(S)) \cap \lambda_1(S) = \emptyset$, as desired. $\square$

Given two disjoint embeddings λ_0, λ_1 of a codimension-one manifold in a manifold M, we use $[\lambda_0, \lambda_1]$ to denote the unique compact region, if such a region exists, having the union of these images as frontier; if λ_0, λ_1 are disjoint close approximations to a two-sided codimension-one manifold S in M, we use the same symbolism $[\lambda_0, \lambda_1]$ to denote the compact region near S their images cobound; the existence of such a compact region is assured by Corollary 7.1.9.

Lemma 7.3.7. *Given $\epsilon > 0$, there exists $\eta > 0$ such that for every pair $\lambda_0, \lambda_1 : S \to M$ of disjoint embeddings within η of* incl_S*, there are strong ϵ-deformations of $[\lambda_0, \lambda_1]$ onto $\lambda_0(S)$ and $\lambda_1(S)$, respectively.*

Proof. Let W_S be a connected neighborhood of S in M that is separated by S. Reduce W_S so it admits an $(\epsilon/6)$-retraction to S. Apply Corollary 7.1.9 and Lemma 7.3.5 to produce $\delta_1 > 0$ such that, for any embedding $\lambda : S \to M$ within δ_1 of incl_S, $\lambda(S)$ separates W_S and W_S admits an $(\epsilon/3)$-retraction $W_S \to \lambda(S)$. In the presence of two δ_1-approximations $\lambda_0, \lambda_1 : S \to W_S$ to incl_S with disjoint images, we have an $(\epsilon/3)$-retraction $R : W_S \to [\lambda_0, \lambda_1]$ sending one of the components of $W_S \smallsetminus [\lambda_0, \lambda_1]$ to $\lambda_0(S)$ and sending the other to $\lambda_1(S)$. Next, find a compact neighborhood $W' \subset W_S$ of S and $\delta_2 > 0$ such that any two maps $f, f' : W' \to W_S$ δ_2-close to $\text{incl} : W' \to W_S$ are $\epsilon/3$-homotopic in W_S. Set $\delta = \min\{\delta_1, \delta_2\}$. Finally, repeat the initial procedure to find $\eta \in (0, \delta)$ such that, for any embedding $\lambda : S \to M$ within η of incl_S, $\lambda(S) \subset W' \subset W_S$ (so it separates both W' and W_S) and W' admits a δ_2-retraction $W' \to \lambda(S)$. Hence, if $\lambda_0, \lambda_1 : S \to M$ are two disjoint embeddings within η of incl_S, there is a homotopy $\mu_t : [\lambda_0, \lambda_1] \to W_S$ such that $\mu_0 = \text{incl}_{[\lambda_0, \lambda_1]}, \mu_t$ is constant on, say, $\lambda_0(S)$, μ_1 is a δ-retraction to $\lambda_0(S)$ and μ_t moves points less than $\epsilon/3$. Then $R\mu_t$ functions as a strong ϵ-deformation of $[\lambda_0, \lambda_1]$ to $\lambda_0(S)$. □

Lemma 7.3.8. *Given $\epsilon > 0$, there exists $\delta > 0$ such that for each pair $\lambda_0, \lambda_1 : S \to M$ of disjoint, locally flat embeddings within δ of* incl_S*, each neighborhood U of $[\lambda_0, \lambda_1] \smallsetminus \lambda_1(S)$, and each neighborhood O of $\lambda_1(S)$ there is an ϵ-push ψ of (M, S) fixed on $\lambda_1(S) \cup (M \smallsetminus U)$ such that $\psi([\lambda_0, \lambda_1]) \subset O$.*

Proof. For the given integer n and positive number ϵ apply Theorem 3.3.7 to get $\epsilon' > 0$ such that whenever one is provided with enough ϵ'-homotopies of $(n-3)$-complexes in any PL n-manifold N, then one also has an $(\epsilon/3)$-isotopy of N engulfing such a complex. Then use Lemma 7.3.7 to obtain $\delta > 0$ corresponding to $\epsilon' > 0$ with the properties mentioned there. Assume δ to be sufficiently small that images of δ-approximations to incl_S separate small neighborhoods of S.

Consider any two locally flat δ-approximations λ_0, λ_1 with disjoint images. As an aid, name another locally flat embedding λ_0^* of S into a bicollar on $\lambda_0(S)$ in U, with λ_0^* very close to λ_0 and $[\lambda_0^*, \lambda_1]$ properly containing $[\lambda_0, \lambda_1]$. Let W denote the interior of $[\lambda_0^*, \lambda_1]$. We will produce an ϵ-push of (M, S) moving $[\lambda_0, \lambda_1]$ into O by obtaining an appropriate, compactly supported isotopy of W.

Find a PL ∂-manifold P such that P is closed in W, $P \cup \lambda_1(S) \supset [\lambda_0, \lambda_1]$ and $\partial P \cap \lambda_0(S) = \emptyset$. Impose a triangulation T on P having mesh less than $\epsilon/3$, with diameters of simplices going to 0 as simplices approach

$\lambda_1(S)$, and let K denote the $(n-3)$-skeleton of P. Since W is the union of $[\lambda_0', \lambda_1']$ where λ_0' varies over close approximations to λ_0^*, and λ_1' varies over close approximations to λ_1 in W, Lemma 7.3.7 promises the existence of ϵ'-homotopies deforming $(n-3)$-complexes into $O_W = O \cap W$. Controlled Engulfing Theorem 3.3.7 provides a compactly supported $(\epsilon/3)$-isotopy ϕ_+ of W to itself such that $\phi_+(O_W) \supset K$.

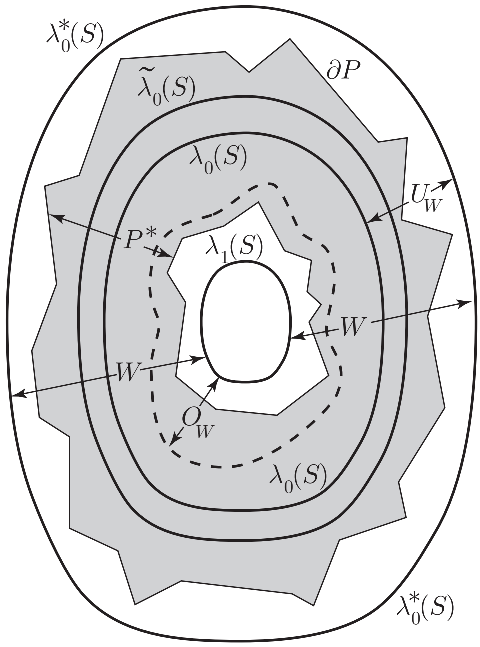

Figure 7.6. Guides for engulfing $[\lambda_0, \lambda_1]$ by O_W.

Let U_W denote $W \smallsetminus [\lambda_0, \lambda_1] = \text{Int}[\lambda_0^*, \lambda_0]$ and let $Sppt_+$ denote the support of ϕ_+. The extra wrinkle in this argument is the observation that, since only a finite part of P extends outside O_W, P contains a finite subcomplex P^* such that $P^* \supset (P \smallsetminus O_W) \cup Sppt_+$. Let K' denote the simplicial complement of $K \cup \overline{P \smallsetminus P^*}$ (subdivided) in the first barycentric subdivision of $T|P^*$. Find another locally flat approximation $\widetilde{\lambda}_0$ to λ_0 in W such that $[\widetilde{\lambda}_0, \lambda_1]$ properly contains $[\lambda_0, \lambda_1]$ and $P \supset [\widetilde{\lambda}_0, \lambda_1] \smallsetminus \lambda_1(S)$. Let $\widetilde{W}$ represent the interior of $[\widetilde{\lambda}_0, \lambda_1]$ and $\widetilde{U}_W = U_W \cap \widetilde{W}$; note that $\widetilde{W} \subset P \subset W$. Again there are ϵ'-homotopies deforming any relative 2-complex in $(\widetilde{W}, \widetilde{U}_W)$ into $\widetilde{U}_W$, so there exists a compactly supported isotopy $\phi_- : \widetilde{W} \to \widetilde{W}$ moving points less than $\epsilon/3$ such that $\phi_-(\widetilde{U}_W) \supset K' \cap \widetilde{W}$. Extend via the identity on $W \smallsetminus \widetilde{W}$ to regard ϕ_- as defined on all of W, with $\phi_-(U_W) \supset K'$ and with ϕ_- fixed on $W \smallsetminus P \subset U_W$. Exploit the usual stretch across the join structure of P^* to produce a third $(\epsilon/3)$-push θ, compactly supported in P^*, such that $\phi_+(O_W) \cup \theta\phi_-(U_W) \supset P^*$. Then

$$W \supset \phi_+(O_W) \cup \theta\phi_-(U_W) \supset (P \setminus P^*) \cup P^* \cup (W \setminus P) = W.$$

Now for $\psi = (\phi_+)^{-1}\theta\phi_-$ we have $O_W \cup \psi(U_W) = W$ and $\psi([\lambda_0, \lambda_1]) \cap \psi(U_W) = \emptyset$. Thus ψ extends via the identity over $M \setminus W$ to an ϵ-push of (M, S) such that $O \supset O_W \supset \psi([\lambda_0, \lambda_1])$. $\square$

As a routine consequence of Lemmas 7.3.6 and 7.3.8 we obtain:

Lemma 7.3.9. *Given $\epsilon > 0$, there exists $\eta > 0$ such that for each pair $\lambda_0, \lambda_1 : S \to M$ of locally flat embeddings within η of* incl_S *and each bicollar $g : S \times [-1, 1] \to M$ on $\lambda_1(S)$, there is an ϵ-push ψ of (M, S) such that $\psi(\lambda_0(S)) \subset g(S \times (0, 1))$.*

Lemma 7.3.10. *Given $n \geq 5$, a compact PL $(n-1)$-manifold S, and $\delta > 0$, there exists $\eta > 0$ such that every locally flat embedding $\lambda : S \times \{0\} \to S \times [-2, 2]$ within η of* $\mathrm{incl}_{S\times\{0\}}$ *extends to an embedding $\Lambda : S \times [-1, 1] \to S \times [-2, 2]$ within δ of* $\mathrm{incl}_{S\times[-1,1]}$.

Proof. Apply Lemma 7.3.8 for the inclusion $S \times \{0\} \hookrightarrow S \times [-2, 2]$ and positive number $\delta/4$ to obtain $\eta' > 0$. Choose a large integer $k > 0$ such that $2/k < \eta'$ and set $\eta = 1/k$.

Consider a locally flat embedding $\lambda : S \times \{0\} \to S \times [-2, 2]$ within η of $\mathrm{incl}_{S\times\{0\}}$. Extend λ to an embedding $g : S \times [-1, 1] \to S \times (-\eta, \eta)$ such that $g(s \times [-1, 1]) \subset B(\langle s, 0\rangle; \eta)$ for all $s \in S$. Arrange the parameterization of the bicollar determined by g so that $g(S \times \{1\})$ separates $g(S \times \{0\})$ from $S \times \{2\}$.

For $i = 0, 1, 2, ..., k$, set $t(i) = (k - i)/k$. The initial choice of η' assures the existence of a controlled $(\delta/4)$-push of $(S \times [-2, 2], S \times \{0\})$ moving $g(S \times \{\pm t(i)\})$ very close to $S \times \{\pm\eta\}$. These pushes will be followed by large moves that change only the $[-2, 2]$ coordinates and effect a precise shuffle repositioning each $g(S \times \{\pm t(i)\})$ very close to $S \times \{\pm t(i)\}$ (respecting $\pm$ signs).

Use $\kappa : S \times \{0\} \to S \times \{\eta\} \subset S \times [-2, 2]$ to denote the obvious embedding, and let g_t denote the embedding sending $\langle s, 0\rangle$ to $g(s, t)$. Note that both κ and g_t are within $2\eta = 2/k < \eta'$ of $\mathrm{incl}_{S\times\{0\}}$. Thus, Lemma 7.3.8 provides a $(\delta/4)$-homeomorphism Ψ'_1 of $S \times [-2, 2]$ to itself fixed outside a small neighborhood of $[g_{t(1)}, \kappa]$—in particular, fixed on $g(S \times [-1, t(2)])$—and moving $g(S \times \{t(1)\})$ into $S \times (\eta/2, \eta)$. Follow Ψ'_1 by another homeomorphism that changes only $[-2, 2]$ coordinates, fixes $\Psi'_1 g(S \times [-1, t(2)]) = g(S \times [-1, t(2)])$ and moves $\Psi'_1 g(S \times \{t(1)\})$ into $S \times [t(1), 1]$. Call the composite Ψ_1. Repeat, obtaining homeomorphisms $\Psi_2, \Psi_3, ..., \Psi_{k-1}$ of $S \times [-2, 2]$ to itself that change S coordinates by less than $\delta/4$ and satisfy $\Psi_i g(S \times \{t(i)\}) \subset$

$S \times [t(i), t(i-1)]$. Restrict supports so that the composite $\Psi = \Psi_{k-1} \cdots \Psi_2 \Psi_1$ satisfies $\Psi(g(S \times \{t(i)\})) \subset S \times [t(i), t(i-1)]$. Also require if a point of $g(S \times [-1,1])$ is fixed by $\Psi_1, ..., \Psi_j$ but moved by Ψ_{j+1}, then its image can be moved by Ψ_{j+2} but cannot be moved by subsequent Ψ_i. Define Ψ by exactly the same process for the other side of the bicollar. One can see that $\Psi g : S \times [-1,1] \to S \times [-2,2]$ changes second coordinates by less than $2/k < \eta' < \delta/4$. Since Ψg changes first coordinates less than $\delta/2$, $\Lambda = \Psi g$ is δ-close to $\mathrm{incl}_{S \times [-1,1]}$. □

Remark. Lemma 7.3.10 is one place in this section where the hypothesis about the codimension-one submanifold S being PL plays a role in the argument, simply by assuring that $S \times [-2,2]$ is PL.

Theorem 7.3.11. *Let S denote a compact PL $(n-1)$-manifold topologically embedded in a PL n-manifold M^n, $n \geq 5$, as a two-sided subset and let ϵ be a positive number. Then there exists $\delta > 0$ such that for any two locally flat embeddings λ_0, λ_1 of S in M^n within δ of the inclusion, where $\lambda_0(S) \cap \lambda_1(S) = \emptyset$, there exists an embedding $\Lambda : S \times [0,1] \to M$ such that $\Lambda_0 = \lambda_0$, $\Lambda_1 = \lambda_1$ and $\operatorname{diam} \Lambda(s \times [0,1]) < \epsilon$ for all $s \in S$.*

Proof. Once the constraint δ is in place and we get to the locally flat approximations, we will extend λ_1 to an embedding $g : S \times [-2,2] \to M$ for which the fiber arcs $g(s \times [-2,2]), s \in S$, have small images. The image bicollar will play the role of $S \times [-2,2]$ in Lemma 7.3.10. The plan is to produce a controlled push ψ of (M, S) that, in spirit, moves $\lambda_0(S)$ into $g(S \times (-1,0))$ extremely close to λ_1. There will be an obvious short product structure on something like $[\lambda_0, \psi\lambda_0]$, Lemma 7.3.10 will provide a short product structure on $[\psi\lambda_0, \lambda_1]$, and these two pieces will fit together as a short product structure on $[\lambda_0, \lambda_1]$.

The crucial issue is size control; it is somewhat delicate due to the need to pass back and forth between the abstract product space $S \times [-2,2]$, where Lemma 7.3.10 applies, and its image under g, where we must operate. To highlight the distinction we use d_M to denote a metric on M and d to denote both the restriction of d_M to S and the product metric on $S \times [-2,2]$. Here are rules for obtaining the required δ. Set $\delta_1 = \epsilon/15$. Apply Corollary 7.3.4 for S and $\epsilon/15$ $(= \delta_1)$ to obtain $\delta_2 > 0$. Take $\eta_1 > 0$ to be the positive number corresponding to S and δ_2 promised in Lemma 7.3.10. Set $\delta_3 = \min\{\delta_1, \eta_1/6\} > 0$, and take δ_4 to be a positive number promised by Lemma 7.3.8 with ϵ replaced by $\min\{\epsilon/3, \delta_3\}$. Finally, let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4/3\}$.

Consider disjoint, locally flat embeddings $\lambda_0, \lambda_1 : S \to M$ within δ of incl_S. Assume δ to be small enough that each $\lambda_j(S)$ is two-sided (Corollary 7.1.8). Specify an embedding $g : S \times [-2,2] \to M$ with $g_0 = \lambda_1$, diam $g(s \times [-2,2]) < \delta$ for all $s \in S$, and $g(S \times [-2,2]) \cap \lambda_0(S) = \emptyset$. For definiteness, parameterize so $g(S \times \{-2\}) \subset [\lambda_0, \lambda_1]$. Note that, since $\rho(\lambda_1, \mathrm{incl}_S) < \delta \leq \delta_1 = \epsilon/15$,

$$x, y \in S, d(x,y) < \delta_1 \Rightarrow d_M(\lambda_1(x), \lambda_1(y)) < 3\epsilon/15.$$

As a result,

$$x', y' \in S \times [-2,2], d(x', y') < \delta_1 \Rightarrow d_M(g(x'), g(y')) < \epsilon/3.$$

This means that for any embedded $S \times I$ in $S \times [-2,2]$ for which fiber arcs have diameter less than δ_1, the image fiber arcs under g have diameter less than $\epsilon/3$.

Let $c : S \times [0,2] \to M$ be an embedding that determines a collar on $\lambda_0(S)$ in $M \setminus g(S \times [-2,2])$, where $c_0 = \lambda_0$, diam $c(s \times [0,2]) < \delta$ for all s, and c_1 separates the two boundary components of $[\lambda_0, \lambda_1]$. Set $\lambda_0' = c_1$.

The choice of $\delta \leq \delta_3$ ensures that

$$x, y \in \lambda_1(S), d_M(x,y) < 4\delta_3 \Rightarrow d(\lambda_1^{-1}(x), \lambda_1^{-1}(y)) < 6\delta_3 \leq \eta_1.$$

Find a neighborhood $O \subset g(S \times (-1,1))$ of $\lambda_1(S)$ so small that

$$x, y \in O,\ d_M(x,y) < 4\delta_3 \Rightarrow d(g^{-1}(x), g^{-1}(y)) < \eta_1.$$

Note that $\rho_M(\lambda_0', \mathrm{incl}_S) < 2\delta$, yielding $\rho_M(\lambda_0', \lambda_1) < 3\delta \leq \delta_4$. Applying Lemma 7.3.8 to $[\lambda_0', \lambda_1]$, we obtain a δ_3-push ψ of (M, S) which is fixed on $\lambda_0(S) \cup \lambda_1(S)$ and satisfies $\psi([\lambda_0', \lambda_1]) \subset O \setminus g(S \times [0,2])$. Then $\psi c(S \times [0,1])$ provides an $S \times I$ structure on $[\lambda_0, \psi\lambda_0']$ for which the fiber arcs have diameter less than $\delta + 2\delta_3 \leq 3\delta_3 \leq 3\delta_1 < \epsilon/3$. Moreover,

$$\rho_M(\psi\lambda_0', \lambda_1) \leq \rho_M(\psi\lambda_0', \lambda_0') + \rho_M(\lambda_0', \lambda_1) < \delta_3 + 3\delta \leq 4\delta_3.$$

Thus, for $s \in S = S \times \{0\} \subset S \times [-2,2]$, $d(g^{-1}\psi\lambda_0'(s), s) < \eta_1$ by the choice of O. Lemma 7.3.10 promises an embedding

$$\Lambda : S \times [-1,1] \to S \times [-2,2]$$

within δ_2 of the inclusion, where $\Lambda_0 = g^{-1}\psi\lambda_0'$. Here $\Lambda_0(S) \cap (S \times [0,2]) = \emptyset$, so Corollary 7.3.4 assures the existence of an $(\epsilon/15 = \delta_1)$-product structure on $[g^{-1}\psi\lambda_0', \mathrm{incl}_{S\times 0}]$, and its image under g is an $(\epsilon/3)$-product structure on $[\psi\lambda_0', \lambda_1]$, as desired. □

Proof of Theorem 7.3.1. First apply Theorem 7.3.11 with positive number $\epsilon/2$ to obtain $\delta \in (0, \epsilon/2)$, and next apply Lemma 7.3.8 with $\delta/2$ to obtain $\eta \in (0, \delta/2)$. Given two locally flat η-approximations λ_0, λ_1 to incl_S, use 7.3.8 to produce a $(\delta/2)$-push ϕ of (M, S) moving $\lambda_0(S)$ off $\lambda_1(S)$. Then $\phi\lambda_0, \lambda_1$ are disjoint δ-approximations to incl_S, so Theorem 7.3.11 yields an $(\epsilon/2)$-push ψ of (M, S) supported close to $[\phi\lambda_0, \lambda_1]$ and sending $\phi\lambda_0$ to λ_1. □

Corollary 7.3.12. *Suppose M is a connected PL n-manifold, $n \geq 5$, S is a compact, connected, PL $(n-1)$-manifold that separates M, and U is a component of $M \setminus S$. Then S is collared in $\overline{U}$ if and only if S can be pointwise approximated by locally flat embeddings in U.*

Corollary 7.3.13. *Suppose M is a PL n-manifold, $n \geq 5$, and S is a compact PL $(n-1)$-manifold 1-LCC embedded in M as a two-sided subset. Also suppose S can be pointwise approximated by locally flat embeddings. Then S is bicollared.*

Proof. 1-LCC Pushoff Proposition 7.2.1 indicates that S can be pointwise approximated by locally flat embeddings on either side. □

Historical Notes. The results of this section, as well as the entire approach, again are due to (Seebeck, 1970).

Edwards and Kirby were not alone in addressing local contractibility of the manifold homeomorphism group. Černavskiĭ (1969c) had an independent, possibly earlier, proof of the main result.

Exercises

7.3.1. Prove Corollary 7.3.12. [Hint: See the proof of Theorem 7.2.3.]

7.3.2. Suppose M is a PL n-manifold ($n \geq 5$) and $S \subset M$ a PL $(n-1)$-manifold that is one-sided and 1-LCC embedded in M. Suppose also that S can be pointwise approximated by locally flat embeddings. Then S has an I-bundle neighborhood.

7.3.3. Suppose M is a PL n-manifold, $n \geq 5$, and S is a closed, PL $(n-1)$-manifold topologically embedded in M as a two-sided subset. Then S is ϵ-tame if and only if it is 1-LCC and it can be pointwise approximated by PL embeddings.

7.4. The Cell-like Approximation Theorem

At this juncture we begin to make use of Edwards's Cell-like Approximation Theorem, stated below. Many methods found in its proof have already been employed in this book, and others are completely accessible to all readers. Nevertheless, we omit the rather lengthy argument and refer readers to (Edwards, 1980) or the more complete exposition in (Daverman, 1986).

Theorem 7.4.1 (Cell-like Approximation)**.** *A proper, surjective, cell-like mapping $f : M^n \to X$ defined on an n-manifold M^n, $n \geq 5$ is a near-homeomorphism if and only if X is a finite-dimensional space with the Disjoint Disks Property.*

Definition. A metric space X has the *Disjoint Disks Property*, abbreviated as DDP, if for every pair of maps $f_1, f_2 : I^2 \to X$ and for every $\epsilon > 0$ there exist maps $F_1, F_2 : I^2 \to X$ such that $\rho(F_i, f_i) < \epsilon$ $(i = 1, 2)$ and $F_1(I^2) \cap F_2(I^2) = \emptyset$.

Corollary 7.4.2. *Every cell-like map $f : M^n \to N^n$ between n-manifolds, $n \geq 5$, is a near-homeomorphism. Specifically, if $\epsilon : N^n \to (0, \infty)$ is continuous, then there exists a homeomorphism $g : M^n \to N^n$ such that $\rho(g(x), f(x)) < \epsilon(x)$ for all $x \in M^n$.*

Corollary 7.4.3. *Let $f : M^n \to N^n$ be a cell-like map between n-manifolds, $n \geq 5$, C a closed subset of N^n such that $f|f^{-1}(C)$ is 1-1, and $\epsilon : N^n \to [0, \infty)$ a continuous function such that $\epsilon^{-1}(0) = C$. Then there exists a homeomorphism $g : M^n \to N^n$ satisfying $\rho(g(x), f(x)) < \epsilon(x)$ for all $x \in M^n \smallsetminus f^{-1}(C)$ and $g|f^{-1}(C) = f|f^{-1}(C)$.*

The intent for the remainder of §7.4 is to develop conditions, for later use, under which a cell-like image of a manifold has the DDP. In support of that aim, the immediate issue is to prove that such a cell-like image is an ANR provided it is finite dimensional.

Proposition 7.4.4. *Suppose $p : Y \to X$ is a closed, cell-like mapping defined on a locally compact ANR Y, (K, L) is a pair of finite simplicial complexes, $\mu : K \to X$ is a map, $\nu : L \to Y$ is a map such that $p\nu = \mu|L$ and $\epsilon > 0$. Then there exists a map $\widetilde{\mu} : K \to Y$ with $\widetilde{\mu}|L = \nu$ and there exists a homotopy $H_t : K \to X$ such that $H_0 = p\widetilde{\mu}$, $H_1 = \mu$, $H_t|L = \mu|L$ and $\rho(H_t, \mu) < \epsilon$ for all $t \in I$.*

The preceding proposition supplements Approximate Lifting Proposition 3.2.10. The next lemma serves as the principal tool, and its proof retraces the one given for 3.2.10.

Lemma 7.4.5. *Under the hypothesis of Proposition 7.4.4, there exists $\delta > 0$ such that, for any two maps $\alpha_0, \alpha_1 : K \to Y$ extending ν with $\rho(p\alpha_e, \mu) < \delta$ for $e = 0, 1$, there is a homotopy $h_t : K \to Y$ such that $h_e = \alpha_e$ and, for all $t \in I$, $h_t|L = \nu$ and $\rho(ph_t, \mu) < \epsilon$.*

With Lemma 7.4.5 in hand, the derivation of Proposition 7.4.4 proceeds like the one showing why every non-isolated point in a locally connected complete metric space can be joined via a path to another point nearby. One constructs a sequence of lifts $\alpha_i : K \to Y$ such that not only do the images $p\alpha_i$ converge to μ but also successive images are connected via shorter and shorter homotopies, by Lemma 7.4.5.

Corollary 7.4.6. *Suppose $p : Y \to X$ is a closed, cell-like mapping defined on a locally compact ANR Y, W is an open subset of X, $w \in p^{-1}(W)$ and $i \geq 0$. Then $p_* : \pi_i(p^{-1}(W), w) \to \pi_i(W, p(w))$ is an isomorphism.*

Corollary 7.4.7. *If $p : Y \to X$ is a closed, cell-like mapping defined on a locally compact ANR Y, then X is LC^k for all integers $k \geq 0$.*

In light of Theorem 0.6.1, we also have:

Corollary 7.4.8. *If $p : Y \to X$ is a closed, cell-like mapping from a locally compact ANR to a finite-dimensional metric space X, then X is an ANR.*

Lemma 7.4.9. *If Y is a locally compact ANR satisfying the DDP, then every map $g : N^2 \to Y$ defined on a compact 2-dimensional ∂-manifold N^2 can be approximated by embeddings. Moreover, if $S \subset Y$ is a closed set that has empty interior and is 0-LCC in Y, then each $g : N^2 \to Y$ can be approximated by an embedding $\lambda : N^2 \to Y$ such that $S \cap \lambda(N^2)$ is 0-dimensional.*

Proof. Find a countable collection $\{(D_i, E_i)\}_{i=1}^{\infty}$ of disjoint 2-cell pairs that separate points of N^2—that is to say, for any two points $x, x' \in N^2$, there exists an integer $i \geq 1$ such that $x \in D_i$ and $x' \in E_i$. In the space $C(N^2, Y)$ of continuous functions from N^2 to Y, let

$$O_i = \{f \in C(N^2, Y) \mid f(D_i) \cap f(E_i) = \emptyset\}.$$

Clearly O_i is open in $C(N^2, Y)$. By the DDP and Estimated Homotopy Extension Theorem 0.6.4, each $f \in C(N^2, Y)$ can be approximated by some $f' \in C(N^2, Y)$ such that $f'(D_i) \cap f'(E_i) = \emptyset$; in other words, O_i is dense in $C(N^2, Y)$. The Baire Category Theorem assures that each $f \in C(N^2, Y)$ can be approximated by $\lambda \in \cap_i O_i$, an embedding.

Let S denote a closed 0-LCC subset of Y. Choose triangulations $T_1, T_2, \ldots$ of I^2 with $\text{mesh} T_i < 1/i$. Let L_i denote the 1-skeleton of T_i and

$$O'_i = \{f \in O_i \mid f(L_i) \cap S = \emptyset\}.$$

Since S is 0-LCC in Y, O'_i is an open dense subset of $C(N^2, Y)$. Each $\lambda \in \cap_i O'_i$ is an embedding for which $\lambda(N^2) \cap S \subset \lambda(N^2 \smallsetminus \cup_i L_i)$, a 0-dimensional set. □

A similar argument yields:

Lemma 7.4.10. *Suppose the space X is a union of locally compact ANRs Y_1 and Y_2, each Y_i is a closed subset of X and has the DDP, $S = Y_1 \cap Y_2$ has empty interior and is 0-LCC in Y_i, and any two maps $f_i : I^2 \to Y_i$ can be approximated, arbitrarily closely, by maps $F_i : I^2 \to Y_i$ such that $F_1(I^2) \cap F_2(I^2) = \emptyset$. Then S contains disjoint, 0-dimensional, σ-compact subsets Z_1, Z_2 such that any map from a compact 2-dimensional ∂-manifold N^2 to Y_i can be approximated by a map $g_i : N^2 \to Y_i$ with $g_i(N^2) \cap S \subset Z_i$.*

Lemma 7.4.11. *Let $p : M \to X$ be a closed, cell-like mapping from a connected n-manifold M onto a metric space X that contains a connected $(n-1)$-manifold S as a closed subset, where $X \smallsetminus S$ is disconnected. Then $X \smallsetminus S$ has precisely two components, and S is 0-LCC in the closure of each.*

Proof. Proposition 3.2.9 promises that p induces a cohomology isomorphism $p^* : H_c^{n-1}(S; \mathbb{Z}_2) \cong \mathbb{Z}_2 \to H_c^{n-1}(p^{-1}(S); \mathbb{Z}_2)$. The duality methods of §7.1 assure that $p^{-1}(S)$ separates M into at most two components, so it must separate into exactly two components, for otherwise $X \smallsetminus S$ would be connected. Hence, $X \smallsetminus S$ has exactly two components. Localization yields the 0-LCC conclusion. □

Proposition 7.4.12. *Let $p : M \to X$ be a cell-like map from an n-manifold M onto an ANR X that contains a connected $(n-1)$-manifold S, $n \geq 5$, as a closed subset, where $X \smallsetminus S$ is the disjoint union of components U_1, U_2 such that each $\overline{U}_j$ satisfies the Disjoint Disks Property and, moreover, any two maps $f_j : I^2 \to \overline{U}_j$ can be approximated, arbitrarily closely, by maps $F_j : I^2 \to \overline{U}_j$ such that $F_1(I^2) \cap F_2(I^2) = \emptyset$. Then X has the Disjoint Disks Property and p is a near-homeomorphism.*

Proof. Apply Lemma 7.4.10 to obtain disjoint 0-dimensional σ-compact sets $Z_1, Z_2 \subset S$ such that any map from a compact 2-dimensional ∂-manifold N^2 to $\overline{U}_j$ can be approximated by a map $g : N^2 \to \overline{U}_j$ with $g(N^2) \cap S \subset Z_j$.

Since S is an ANR there exists a neighborhood V of S in X and a retraction $r : V \to S$. Define retractions $r_j : V \cup U_j \to \overline{U}_j$ by $r_j|\overline{U}_j = \text{incl}$ and $r_j|V \smallsetminus U_j = r|V \smallsetminus U_j$. Choose a neighborhood V' of S so that $r|V'$ is homotopic to the inclusion in V. Choose a third neighborhood V'' of S so that the closure of V'' is contained in V'. By the proof of the Estimated Homotopy Extension Theorem (Theorem 0.6.4) there is a map $r' : X \to X$ such that $r'|V'' = r|V''$, $r'|X \smallsetminus V = \text{incl}$, and $r'(V) \subset V$. Define $r'' : X \to X$ by $r''|\overline{U}_j = r_i r'|\overline{U}_j$; then $r''|S = \text{incl}$, $r''(V'') \subset S$, and $r''(\overline{U}_j) = \overline{U}_j$. Observe that r'' can be made arbitrarily close to the identity.

Given maps $f_1, f_2 : I^2 \to X$, choose compact ∂-manifolds $A_i \subset I^2$ such that $f_i^{-1}(S) \subset A_i \subset f_i^{-1}(V'')$. Define $f_i' = r'' f_i$. Then $f_i'(A_i) \subset S$ and each component of $I^2 \smallsetminus A_i$ is mapped by f_i' into either $\overline{U}_1$ or $\overline{U}_2$. Add the components that are mapped to $\overline{U}_1$ to A_i and define B_i to be the union of the closures of the remaining components. Then A_i and B_i are compact boundary submanifolds of I^2 that satisfy the following conditions.

(1) $A_i \cup B_i = I^2$,

(2) $A_i \cap B_i \subset \partial A_i \cap \partial B_i$, and

(3) $f_i'(A_i) \subset \overline{U}_1$ and $f_i'(B_i) \subset \overline{U}_2$.

Since S is a PL manifold, we can make a further adjustment so that

(4) $f_i'(A_i \cap B_i) \subset S \setminus (Z_1 \cup Z_2)$ and

(5) $f_1'(A_1 \cap B_1) \cap f_2'(A_2 \cap B_2) = \emptyset$.

Invoking the hypothesis that U_i has the DDP, one can obtain further approximations f_i'' to f_i' satisfying analogs of conditions (1)–(5), as well as

(6) $f_i''|A_i \cap B_i = f_i'|A_i \cap B_i$,

(7) $f_i''(A_i) \subset \overline{U}_1$ and $f_i''(B_i) \subset \overline{U}_2$, and

(8) $f_1''(A_1) \cap f_2''(A_2) = \emptyset = f_1''(B_1) \cap f_2''(B_2)$.

Then by choice of Z_1, Z_2 one can produce yet another set of approximations F_i satisfying the analogs of (1)–(8), as well as

(9) $S \cap (F_1(\operatorname{Int} A_1) \cup F_2(\operatorname{Int} A_2)) \subset Z_1$ and

(10) $S \cap (F_1(\operatorname{Int} B_1) \cup F_2(\operatorname{Int} B_2)) \subset Z_2$.

It follows from the prearranged (see (4) and (6))

$$Z_1 \cap Z_2 = \emptyset = (Z_1 \cup Z_2) \cap (F_1(A_1 \cap B_1) \cup F_2(A_2 \cap B_2))$$

that $F_1(I^2) \cap F_2(I^2) = \emptyset$. □

Corollary 7.4.13. *Let $p : M \to X$ be a cell-like map defined on an n-manifold M, $n \geq 5$, onto a metric space X containing an $(n-1)$-manifold S embedded in M as a closed, 1-LCC subset, where $X \setminus S$ is an n-manifold. Then X is an n-manifold and p is a near-homeomorphism.*

Proof. Since S locally separates X and the desired conclusion is local, it suffices to consider the case where X and S are connected and $X \setminus S$ has two components, U_1 and U_2. As S is LCC^1 in $\overline{U}_i = S \cup U_i$ ($i \in \{1, 2\}$) and U_i is an n-manifold, $\overline{U}_i$ has the DDP and the hypotheses of 7.4.12 are satisfied. □

Proposition 7.4.14. *Let $p : S^n \to X$ be a cell-like map onto a metric space X that contains an $(n-1)$-sphere S, $n \geq 5$, and $X \setminus S$ is the disjoint union of components U, V where $\overline{U} = A$ embeds in S^n and $\overline{V} = B$ is an n-cell. Then X has the Disjoint Disks Property and p is a near-homeomorphism.*

Proof. As in Proposition 7.4.12, given maps $f_1, f_2 : I^2 \to X$, for $\epsilon > 0$ and $i = 1, 2$ produce ϵ-approximations f_i' to f_i and ∂-manifolds A_i, B_i in I^2 satisfying

(1) $I^2 = A_i \cup B_i$,

(2) $A_i \cap B_i = \partial A_i \cap \partial B_i$,

(3) $f_1'(A_1) \cup f_2'(A_2) \subset A$ and $f_1'(B_1) \cup f_2'(B_2) \subset B$, and

(4) $f_1'(B_1) \cap f_2'(B_2) = \emptyset$ (since B is an n-cell).

Temporarily regard A as a subset of S^n. Identify a neighborhood W of A and retraction $R : W \to A$ such that $R(W \setminus A) \subset \operatorname{Bd} A = S$. Restrict W so

R moves points less than ϵ. Then approximate $f_i'|A_i \to A \subset S^n$ by $g_i : A_i \to W$, with g_i ϵ-close to f_i', $g_i|A_i \cap B_i = f_i'|A_i \cap B_i$, and $g_1(A_1) \cap g(A_2) = \emptyset$.

Once again treating A as a subset of X, exploit the n-cell structure of B to adjust Rg_i to a map $g_i' : A_i \to X$ such that $\rho(g_i', g_i) < \epsilon$, $g_i'|g_i^{-1}(A) = g_i'|g_i^{-1}(A)$ and $g_i'(g_i^{-1}(W \smallsetminus A)) \subset \operatorname{Int} B$. Define $F_i : I^2 \to X$ as $F_i|A_i = g_i'$ and $F_i|B_i = f_i'|B_i$. Note $\rho(F_i, f_i) < 4\epsilon$. Intersections between $F_1(I^2)$ and $F_2(I^2)$ occur at points of $F_1(A_1) \cap F_2(A_2) \cap \operatorname{Int} B$ or of $F_i(A_i) \cap F_j(B_j) \subset \operatorname{Int} B$, $i \neq j$, and can be removed easily by general position. Hence, X has the DDP. $\square$

Historical Notes. R. D. Edwards outlined a proof of the Cell-like Approximation Theorem in his ICM 1978 article (Edwards, 1980). Details of Edwards's proof for $n \geq 6$ are presented in (Daverman, 1986); the 5-dimensional case is treated in (Daverman and Halverson, 2007). Corollary 7.4.2 is originally due to L. C. Siebenmann (1972); its analog in dimension $n = 3$ was done by S. Armentrout (1971) and in dimension $n = 4$ by M. H. Freedman and F. S. Quinn (1990).

Cannon (1978), (1979) introduced the Disjoint Disks Property and early on he conjectured its fundamental role for the Cell-like Approximation Theorem.

The hypothesis about finite-dimensional-image in the Cell-like Approximation Theorem is a necessary one. A. N. Dranishnikov (1989) established the existence of a cell-like map on a 3-dimensional compactum with infinite-dimensional image; this automatically gave a dimension-raising cell-like map defined on S^n, $n \geq 7$. Improving upon Dranishnikov's example slightly, J. Dydak and J. J. Walsh (1993) produced dimension-raising cell-like maps on 2-dimensional compacta and, hence, on S^5. In contrast, work of G. Kozlowski and J. J. Walsh (1983) certifies that cell-like maps defined on 3-manifolds have 3-dimensional images.

Exercise

7.4.1. If Y is a locally compact ANR with the DDP, then each map $f : I^2 \to Y$ can be approximated by a 1-LCC embedding.

7.5. Determining n-cells by embeddings of M_n^{n-1} in S^n

The combined aim of this section and the next is to characterize local flatness of codimension-one manifold embeddings in terms of the 1-LCC condition. Taking a step in that direction, this section establishes (Corollary 7.5.10) that, given an $(n-1)$-sphere $\Sigma \subset S^n$ and component W of $S^n \smallsetminus \Sigma$, $\overline{W}$ is an n-cell if and only if there is an embedded Menger continuum $e(M_n^{n-1})$ in S^n with $\Sigma \subset e(M_n^{n-1}) \subset \overline{W}$. Rounding this out, the next section demonstrates that for any 1-LCC embedded $(n-1)$-sphere $\Sigma \subset S^n$ and complementary domain W, there exists such an embedding $e : M_n^{n-1} \to \overline{W}$.

A secondary goal of the section at hand is a positional characterization of the $(n-1)$-dimensional Menger space in S^n. To that end, we present an *ad hoc* definition of objects called $\mathcal{S}$-curves, which include the standard Menger space M_n^{n-1}, and ultimately (Theorem 7.5.7) we show that any two such $\mathcal{S}$-curves are homeomorphic. This topological analysis of M_n^{n-1} is not essential to the primary purpose: all the lemmas developed in this section lead to 7.5.7 and can be ignored, provided one broadens Theorem 7.5.8 (using the same proof presented here) to detect the n-cell though embeddings of arbitrary $\mathcal{S}$-curves in S^n, not simply through embedded copies of M_n^{n-1}.

The starting point is an elementary result from decomposition theory. A sequence of sets $X_1, X_2, \ldots$ in a metric space is called a *null sequence* if $\operatorname{diam} X_i \to 0$ as $i \to \infty$.

Proposition 7.5.1 (Null sequence decompositions into flat n-cells)**.** *Let $B_1, B_2, \ldots$ be a null sequence of pairwise disjoint, flat n-cells in S^n, U an open subset of S^n containing $\cup_i B_i$, and G the decomposition of S^n having the sets B_i as nondegenerate elements. Then G is shrinkable fixing $S^n \setminus U$. In particular, there exists a surjective map $f : S^n \to S^n$ such that the nondegenerate point preimages of f are the cells $B_1, B_2, \ldots$ and $f|S^n \setminus U =$ Id.*

Proof. Consider any nondegenerate $g_0 \in G$. Since g_0 is flat, regard it as the standard ball of radius 1 centered at O in $\mathbb{R}^n = S^n \setminus \{\infty\}$. Given $\delta > 0$, restrict further, if necessary, so $B(g_0; \delta) \subset U$ and let k denote the smallest positive integer such that $k\delta/3 > 1$. We will produce a homeomorphism $\Theta : S^n \to S^n$ such that

(a) $\Theta|S^n \setminus B(g_0; \delta) = \text{Id}$,

(b) $\operatorname{diam} \Theta(g_0) < \delta$, and

(c) for $g \in G$ either $\operatorname{diam} \Theta(g) < \delta$ or $\Theta(g) = g$.

The homeomorphism Θ will be expressed as a composition $\Theta = \theta_{k-1} \cdots \theta_1$, of $k-1$ homeomorphisms, where each θ_j moves points less than $\delta/3$ and compresses $\theta_{j-1} \cdots \theta_1(g_0)$ radially into the ball of radius $(k-j)\delta/3$.

Using the nullity of the nondegenerate elements, require θ_1 to be the identity outside an open set $U_1 \subset U$ so near g_0 that all $g \in G$ meeting U_1 have diameter less than $\delta/3$. Similarly, after $\theta_{j-1}, \ldots, \theta_1$ have been defined, require θ_j to be the identity outside an open set $U_j \subset U_{j-1}$ so close to $\theta_{j-1} \cdots \theta_1(g_0)$ that, for any other $g \in G$ whose image under $\theta_{j-1} \cdots \theta_1$ meets U_j, $\operatorname{diam} \theta_{j-1} \cdots \theta_1(g) < \delta/3$.

For $i = -1, 0, 1, \ldots, k$, set $\alpha_i = (k-i)/k$. Each θ_j can be defined so as to have similar effect on all rays R emanating from O: there will be a positive number $\xi_j \in (\alpha_{j-1}, \alpha_{j-2})$ such that the segment of R of length ξ_j based at O lies in U_j. There will be four special points on R: the points P_j, Q_j, S_j, T_j at distances $\alpha_{j+1}, \alpha_j, \alpha_{j-1}, \xi_j$, respectively, from O; θ_j will move only those points between P_j and T_j, will send S_j to Q_j and will be linear on the intervals $[P_j, S_j]$ and $[S_j, T_j]$. Thus, each θ_j will move points at most $1/k < \delta/3$ and will compress $\overline{B(O;\alpha_{j-1})}$ to $\overline{B(O;\alpha_j)}$.

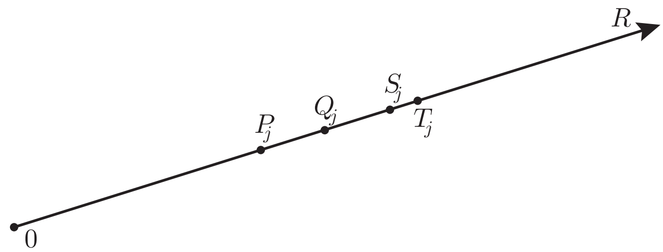

Figure 7.7. The action of θ_j on R

Accordingly, for $g \in G$, $g \neq g_0$, and $j \in \{2, 3, \dots, k-1\}$, either $\theta_j \cdots \theta_1(g) = \theta_{j-1} \cdots \theta_1(g)$ or $\operatorname{diam} \theta_j \cdots \theta_1(g) < \delta$; moreover, if after the jth compression, $\operatorname{diam} \theta_j \cdots \theta_1(g) \geq \delta/3$, then $\operatorname{diam} \theta_j \cdots \theta_1(g) < \delta$ and $\Theta(g) = \theta_j \cdots \theta_1(g)$. Finally, since $\alpha_{k-1} = 1/k < \delta/3$ and

$$\Theta(g_0) = \theta_{k-1} \cdots \theta_1(g_0) = \overline{B(O;\alpha_{k-1})} \subset B(O;\delta/3),$$

Θ shrinks g_0 to sufficiently small size. By construction Θ moves no point of $S^n \setminus U_1$.

Upon performing this shrinking in pairwise disjoint neighborhoods of each of the finitely many large n-cells in the collection G, we see that G itself is shrinkable fixing $S^n \setminus U$. $\square$

Remark. While the statement and proof of the last result are elementary, they are still quite delicate. For example, a decomposition into points and a null sequence of cellular arcs need not be shrinkable (Daverman and Walsh, 1982).

Corollary 7.5.2. *If $B_1, B_2, \dots$ is a null sequence of pairwise disjoint, flat n-cells in the interior of B^n and G is the decomposition of B^n having the sets B_i as nondegenerate elements, then the decomposition space B^n/G is an n-cell. Furthermore, if G_m is the decomposition of the ∂-manifold $D_m^n = B^n \setminus \cup_{i=1}^m \operatorname{Int} B_i$ having $B_{m+1}, B_{m+2}, \dots$ as nondegenerate elements, then the associated decomposition space is homeomorphic to D_m^n.*

Proof. Regard D_m^n as a subset of S^n. Apply Proposition 7.5.1 and Theorem 2.3.4 with $U = \operatorname{Int} D_m^n$. $\square$

We will make use, without proof, of the fundamental Annulus Theorem. This will be discussed more extensively in §8.8.

Theorem 7.5.3 (Annulus Theorem)**.** *Let B' denote a flat n-cell in the interior of an n-cell B $(n > 4)$. Then $B \setminus \operatorname{Int} B'$ is homeomorphic to $S^{n-1} \times I$.*

Let $B_1, \dots, B_k$ be pairwise disjoint, flat n-cells in the interior of an n-cell B. The ∂-manifold $B \setminus \cup_i \operatorname{Int} B_i$ is called an *n-cell with k holes.* The ∂-manifold D^n_m of Corollary 7.5.2 is a relevant example of an n-cell with m holes.

Corollary 7.5.4 (Generalized Annulus Theorem)**.** *If B^* and C^* are n-cells with k holes, $n > 4$, then every homeomorphism h from a component of ∂B^* to a component of ∂C^* extends to a homeomorphism $H : B^* \to C^*$.*

The proof is an exercise.

Definition. An $(n-1)$-*dimensional Sierpiński curve* is a compact metric continuum X which admits an embedding h in S^n such that the components of $S^n \setminus h(X)$ form a null sequence $U_1, U_2, \dots$ satisfying: (1) each $S^n \setminus U_i$ is an n-cell, (2) $\overline{U}_i \cap \overline{U}_j = \emptyset$ whenever $i \neq j$, and (3) $\overline{\cup_i U_i} = S^n$. For brevity we will say that a compact continuum $X' \subset S^n$ is an $\mathcal{S}$-curve if it is the image of an embedding $h : X \to S^n$, where X satisfies conditions (1)–(3).

The prototypical $\mathcal{S}$-curve is the standard Menger space M^{n-1}_n. The immediate goal is to prove that any two such $\mathcal{S}$-curves are topologically equivalent.

Lemma 7.5.5. *If X is an $(n-1)$-dimensional $\mathcal{S}$-curve in S^n, then for each $\epsilon > 0$ there is an embedding $e : X \to S^n$ such that $\rho(e, \mathrm{incl}_X) < \epsilon$ and the components of $S^n \setminus e(X)$ are bounded by flat $(n-1)$-spheres.*

Proof. For any component U of $S^n \setminus X$, $S^n \setminus U$ is an n-cell which can be re-embedded in its own interior so the image of $\partial(S^n \setminus U)$ is bicollared and, hence, flat. It follows almost automatically that the image of X under this re-embedding is an $\mathcal{S}$-curve. The re-embedding can be controlled to move points only a short distance and to have support very close to $\overline{U}$. Infinite repetition, with increasingly strict motion controls, yields the lemma. $\square$

Definitions. Say that an $(n-1)$-dimensional $\mathcal{S}$-curve $X \subset S^n$ is *special* if each of the components of $S^n \setminus X$ is bounded by a flat sphere. Let X be a special $(n-1)$-dimensional $\mathcal{S}$-curve in S^n and $U_0, U_1, U_2 \dots$ the components of $S^n \setminus X$. A *subdivision* of X is a division of X into a finite number of such $\mathcal{S}$-curves, brought about by taking a simplicial subdivision T of the compact ∂-manifold R obtained by adding to X all but a finite number $U_0, \dots, U_m$ of its complementary domains in such a way that the $(n-1)$-skeleton of T lies entirely in X, contains the boundary of R, and does not meet the boundary of any component of $S^n \smallsetminus X$ other than $U_0, \dots, U_m$. The intersection of the n-cells of T with X gives a collection of $(n-1)$-dimensional $\mathcal{S}$-curves. The subdivision is said to have *mesh less than* ϵ if each n-cell in T has diameter less than ϵ.

Lemma 7.5.6. *Suppose X and Y are special $(n-1)$-dimensional $\mathcal{S}$-curves in S^n $(n \neq 4)$, U and V are components of $S^n \smallsetminus X$ and $S^n \smallsetminus Y$, respectively, h is any homeomorphism of* $\operatorname{Bd} U$ *onto* $\operatorname{Bd} V$, *and $\epsilon > 0$. Then there exist ϵ-subdivisions of X and Y whose $(n-1)$-dimensional skeleta correspond under a homeomorphism h' that extends h.*

Proof. List the components $U_0 = U, U_1, U_2, \dots$ of $S^n \smallsetminus X$ and, similarly, the components $V_0 = V, V_1, V_2, \dots$ of $S^n \smallsetminus Y$. Choose an integer $m > 0$ such that all U_i and V_i, $i > m$, have diameter less than ϵ. Form the decomposition space A_X of $S^n \smallsetminus \cup_{i=0}^m U_i$ determined by the nondegenerate elements $\overline{U}_{m+1}, \overline{U}_{m+2}, \dots$ and, similarly, the decomposition space A_Y of $S^n \smallsetminus \cup_{i=0}^m V_i$ determined by $\overline{V}_{m+1}, \overline{V}_{m+2}, \dots$. Let $\pi_X : S^n \smallsetminus \cup_{i=0}^m Ui \to A_X$ and $\pi_Y : S^n \smallsetminus \cup_{i=0}^M Vi \to A_Y$ denote the associated decomposition maps. Here A_X and A_Y are n-cells with holes—an equal number of holes, by design. Corollary 7.5.4 assures that the homeomorphism $\pi_Y h(\pi_X)^{-1} : \pi_X(\operatorname{Bd} U) \to \pi_Y(\operatorname{Bd} V)$ extends to a homeomorphism $H : A_X \to A_Y$.

For each $\delta > 0$ there exists a simplicial triangulation T of W' of mesh less than δ whose $(n-1)$-skeleton Σ intersects none of the countably many points having nondegenerate preimages under either $H\pi_X$ or π_Y. Then the sets $K = (H\pi_Z)^{-1}(\Sigma)$ and $K' = (\pi_Y)^{-1}(\Sigma)$ each correspond in 1-1 fashion with Σ. Moreover, when σ is an n-simplex of T, then $(\pi_Y)^{-1}(\partial\sigma) \subset Y$ and $(H\pi_X)^{-1}(\partial\sigma) \subset X$ are flat $(n-1)$-spheres in S^n, by Corollary 7.4.3 to the Cell-like Approximation Theorem; as a result, K and K' effect subdivisions of X and Y, respectively. Since point preimages under $H\pi_X$ and π_Y have diameter less than ϵ, one can choose T of sufficiently small mesh that K and K' have mesh less than ϵ. The desired homeomorphism $h' : K \to K'$ can be defined as the restriction of $(\pi_Y)^{-1} H\pi_X$. □

Theorem 7.5.7. *Any two $(n-1)$-dimensional $\mathcal{S}$-curves in S^n are homeomorphic.*

Proof. Consider any two special $\mathcal{S}$-curves X and Y in S^n. For $i = 1, 2, \dots$ Lemma 7.5.6 promises an embedding $e_i : X \to B(Y; 1/i)$ such that $Y \subset B(e_i(X); 1/i)$. These embeddings submit to controls ensuring that $\{e_i\}$ forms a Cauchy sequence. Moreover, given any two points $x_1, x_2 \in X$ there exist disjoint n-cells $C_1, C_2 \subset S^n$ such that $e_j(x_1) \in C_1$ and $e_j(x_2) \in C_2$ for sufficiently large j. Hence, the sequence $\{e_i\}$ converges to a homeomorphism $X \to Y$. $\square$

Theorem 7.5.8. *Let e denote an embedding of the $(n-1)$-dimensional Menger space M_n^{n-1} into S^n $(n \geq 5)$ and V a component of $S^n \smallsetminus e(M_n^{n-1})$. Then $S^n \smallsetminus V$ is an n-cell.*

Proof. List the components $U_1, U_2, \ldots$ of $S^n \smallsetminus M_n^{n-1}$ and also the components $V = V_1, V_2 \ldots$ of $S^n \smallsetminus e(M_n^{n-1})$. Choose these indices so that $\operatorname{Bd} V_i = e(\operatorname{Bd} U_i)$ for each i.

Every $\overline{V}_i$ is contractible, since it is a simply connected, homologically trivial (by duality) ANR.

Examine the decompositions G and G' of $S^n \smallsetminus U_1$ and $S^n \smallsetminus V_1$ having $\overline{U}_2, \overline{U}_3, \ldots$ and $\overline{V}_2, \overline{V}_3, \ldots$ as their respective nondegenerate elements. Both G and G' are cell-like, upper semicontinuous decompositions; the nondegenerate elements of G form a null sequence of flat n-cells in the interior of the n-cell $S^n \smallsetminus U_1$. Let $\varphi : S^n \smallsetminus U_1 \to A$ and $\varphi' : S^n \smallsetminus V_1 \to A'$ denote the associated decomposition maps. Obviously there is a unique homeomorphism $e' : A \to A'$ such that the following diagram is commutative:

$$\begin{tikzcd} M_n^{n-1} \arrow[r, "\subset"] \arrow[d, "e"] & S^n \smallsetminus U_1 \arrow[r, "\varphi"] & A \arrow[d, "e'"] \\ e(M_n^{n-1}) \arrow[r, "\subset"] & S^n \smallsetminus V_1 \arrow[r, "\varphi'"] & A' \end{tikzcd}$$

By Corollary 7.5.2, A is an n-cell, so A' is an n-cell as well. A minor modification of Corollary 7.4.3 provides a homeomorphism $\Phi' : S^n \smallsetminus V_1 \to A'$ that agrees with φ' on $\operatorname{Bd} V_1$. Hence, $S^n \smallsetminus V_1$ is also an n-cell. $\square$

Corollary 7.5.9. *For any embedding $e : M_n^{n-1} \to S^n$ $(n \geq 5)$ of the $(n-1)$-dimensional Menger space, $e(M_n^{n-1})$ is an $\mathcal{S}$-curve.*

Corollary 7.5.10. *Let $\lambda : \partial I^n \to S^n$ $(n \geq 5)$ be an embedding and W a component of $S^n \smallsetminus \lambda(\partial I^n)$. Then $\overline{W}$ is an n-cell if and only if λ can be extended to an embedding $\Lambda : M_n^{n-1} \to \overline{W}$.*

Historical Notes. The positional characterization of Sierpiński curves in S^n presented in Theorem 7.5.7 is due to Cannon (1973b), who based his argument on that given by G. T. Whyburn for the 2-dimensional case.

The proof of the Annulus Conjecture was a sweeping breakthrough, by Kirby (1969); the key idea, usually referred to as the torus trick, had profound implications, including the deep analysis of PL and DIFF structures of manifolds (Kirby and Siebenmann, 1977). More about the momentous importance of tori comes up in Chapter 8.

Proposition 7.5.1, in more general form, was proved by R. J. Bean (1967), who credited Bing for the technique.

Exercises

7.5.1. Let G denote an upper semicontinuous decomposition of an n-dimensional ∂-manifold N such that for each nondegenerate element $g_0 \in G$ and each $\epsilon > 0$ there exists an n-cell B with $g_0 \subset B \subset B(g_0; \epsilon)$, where all nondegenerate elements of G that meet B lie in $\operatorname{Int} B$. Then G is shrinkable.

7.5.2. Show that every homeomorphism between two special $\mathcal{S}$-curves in S^n extends to a homeomorphism of S^n.

7.5.3. Prove Corollary 7.5.4.

7.6. The 1-LCC characterization of local flatness

All the groundwork now has been laid for the foundational characterization, in Theorem 7.6.1 below, of locally flat codimension-one manifold embeddings in terms of the 1-LCC condition. The initial steps reduce the issue to the 1-LCC characterization of flat codimension-one spheres in S^n, which is treated in Theorem 7.6.5; its proof, in turn, capitalizes on the Menger space technology of the preceding section (Corollary 7.5.10).

Theorem 7.6.1. *Every 1-LCC embedding of an $(n-1)$-manifold S in an n-manifold M $(n \geq 5)$ is locally flat.*

No hypothesis about M being PL is needed here; all constructions can be localized to Euclidean patches in M.

The indispensable tool is the following Bubble Lemma. Its proof retraces that of 1-LCC Push-off Proposition 7.2.1, using infinite controlled engulfing.

Lemma 7.6.2 (Bubble Lemma)**.** *Suppose S is an $(n-1)$-manifold in a connected PL n-manifold M $(n \geq 5)$, $D \subset S$ is an $(n-1)$-cell such that S is 1-LCC at each point of* $\operatorname{Int} D$*, U is a component of $M \smallsetminus S$ and $\epsilon : \operatorname{Int} D \to (0,1)$ is a continuous function. Then there exists a 1-LCC embedding $e : \operatorname{Int} D \to U$ such that $d(s, e(s)) < \epsilon(s)$ for all $s \in \operatorname{Int} D$.*

Proof. The embedding e will be $\psi|\operatorname{Int} D$, where ψ is a controlled push of M such that $\psi(\operatorname{Int} D) \subset U$ and $\psi|\partial D = \mathrm{incl}_{\partial D}$. The existence of this ψ stems from an engulfing program establishing that infinite codimension-three complexes near $\operatorname{Int} D$ can be pushed into a preassigned component of $M \smallsetminus S$.

Determine a small connected neighborhood W of $\operatorname{Int} D$ such that W intersects S at $\operatorname{Int} D$, and let W_+ be a component of $W \smallsetminus D$. The claim is that, for $k \leq n-3$ and a sufficiently small neighborhood $W' \subset W$ of $\operatorname{Int} D$, any infinite k-complex in W' admits a controlled push φ into W_+, where $\varphi|M \smallsetminus W = \text{incl}|M \smallsetminus W$. To complete the argument, one works with a PL neighborhood N of $\operatorname{Int} D$ in W', pushes the $(n-3)$-skeleton of N to one side of $W \smallsetminus D$, pushes the dual 2-skeleton of N to the other side of $W \smallsetminus D$, and stretches across the join structure of N to obtain ψ, just as in 7.2.1. Of course, controls on the pushes and, more automatically, on the stretch are necessary to assure that all three adjustments operating in W extend over the rest of M via the Identity.

One way to nail down the engulfing claim is to produce ∂-manifolds A and B whose interiors cover $\operatorname{Int} D$ and whose components are compact, and then to use the usual controlled engulfing methodology, applied component by component, to obtain that complexes near either A or B can be pushed into W_+ with control. Given a k-complex K in W', express it as a union of closed subpolyhedra K_A and K_B, where K_A, K_B are near A, B, respectively. Push K_A into W' with enough control that image of K_B is still near B. Then push that image into W', fixing the image of K_A. Details are left to the reader. □

Lemma 7.6.3. *Suppose the $(n-1)$-sphere $\Sigma \subset S^n$ is the union of two $(n-1)$-cells D and D' such that $\partial D = D \cap D' = \partial D'$, D is 1-LCC in S^n and $\operatorname{Int} D'$ is 1-LCC in S^n. Then Σ is 1-LCC in S^n.*

Proof. Focus on $s \in \partial D$; the conclusion is obvious for other points of Σ. Given any neighborhood N_1 of s, find a smaller neighborhood N_2 such that $N_2 \cap (D' \smallsetminus D)$ is simply connected (and connected). Use the hypothesis about D being 1-LCC in S^n to locate another neighborhood $N_3 \subset N_2$ such that all loops in $N_3 \smallsetminus D$ are null-homotopic in $N_2 \smallsetminus D$.

Hence, each loop $f : \partial I^2 \to N_3 \smallsetminus \Sigma$ extends to a map $F_1 : I^2 \to N_2 \smallsetminus D$. Let Z denote the component of $I^2 \smallsetminus (F_1)^{-1}(D')$ containing ∂I^2. Since $D' \cap N_2$ is an ANR, $F_1|\operatorname{Fr} Z$ extends to a map sending a small neighborhood of $\operatorname{Fr} Z$ in $I^2 \smallsetminus Z$ into $D' \cap N_2$. Thus, there exist a compact, connected ∂-manifold Q, $\partial I^2 \subset Z \subset Q \subset I^2$, and map $F_2 : Q \to N_2 \smallsetminus D$ such that $F_2|Z = F_1|Z$ and $F_2(Q \smallsetminus Z) \subset N_2 \cap (D' \smallsetminus D)$. The connectedness of Q implies each component of $I^2 \smallsetminus Q$ is bounded by a simple closed curve, so by the simple-connectedness of $N_2 \cap (D' \smallsetminus D)$, $F_2|Z$ extends to a map $F_3 : I^2 \to N_2 \smallsetminus D$ with $(F_3)^{-1}(\Sigma) = I^2 \smallsetminus Z$.

As $N_2 \cap D'$ is two-sided in N_2 (Corollary 7.1.7), choose a connected open set $U_{D'}$ such that $N_2 \cap D' \subset U_{D'} \subset N_2$ and $U_{D'} \smallsetminus D'$ has two components; $N_2 \cap D'$ is LCC^1 in the closure (rel N_2) of each of these components, by Proposition 7.1.11. Cover $(F_3)^{-1}(D')$ by another compact ∂-manifold $Q' \subset (F_3)^{-1}(U_{D'}) \cap \operatorname{Int} I^2$.

We claim that the image of each component C of Q' under F_3 meets the closure of only one component of $U_{D'} \smallsetminus D'$. To examine that image, let Z^c denote $I^2 \smallsetminus Z$. By duality,

$$\check{H}^1(Z^c) \cong H_1(\operatorname{Int} I^2, \operatorname{Int} Z) \cong \widetilde{H}_0(\operatorname{Int} Z) \cong 0.$$

Since Z^c splits into the disjoint union of the compact sets $Z^c \cap C$ and $Z^c \smallsetminus C$, $\check{H}^1(Z^c \cap C) \cong 0$. Consequently,

$$0 \cong \check{H}^1(Z^c \cap \operatorname{Int} C) \cong H_1(\operatorname{Int} C, \operatorname{Int} C \smallsetminus Z^c) \to \widetilde{H}_0(\operatorname{Int} C \smallsetminus Z^c) \to 0,$$

so $\operatorname{Int} C \smallsetminus Z^c$ is connected, and its image under F_3 meets exactly one of the components of $U_{D'} \smallsetminus D'$. Applying Lemma 3.3.3 to $F_3|C$ for each C, we obtain a map $F_4 : I^2 \to N_2 \smallsetminus \Sigma$ such that $F_4|I^2 \smallsetminus Q' = F_3|I^2 \smallsetminus Q'$ (and $F_4(C) \subset U_{D'} \smallsetminus D'$); in particular, $F_4|\partial I^2 = f$. $\square$

Corollary 7.6.4. *Suppose S is an $(n-1)$-manifold 1-LCC embedded in an n-manifold M $(n \geq 5)$ and $s \in S$. Then there exist a neighborhood N_s of s in M and a 1-LCC embedded $(n-1)$-sphere $\Sigma \subset N_s$ such that $N_s \approx \mathbb{R}^n$ and $\Sigma \cap S$ contains a neighborhood of s in S.*

As a result, Theorem 7.6.1 reduces to the following:

Theorem 7.6.5. *An $(n-1)$-sphere Σ in S^n $(n \geq 5)$ is flat if and only if it is 1-LCC embedded.*

Proof. Let $\lambda : \partial I^n \to \Sigma$ be a homeomorphism and W a component of $S^n \smallsetminus \Sigma$. The goal will be to extend λ to an embedding $e : M_n^{n-1} \to \overline{W}$ and to apply Corollary 7.5.10.

Let $\kappa = \{k_1, k_2, \dots\}$ be a sequence of integers, $k_i \geq 3$. Associated with κ is an $(n-1)$-dimensional $\mathcal{S}$-curve, X_κ, constructed in a manner modelled on that of M_n^{n-1} in §3.5. Let T_0 be the trivial subdivision of I, just as in that construction. Let T_1 be the subdivision of I into k_1 subintervals of equal length. Assuming T_j to be a subdivision of I into intervals of equal length $1/k_1 \cdots k_j$, let T_{j+1} denote the subdivision obtained by sectioning each interval of T_j into k_{j+1} subintervals of equal length. As a result, T_{j+1} induces a subdivision T_{j+1}^n of I^n into a multitude of isometric subcubes. Set $P_0 = I^n$ and let P_{j+1} denote the union of all n-dimensional subcubes of T_{j+1}^n that lie in P_j and intersect its $(n-1)$-skeleton L_j (as determined by T_j^n). Then $X_\kappa = \cap_j P_j$.

By definition $X_\kappa = M_n^{n-1}$ in the special case $\kappa = \{3, 3, 3, \dots\}$.

For $j = 1, 2, \dots$ there exists a retraction $r_j : P_j \to L_{j-1}$ (since P_j fills no n-cube of T_{j-1}^n), where r_j moves no point more than d_{j-1}, the diameter of the n-cubes from T_{j-1}^n.

The proofs of the next two results are based on routine inverse limit arguments.

Lemma 7.6.6. *The inverse limit of the sequence* $\{L_j, r_j|L_j\}$ *is* X_κ.

Lemma 7.6.7. *Suppose* $\{L_j, r_j|L_j\}$ *as above, and suppose* $\{\lambda_j : L_j \to S^n\}$ *is a sequence of embeddings such that* $\lambda_{j+1}|L_j = \lambda_j$ *for all* $j \geq 1$. *Suppose also that for each* $\epsilon > 0$ *there exists an index* m_ϵ *such that diam* $\lambda_j(\partial\sigma) < \epsilon$ *for all* $j \geq m_\epsilon$ *and all* n*-cells* $\sigma \subset P_j$ *in* T_j^n. *Then there exists an embedding* $\Lambda : X_\kappa \to S^n$ *such that* $\Lambda|L_j = \lambda_j$ *for all* $j \geq 0$.

Continuing with the proof of 7.6.5, we choose an integer $k_1 \geq 3$ such that, for the subdivision T_1 of I into k_1 intervals of equal length and T_1^n the associated subdivision of I^n determined by the product of n copies of T_1, we have $\operatorname{diam} \lambda(\sigma \cap L_0) < 1/(3^{n-1} \cdot 2) = \delta_1$ for all $\sigma \in T_1^n$. Order the n-cubes $\sigma_1, \sigma_2, \ldots, \sigma_{m(1)}$ of T_1^n in P_1 so that $\partial\sigma_i \cap L_0^{i-1}$ is a PL $(n-1)$-cell E_i, where $L_0^{i-1} = L_0 \cup \cup_{t=1}^{i-1} \partial\sigma_t$. We will apply Lemma 7.6.2 recursively to extend λ to an embedding $\lambda : L_0^i \to \overline{W}$ such that $\operatorname{diam} \lambda(\partial\sigma_i) < 1/2$ for all σ_i. In addition, we will have assured that $\lambda(E_i)$ is 1-LCC, so $\lambda(\partial\sigma_i)$ will be 1-LCC by Lemma 7.6.3. At the end of the process, when $i = m(1)$, we will have an embedding $\lambda_1 : L_1 \to \overline{W}$ with $\lambda_1(\partial\sigma_i)$ being a 1-LCC embedded sphere and $\operatorname{diam} \lambda_1(\partial\sigma_i) < 1/2$ for $i = 1, 2, \ldots, m(1)$.

For the promised assurance that $\lambda(E_i)$ is 1-LCC, we will employ the following variation on Lemma 7.6.3. The proof is an exercise.

Lemma 7.6.8. *Suppose the* $(n-1)$*-cell* $E \subset S^n$ *is the union of two* $(n-1)$*-cells* E' *and* E'' *such that* $E' \cap E''$ *is an* $(n-2)$*-cell in the boundary of each,* E *is 1-LCC in* S^n *and* $E \setminus E'$ *is 1-LCC in* S^n. *Then* E *is 1-LCC in* S^n.

The recursive process runs as follows. Start with $S_0 = \lambda(\partial I^n)$. We produce $(n-1)$-spheres S_i $(i = 1, \ldots, m(1))$ and extensions of λ over L_i so $S_i \subset \lambda(L_i)$ using the Bubble Lemma. It gives approximations $\lambda|\operatorname{Int} E_i$ by new 1-LCC embeddings λ_i' with image very close to $\lambda(E_i)$; since each $\partial\sigma_i \setminus \operatorname{Int} E_i$ is a PL $(n-1)$-cell E_i', $\lambda|\partial E_i'$ extends to a homeomorphism (still called λ) of E_i' to the closure of $\lambda_i'(\operatorname{Int} E_i)$. The sphere S_i is obtained from S_{i-1} by replacing $\lambda(E_i)$ with $\lambda(E_i') = \lambda_i(E_i')$. Let W_j denote the component of $S^n \setminus S_j$ contained in W. In the successive applications of the Bubble Lemma require that $\lambda_i'(E_i) \subset W_i$. Inductively assuming that $\lambda(L_{i-1}) \cap W_{i-1} = \emptyset$, we see that $\lambda|L_i = L_{i-1} \cup E_i'$ is 1-1 and $\lambda(L_i) \cap W_i = \emptyset$, as required.

To control sizes of the $\lambda(\sigma_j)$, one can partition the cubes σ_i into n pairwise disjoint groups. Begin with the cubes from the first group, proceed to those in the second group, and so on. One way to obtain an acceptable grouping is to insist that k_1 be odd and to partition the intervals of T_1 into two pairwise disjoint groups, designated as black and white, with the two intervals containing the endpoints of I being black. The first group of cubes from T_1^n consists of all products of black intervals; generally, the jth group ($j > 1$) consists of all n-cubes from T_1^n expressed as a product involving exactly $j-1$ white intervals. There are exactly n such groups, not $n+1$, since no product determined by n white intervals meets L_0. When using the Bubble Lemma with the first group, insist that $\operatorname{diam} \lambda(\sigma_i) < 1/(3^{n-1} \cdot 2) = \delta_1$. Consequently, when treating cubes from the second group, we see that $\lambda(E_i)$ has diameter at most $3\delta_1$, and we can extend λ over E_i' so $\operatorname{diam} \lambda(\sigma_i) < 3\delta_1$. Each time we progress from one group to the next the diameters of the $\lambda(E_i)$ can triple. Thus, at the end, the extension λ satisfies $\operatorname{diam} \lambda(\sigma_i) < 3^{n-1} \cdot \delta_1 = 1/2$ for all i.

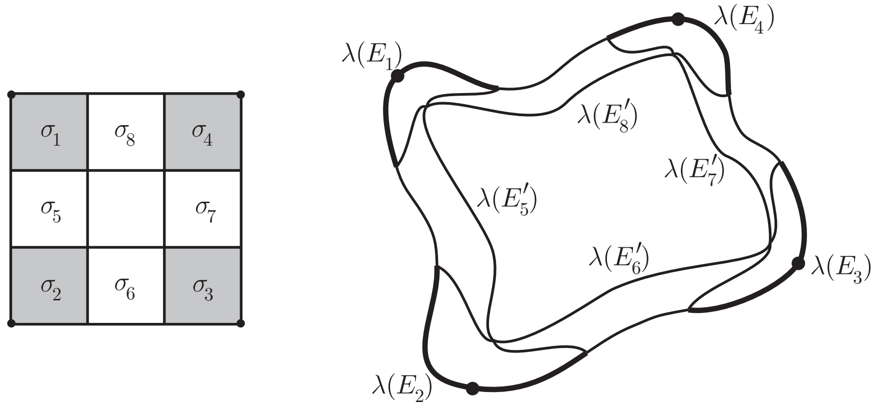

Figure 7.8. P_1 and $\lambda(L_1)$ for $n = 2$, $k_1 = 3$

The next step simply repeats this procedure, except that the role of $\lambda|\partial I^n$ now is taken over successively by $\lambda_1|\partial\sigma_i$, and the role of W by the component of $S^n \setminus \lambda(\partial\sigma_i)$ contained in W. Critical size control is imposed by choosing an integer $k_2 \geq 3$ such that for the subdivision T_2 of I into k_1k_2 intervals of equal length and T_1^n the associated iterated product subdivision of I^n, we have $\operatorname{diam} \lambda_1(\sigma \cap L_1) < 1/(3^{n-1} \cdot 4)$ for all $\sigma \in T_2^n$. Fix an n-cube C_j of T_1^n in P_1 and order the n-cubes $\sigma_1, \sigma_2, \dots, \sigma_{m(2)}$ of T_2^n in C_j so that $\partial\sigma_i \cap C_jL_1^{i-1}$ is an $(n-1)$-cell, where $C_jL_1^{i-1} = \partial C_j \cup \cup_{t=1}^{i-1} \partial\sigma_i$. Apply Lemma 7.6.2 to extend λ_1 to an embedding $\lambda_1 : C_jL_1^i \to \overline{V}$ such that each $\lambda_1(\partial\sigma_i)$ is a 1-LCC embedded $(n-1)$-sphere of diameter less than 1/4. At the end of the second stage of this process we have an extension of λ_1 to an embedding $\lambda_2 : L_2 \to \overline{V}$ such that $\lambda_2(\partial\sigma)$ is a 1-LCC embedded sphere of diameter less than 1/4 for all n-cubes $\sigma \subset P_2$ from T_2^n.

Continue in the same way, thereby generating infinite sequences $\kappa = \{k_1, k_2, k_3, \dots\}$ of positive integers, with each $k_i \geq 3$, $\{T_1, T_2, T_3, \dots\}$ of subdivisions of I, with T_i determining $k_1k_2 \cdots k_i$ equal length subintervals, and $\{\lambda_i : L_i \to \overline{W}\}$ of embeddings such that $\lambda_{i+1}|L_i = \lambda_i$ and, for each

n-cube $\sigma \subset P_i$ from T_i^n, $\lambda_i(\sigma)$ is a 1-LCC embedded sphere of diameter less than $1/2^i$.

Lemma 7.6.7 furnishes an embedding $\Lambda : X_\kappa \to \overline{W}$ of the associated $\mathcal{S}$-curve X_κ such that $\Lambda|\partial I^n = \lambda|\partial I^n$. By Theorem 7.5.7, X_κ is topologically equivalent to the standard Menger space M_n^{n-1}, and then Corollary 7.5.10 assures that $\overline{W}$ is an n-cell. Finally, since W could be either component of $S^n \setminus \Sigma$, Σ is flat. □

Corollary 7.6.9. *Every 1-LCC embedding of an $(n-1)$-dimensional ∂-manifold S in an n-manifold M $(n \geq 5)$ is locally flat.*

Proof. The Bubble Lemma and Lemma 7.6.3 indicate any sufficiently small $(n-1)$-disk D in S lies on a 1-LCC embedded $(n-1)$-sphere Σ_D that lives in a coordinate chart of M, and D can be assumed to be standardly embedded in Σ_D. □

Corollary 7.6.10. *Every $(n-1)$-cell $E \subset S^n$ as in Lemma 7.6.8 is flat.*

Corollary 7.6.11. *An $(n-1)$-cell E in S^n $(n \geq 5)$ is flat if and only if E is 1-LCC at points of* $\operatorname{Int} E$ *and ∂E is locally homotopically unknotted.*

Historical Notes. The approach to Theorem 7.6.1 presented here was developed by Černavskiĭ (1973). Another argument was given by Daverman (1973b).

The 3-dimensional analogs of the main results of this section were developed by Bing (1961b), and the 4-dimensional analogs were produced by Freedman and Quinn (1990).

On a topic related to Corollary 7.6.10, Černavskiĭ (1967) showed the union of two locally flat $(n-1)$-cells that intersect in an $(n-2)$-cell standardly embedded in the boundary of each to be flat itself, and Kirby (1968b) did the same for $n = 4$. Earlier, P. H. Doyle (1960) established the 3-dimensional version. More recently, Černavskiĭ (2006) provided a new proof of the result.

Exercises

7.6.1. Show that if $\sigma_i \in T_1^n$ in the proof of 7.6.5 belongs to the $(j+1)$st group and contains no point with coordinate 0 or 1, then E_i is congruent to a rescaled version of $[I^j \cup (\partial I^j \times I)] \times I^{n-j}$.

7.6.2. Prove Lemma 7.6.8.

7.6.3. Prove Corollary 7.6.11.

7.7. Locally flat approximations

Theorem 7.7.1 (Locally Flat Approximation). *Let M be an n-manifold $(n \geq 5)$, Q an $(n-1)$-manifold topologically embedded in M as a closed* *subset and $\epsilon : Q \to (0,1)$ a continuous function. Then there exists a locally flat embedding $\lambda : Q \to M$ such that $\rho(\lambda(q), q) < \epsilon(q)$ for all $q \in Q$.*

To establish this foundational result, we will follow the lead of Ancel and Cannon (1979) who exploited a notion of embedding relation originally introduced in (Cannon, 1975). Embedding relations involve considerable new terminology, which will be laid out in what immediately follows, to provide context for statements of some forthcoming rather technical results. The conclusion of this section contains a brief appendix in which basic properties of embedding relations are developed.

A *relation* $R : X \to Y$ is simply a subset of $X \times Y$ whose projection to the first factor is surjective; in other words, R is a multi-valued function. Its *image* $R(X)$ has the usual meaning, and its *inverse* $R^{-1} : R(X) \to X$ is the relation $\{\langle y, x\rangle \in R(X) \times X \mid \langle x, y\rangle \in R\}$. A relation $R : X \to Y$ is *continuous* if for each closed subset C of Y, $R^{-1}(C)$ is closed in X (expecting inverses of open sets to be open would be unreasonable), and it is an *embedding relation* if any two of its point images are disjoint.

For simplicity, assume that all spaces are locally compact, separable and metrizable. A relation $R : X \to Y$ is *proper* if both R and R^{-1} are continuous with compact point images.

Given a metric space (X, d) and $\epsilon > 0$, (ϵ) denotes the relation $(\epsilon) : X \to X$ defined as $\{\langle x, y\rangle \in X \times X \mid d(x, y) < \epsilon\}$. When such expressions appear as terms in a composition of relations, we occasionally omit the parentheses to shorten the formulae. It should be noted that, if $R : X \to Y$ is a relation, even when the metric on $X \times Y$ is, say, the sum of the metrics on its factors, $(\epsilon) \circ R \subset X \times Y$ is not identical to the ϵ-neighborhood of R but, instead, is a subset. Also, given two functions $f, f' : Y \to X$, $\rho(f, f') < \epsilon$ if and only if $f' \subset (\epsilon) \circ f$ as relations.

A relation $R : X \to Y$ is 1-LCC if for each $x \in X$ and neighborhood U of $R(x)$ in Y there exists a neighborhood $V \subset U$ of $R(x)$ such that loops in $V \setminus \operatorname{Im} R$ are null-homotopic in $U \setminus \operatorname{Im} R$. Quite obviously, when R is an embedding relation, $R(X)$ is closed in Y and $\pi : Y \to Y/R$ is the quotient map determined by the decomposition of Y into singletons from $Y \setminus R(X)$ and the sets $\{R(x) \mid x \in X\}$, then R is 1-LCC if and only if $\pi R(X)$ (which is an embedded copy of X) is a 1-LCC subset of Y/R, in the usual sense.

A *cell-like embedding relation* is a proper embedding relation $R : X \to Y$ from a locally compact metric space X to an ANR Y such that the point images under R are non-empty, disjoint, cell-like sets. By definition, continuous embedding relations between compact ANRs are necessarily proper if point images are cell-like. Our attention will focus on cell-like embedding relations $R : S^{n-1} \to S^n$; it follows easily that then $R(S^{n-1})$ is compact and $R^{-1} : R(S^{n-1}) \to S^{n-1}$ is a genuine cell-like mapping. It is perfectly appropriate to regard a cell-like embedding relation simply as the inverse of a cell-like mapping. Approximation of one embedding relation by another permits the domain of the inverse cell-like map to change, in a controlled way, while preserving the target. In the strategy employed here, given a cell-like embedding relation $R : S^{n-1} \to S^n$, one will find a better approximating relation R'; the associated image $R'(S^{n-1})$ then will be close, in a reasonably rich sense, to $R(S^{n-1})$ and both images will admit cell-like maps (R^{-1} and $(R')^{-1}$) to S^{n-1}. Ultimately, upon passage to a limit, the resulting cell-like embedding relation will admit an approximating 1-LCC embedding of S^{n-1}, which will be flat, by Theorem 7.6.1.

In partial compensation for the introduction of unfamiliar concepts, we will address only the most familiar case of 7.7.1: an embedding of the $(n-1)$-sphere in S^n. Restated in the language of embedding relations, the precise aim of §7.7 is to establish the following 1-LCC variation of Theorem 7.7.1.

Theorem 7.7.2 (1-LCC Approximation of Relations)**.** *Suppose $R : S^{n-1} \to S^n$ $(n \geq 5)$ is a cell-like embedding relation and L is a neighborhood of R in $S^{n-1} \times S^n$. Then L contains a 1-LCC cell-like embedding relation $R' : S^{n-1} \to S^n$.*

Given a cell-like embedding relation $R : S^{n-1} \to S^n$, we will denote by $\pi_R : S^n \to S^n/R$ the quotient map associated with the decomposition of S^n into the sets $R(x)$, $x \in S^{n-1}$, and the singletons of $S^n \smallsetminus R(S^{n-1})$. A central difficulty is that, generally, S^n/R need not be a manifold.

Corollary 7.7.3. *Under the hypotheses of Theorem 7.7.2, L contains a locally flat embedding $\lambda : S^{n-1} \to S^n$.*

Proof. Given $R : S^{n-1} \to S^n$ and L, use 7.7.2 to obtain a 1-LCC embedding relation $R' : S^{n-1} \to S^n$ in L. Find $\epsilon > 0$ such that the ϵ-neighborhood N_ϵ of R' is a subset of L. Proposition 7.4.13 assures that S^n/R' has the DDP and, hence, in view of the Cell-like Approximation Theorem, that the decomposition induced by R' is shrinkable. Choose $\delta > 0$ such that

$$(\pi')^{-1} \circ (\delta) \circ (\pi R') \subset N_\epsilon,$$

where $\pi' : S^n \to S^n/R'$ is the decomposition map, and apply Theorem 2.3.3 to obtain a map $\mu : S^n \to S^n$ realizing that decomposition—in other words, μ satisfies

$$\{\mu^{-1}(s) \mid s \in S^n\} = \{(\pi')^{-1}(x) \mid x \in S^n/R'\}$$

—and require $\rho(\pi'\mu, \pi') < \delta$ as well. It follows that $\lambda = \mu R' \subset N_\epsilon \subset L$ is an embedding and a 1-LCC approximation to R. Hence, $\lambda(S^{n-1})$ is (locally) flat, by Theorem 7.6.1. □

Application of Corollary 7.7.3 to an arbitrary embedding $\lambda' : S^{n-1} \to S^n$ and to the neighborhood $L = (\epsilon) \circ \lambda'$ in $S^{n-1} \times S^n$ immediately yields Theorem 7.7.1 for codimension-one spheres in S^n.

Lemma 7.7.4. *For any cell-like embedding relation $R : S^{n-1} \to S^n$, $\operatorname{Im} R$ separates S^n into two components.*

Proof. By Proposition 3.2.9, $R(S^{n-1})$ has the Čech cohomology of S^{n-1}. Repeat the analysis given in Proposition 7.1.1. $\square$

Let $f : \partial I^2 \to S^n \smallsetminus \operatorname{Im} R$ be a loop, and let $f^* : I^2 \to S^n/R$ be a map extending $\pi_R \circ f$ such that $\operatorname{Im} f^*$ misses one of the complementary domains of the $(n-1)$-sphere $\operatorname{Im}(\pi_R \circ R)$ in S^n/R. Then the relation $\hat{F} = \pi_R^{-1} \circ f^* : I^2 \to S^n$ is called an *R-disk bounded by f*. In the proof of the 1-LCC Approximation Theorem of Relations we shall show that every such loop f near a point image of R bounds a "small" R-disk $\hat{F}$. That notion of smallness is measured as R-diameter, where the *R-diameter* of a set $X \subset S^n$ is defined as

$$R\text{-}\operatorname{diam}(X) = \inf\{\epsilon > 0 \mid \text{for some } s \in S^{n-1}, X \subset \epsilon \circ R \circ \epsilon(s)\};$$

for simplicity, we also define $R\text{-diam}(\hat{F}) = R\text{-diam}(\operatorname{Im} \hat{F})$.

Lemma 7.7.5 (Basic Lemma). *Suppose $R : S^{n-1} \to S^n$ $(n \geq 5)$ is a cell-like embedding relation, $\hat{F} : I^2 \to S^n$ is an R-disk, and L, O are neighborhoods of $R, \hat{F}$, respectively. Then L contains a cell-like embedding relation $R''' : S^{n-1} \to S^n$ and O contains a continuous function $F^* : I^2 \to S^n$ such that $R'''(S^{n-1}) \cap F^*(I^2) = \emptyset$.*

Proof that Basic Lemma 7.7.5 implies Theorem 7.7.2. For purposes of this argument, given two relations $L', L'' : S^{n-1} \to S^n$, we will say that L' is *slice-trivial* in L'' if $L' \subset L''$ and $L'(s)$ is null-homotopic in $L''(s)$ for each $s \in S^{n-1}$.

Let $f_1, f_2, \ldots : S^1 \to S^n$ denote a countable set of embeddings dense in the space of all loops in S^n.

Set $R_0 = R$ and let $L_0 \subset L$ be a compact neighborhood of R_0. Assume inductively that cell-like embedding relations $R_0, \ldots, R_{i-1} : S^{n-1} \to S^n$, compact neighborhoods $L_0 \supset R_0, \cdots, L_{i-1} \supset R_{i-1}$ in $S^{n-1} \times S^n$, and continuous functions $F_1, \ldots, F_{i-1} : I^2 \to S^n$ bounded by $f_1, \ldots, f_{i-1}$ have been determined satisfying the following four conditions for $j = 0, 1, \ldots, i-1$:

(1_j) $R_j \subset \operatorname{Int} L_j \subset L_j \subset (1/j) \circ R_j \circ (1/j)$;

(2_j) L_j is slice-trivial in L_{j-1};

(3_j) $L_j^{-1} \circ L_j \subset (1/j)$ (the $j = 0$ case is vacuous); and

(4_j) if f_j bounds an R_{j-1}-disk and if

$$\epsilon_j = \inf\{R_{j-1}\text{- diam}(\hat{F}) \mid \hat{F} \text{ is an } R_{j-1}\text{-disk bounded by } f_j\},$$

then R_{j-1}- diam$(\operatorname{Im} F_j) < 2\epsilon_j$ and $\operatorname{Im} L_j \cap \operatorname{Im} F_j = \emptyset$.

Choose R_i, L_i and F_i as follows. In case f_i bounds an R_{i-1}-disk and ϵ_i is the infimum defined in (4_i), an easy consequence of the Basic Lemma gives a cell-like embedding relation $(R_i : S^{n-1} \to S^n) \subset \operatorname{Int} L_{i-1}$ and a continuous function $F_i : I^2 \to S^n$ bounded by f_i such that $\operatorname{Im} R_i \cap \operatorname{Im} F_i = \emptyset$ and R_{i-1}-diam$(\operatorname{Im} F_i) < 2\epsilon_i$. That Condition (4_i) holds is obvious. There is a compact neighborhood L_i of R_i in $\operatorname{Int} L_{i-1} \cap [(1/i) \circ R_i \circ (1/i)]$ by (5) in the Appendix on Continuous Relations; with this choice of L_i, (1_i) will be satisfied. Condition (2_i) can be obtained using (9) of the Appendix. Since $R_i^{-1} \circ R_i = \operatorname{Id} \subset (1/i)$, Condition (3_i) can be obtained by Composition Theorem (6) of the Appendix.

In the other case, where f_i bounds no R_{i-1}-disk, Condition (4_i) is vacuous; one then can take $R_i = R_{i-1}, F_i$ arbitrary and L_i satisfying $(1_i) - (3_i)$, in order to complete the inductive step.

Define $R' : S^{n-1} \to S^n$ as $R' = \cap_{i=0}^{\infty} L_i$. We claim that R' is a 1-LCC cell-like embedding relation. Clearly it is contained in $L_0 \subset L$ and clearly $\operatorname{Im} R' \cap \operatorname{Im} F_i = \emptyset$ for all i such that f_i bounds an R_{i-1}-disk.

(i) R' is a proper relation. Being an intersection of compact sets, R' itself is compact and, thus, proper, by (5) in the Appendix.

(ii) R' is cell-like. For each $x \in S^{n-1}$, $R'(x)$ has a neighborhood system $L_0(x) \supset L_1(x) \supset \cdots$, with $L_i(x)$ compact, nonvoid, and contractible in $L_{i-1}(x)$, by (2_i). As a result, $R'(x) = \cap_i L_i(X)$ is cell-like.

(iii) R' is 1-1. Indeed,

$$(R')^{-1} \circ R' \subset \cap_i (L_i^{-1} \circ L_i) \subset \cap_i (1/i) = \operatorname{Id}_{S^{n-1}};$$

hence, point images under R' are disjoint.

Consequently, R' is a cell-like embedding relation. Showing it to be 1-LCC is the only remaining issue.

(iv) R' is 1-LCC. Consider any point $x \in S^{n-1}$ and any neighborhood U of $R'(x)$ in S^n. The task ahead is to find a neighborhood V of $R'(x)$ in $U \subset S^n$ such that loops in $V \smallsetminus \operatorname{Im} R'$ are null-homotopic in $U \smallsetminus \operatorname{Im} R'$.

We rely on (6) of the Appendix again to supply technical estimates. Since

$$U \supset R'(x) = (\operatorname{Id} \circ R' \circ \operatorname{Id}) \circ (\operatorname{Id} \circ R'^{-1} \circ \operatorname{Id}) \circ (\operatorname{Id} \circ R' \circ \operatorname{Id})(x),$$

Composition Theorem (6) assures the existence of an $\alpha > 0$ and an integer $I > 0$ satisfying:

(1) $U \supset (2\alpha \circ L_I \circ 2\alpha) \circ (2\alpha \circ L_I^{-1} \circ 2\alpha) \circ (\alpha \circ R' \circ \alpha)(x)$.

Set $\beta = \alpha/2$ and choose an integer $J > 2/\alpha$. Then $i > J$ implies

(2) $\beta \circ R' \circ \beta(x) \subset (\alpha/2) \circ L_{i-1} \circ (\alpha/2)(x)$
$\subset (\alpha/2) \circ [(\alpha/2) \circ R_{i-1} \circ (\alpha/2)] \circ (\alpha/2)(x)$, by (1_{i-1})
$= \alpha \circ R_{i-1} \circ \alpha(x)$.

Having chosen α, β, I, and J, we specify an $(n-1)$-cell neighborhood D of x in $S^{n-1} \cap \beta(x)$, a compact neighborhood V' of $R'(x)$ in $\beta \circ R' \circ \beta(x)$ intersecting $\operatorname{Im} R'$ only in $R'(\operatorname{Int} D)$ and a compact neighborhood V of $R'(x)$ which contracts in V' (recall that $R'(x)$ is cell-like).

We show that each loop $f : S^1 \to S^n$ in $V \smallsetminus \operatorname{Im} R'$ contracts in $U \smallsetminus \operatorname{Im} R'$. Pick $K >$Max$\{I, J\}$ so large that, when $i > K$,

(3) $R_{i-1}(D) \subset \beta \circ R' \circ \beta(x)$,

(4) $R_{i-1}(S^{n-1} \smallsetminus \operatorname{Int} D) \cap V' = \emptyset$, and

(5) $\operatorname{Im} f \cap \operatorname{Im} R_{i-1} = \emptyset$.

Using (5) and the density of $\{f_i\}$, pick $i > K$ such that the loop f_i is homotopic to f in $V \smallsetminus \operatorname{Im}(R' \cup R_{i-1})$. We now explain why the associated extension $F_i : I^2 \to S^n$ given by (4_i) has image in $U \smallsetminus \operatorname{Im} R'$, which will complete the proof.

Since V contracts in V', f admits a continuous extension $g : I^2 \to V'$. Let π denote the decomposition map $S^n \to S^n/R_{i-1}$ associated with R_{i-1}, and identify S^{n-1} with its image under the embedding $\pi \circ R_{i-1}$. The set $\pi \circ f(S^1)$ lies in one of the two components of $(S^n/R_{i-1}) \smallsetminus S^{n-1}$, and the set $\pi \circ g(I^2)$ intersects S^{n-1} only in the $(n-1)$-cell $\pi \circ R_{i-1}(D)$ by (4). By the Tietze Extension Theorem, there is a continuous function $f^* : I^2 \to S^{n-1}/R_{i-1}$ extending $\pi \circ f$, the image of which lies in $\pi \circ g(I^2) \cup \pi \circ R_{i-1}(D)$ and misses one component of $(S^n/R_{i-1}) \smallsetminus S^{n-1}$, namely, the component not containing $f(S^1)$.

The relation $\hat{F} = \pi^{-1} \circ f^* : I^2 \to S^n$ is an R_{i-1}-disk bounded by f. Since $\operatorname{Im} f^* \subset \pi \circ g(I^2) \cup \pi \circ R_{i-1}(D)$, it follows that

$$\operatorname{Im} \hat{F} \subset g(I^2) \cup R_{i-1}(D) \subset \beta \circ R' \circ \beta(x)$$

(by (3) and the choice of $V' \supset g(I^2)$). But $\beta \circ R' \circ \beta(x) \subset \alpha \circ R_{i-1} \circ \alpha(x)$ (by (2)). Hence, by definition of R_i-diam$(\hat{F})$ and rules governing the choice of F_i, F_i is a singular disk in $S^n \smallsetminus L_i$ bounded by f_i and lying, for some $y \in S^{n-1}$, in the set $(2\alpha) \circ R_{i-1} \circ (2\alpha)(y)$, by (4_{i-1}). The only issue remaining is to show that $2\alpha \circ R_{i-1} \circ 2\alpha(y) \subset U$.

Since $\operatorname{Im} F_i \subset \alpha \circ R' \circ \alpha(x) \cap (2\alpha) \circ R_{i-1} \circ (2\alpha)(y)$ and is nonempty, we have $\operatorname{Im} F_i$ contained in

$$(2\alpha) \circ R_{i-1} \circ (2\alpha)(y) \subset [(2\alpha) \circ R_{i-1} \circ (2\alpha)][(2\alpha) \circ R_{i-1}^{-1} \circ (2\alpha)][\alpha \circ R' \circ \alpha](x),$$

and this latter set lies in U, by (1). □

Before proceeding it might be beneficial to review the extensive collection of definitions and notation from §5.5 concerning Štan′ko moves, the template (A, B, C, D, e) in $\hat{I}^2$, the n-dimensional expansions

$$\mathscr{A} = A \times \hat{I}^{n-2}, \quad \mathscr{B} = B \times \hat{I}^{n-2}, \quad \mathscr{C} = C \times [-1,1] \times \hat{I}^{n-3},$$

the 2-cells $\mathscr{D} = D \times \mathbf{0} \subset \hat{I}^n$ and $e \times \hat{I} = e \times \{0\} \times \hat{I} \times \mathbf{0} \subset \hat{I}^n$, semi-capped surfaces, Delta structures, branching systems, Štan′ko complexes, and the special homeomorphism $\Phi_n : \hat{I}^n \to \hat{I}^n$.

Here is the setting for the Basic Lemma, the proof of which occupies most of the remainder of this section. All this data and notation is presumed to be in place until the completion of that proof.

$R : S^{n-1} \to S^n$, a cell-like embedding relation, with associated cell-like decomposition map: $\pi = \pi_R : S^n \to S^n/R$;

$(\hat{F} = \pi^{-1} \circ f^*) : I^2 \to S^n$, an R-disk with $f^* : I^2 \to S^n/R$ a continuous function and with $\hat{F}|\partial I^2$ an embedding;

L, a neighborhood of R in $S^{n-1} \times S^n$;

O, a neighborhood of $\hat{F}$ in $I^2 \times S^n$.

We identify S^{n-1} with $\mathrm{Im}(\pi \circ R)$ via the homeomorphism $\pi \circ R : S^{n-1} \to \mathrm{Im}(\pi \circ R)$ so that $R = \pi^{-1}|S^{n-1}$, and we also make use of the notation:

W, a component of $S^n/R \smallsetminus S^{n-1}$ containing $f^*(\partial I^2)$;

L_0, a neighborhood of π^{-1} in $S^n/R \times S^n$ whose restriction to S^{n-1} is L.

Anticipating the verifications to be made near the end of the proof of the Basic Lemma, we now impose certain essential controls using the Composition Theorem (6): since

$$[\mathrm{Id} \circ \mathrm{Id} \circ \pi^{-1} \circ \mathrm{Id}_{S^n/R} \circ \pi \circ \mathrm{Id} \circ \mathrm{Id}] \circ \mathrm{Id} \circ \pi^{-1} = \pi^{-1} \subset L_0$$

(where all unsubscripted "Id" denote $\mathrm{Id} : S^n \to S^n$), and since

$$\mathrm{Id} \circ \pi^{-1} \circ \mathrm{Id}_{S^n/R} \circ f^* = \hat{F} \subset O,$$

there is an $\epsilon > 0$ such that

(S†) $(\epsilon \circ \epsilon \circ \pi^{-1} \circ (2\epsilon) \circ \pi \circ \epsilon \circ \epsilon) \circ \epsilon \circ \pi^{-1} \subset L_0$ and

(S‡) $\epsilon \circ \pi^{-1} \circ \epsilon \circ f^* \subset O$.

Theorem 7.7.6 (Štan′ko Complex Mapping)**.** *For each $\epsilon > 0$ there exist a branching system $\Delta : \Delta_0 \to \Delta_1 \to \Delta_2 \to \cdots$, with D_0 of $\Delta_0 = (D_0, E_0, \Gamma_0)$ equal to I^2, and a continuous function $h : C(\Delta) \to S^n/R$ satisfying:*

(1) $h(C(\Delta) \smallsetminus \mathrm{Int}\, E^*) \subset W \subset S^n/R \smallsetminus S^{n-1}$,

(2) $h(D_i^* \cup D_{i+1}^* \cup D_{i+2}^* \cup \cdots) \subset B(S^{n-1}; \epsilon/2^{i-1})$ *for* $i > 0$,

(3) $\rho(h \circ (^*)|D_0, f^*) < \epsilon$,

(4) $h \circ (^*)|\partial D_0 = f^*|(\partial D_0 = \partial I^2)$,

(5) $\operatorname{diam} h(P_i) < \epsilon/2^i$ *for* $i > 0$ *and each component* P_i *of* $D_i^* \cup E_{i-1}^*$. *In addition, the map* h *may be chosen so there exists a PL injective map* $h' : C(\Delta) \to S^n$ *with* $\pi \circ h' = h$.

Remark. Unlike the output of the related codimension-three Štan′ko Embedding Theorem (5.5.5), h' need not be a homeomorphism: a sequence $\{x_1, x_2, x_3, \ldots \mid x_i \in h'(D_i^*)\}$ can accumulate at a point of $h'(E_0^*)$, causing $(h')^{-1}$ to be discontinuous. An example will be provided later, in §7.10, of an actual embedding $R : S^{n-1} \to S^n$ and disjoint simple closed curves J_1 and J_2 in $\pi^{-1}(W)$ such that for every singular disk D_1 in $\mathrm{Cl}(\pi^{-1}(W))$ bounded by J_1 and every singular disk D_2 in S^n bounded by J_2, $D_1 \cap D_2 \neq \emptyset$. If $\hat{F}$ would be an R-disk sending ∂D homeomorphically onto J_1 and if J_2 would bound a component of $h(E_i^*)$, then points of $h'(C(\Delta) \smallsetminus E^*)$ necessarily would accumulate at $h'(\operatorname{Int} E_i^*)$. This possibility accounts for a thickening procedure to be employed later, in Štan′ko Complex Embedding Theorem 7.7.8, and, to a large extent, for the prolonged diversion through the realm of embedding relations. To establish the Embedding Theorem we will employ a carefully constructed cell-like embedding relation $R' : S^n \to S^n$, will replace R by $R' \circ R$, and will obtain a PL embedding of $C(\Delta) \to S^n$ for the modified R.

As in §5.5, the Mapping Theorem is a consequence of iterated applications of the following, which in turn is simply a variation on Lemma 5.5.6 for the slightly more general context to be faced.

Lemma 7.7.7. *Suppose* $g : (D, \partial D) \to (\overline{W}, W)$ *is a map of pairs, where* D *is a disk, and* $\delta > 0$. *Then there exist a Delta structure* $\Delta = (D, E, \Gamma)$ *and a map* $f : D^* \to \overline{W}$ *satisfying:*

(1′) $f(D^* \smallsetminus \operatorname{Int} E^*) \subset W$,

(2′) $f(E^*) \subset B(S^{n-1}; \delta)$,

(3′) $\rho(f \circ (^*)|D, g) < \delta$,

(4′) $f \circ (^*)|\partial D = g|\partial D$, *and*

(5′) $\operatorname{diam} f(P) < \delta$ *for each component* P *of* E^* *or* Γ^*.

Proof. Being the finite-dimensional image of S^n under a cell-like map, S^n/R is an ANR (Corollary 7.4.8). Since $\overline{W}$ is a closed subset of S^n/R bounded by the $(n-1)$-sphere $\pi R(S^{n-1})$, $\overline{W}$ is also an ANR.

Claim: each $s \in \pi R(S^{n-1})$ has arbitrarily small pairs of neighborhoods $N' \subset N$ such that $H_1(N' \cap W; \mathbb{Z}) \to H_1(N \cap W; \mathbb{Z})$ is trivial. Given N with $N \cap \pi R(S^{n-1})$ contractible, choose $N' \subset N$ with $\mathrm{incl} : N' \to N$ homotopically trivial; by Proposition 3.2.9 it suffices to show that $H_1(\pi^{-1}(N' \cap W); \mathbb{Z}) \to H_1(\pi^{-1}(N \cap W); \mathbb{Z})$ is trivial, and that property holds just as in the proof of Proposition 7.1.11.

Hence, there exist positive numbers $\alpha < \beta < \eta < \delta/4$ such that

(η) η-loops in W bound singular $(\delta/4)$-disks in $\overline{W}$,

(β) β-loops in W bound singular, orientable disks with handles of diameter less than η in W (recall Lemma 5.5.3), and

(α) any two points of W within α of each other are joined by $(\beta/2)$-arcs in W.

Triangulate D with mesh so small that the image under g of each simplex has diameter less than α, and let $D^{(0)}, D^{(1)}$ and $D^{(2)}$ denote the successive skeleta of this triangulation. Since $g(\partial D) \subset W$ we may define f on ∂D as $g|\partial D$ and define f for any other vertex v of $D^{(0)}$ as a point in W so close to $g(v)$ that vertices of the same simplex have images in W within α of one another and within α of their image under f.

Apply (α) to extend f over $|D^{(1)}|$ so that the image of each 1-simplex lies in W and has diameter less than $\beta/2$.

Apply (β) for each $\sigma \in D^{(2)}$ to obtain an orientable disk-with-handles Q_σ bounded by $\partial\sigma$ and a continuous extension $f_\sigma : Q_\sigma \to W$ of $f|\partial\sigma$ sending Q_σ to a set of diameter less than η.

In the interior of each Q_σ identify complete sets $\Gamma_\sigma, \Gamma'_\sigma$ of handle curves, as before.

By (η) there exists for each $\sigma \in D^{(2)}$ a finite disjoint union E_σ of disks whose boundaries equal Γ'_σ and whose interior points have no intersection with $\cup_\sigma Q_\sigma$ as well as a continuous extension $f : E_\sigma \to \overline{W}$ taking each component of E_σ to a set of diameter less than $\delta/4$. Define

$$D^* = |D^{(1)}| \cup [\cup_\sigma (Q_\sigma \cup E_\sigma)], \quad E^* = \cup_\sigma E_\sigma, \quad \Gamma^* = \cup_\sigma \Gamma_\sigma.$$

It should be clear that D^* is the semi-capped surface of a Delta structure $\Delta = (D, E, \Gamma)$, where $(^*)|\ |D^{(1)}| = \mathrm{Id}$ and $(^*){:}\sigma \to Q_\sigma \cup E_\sigma$. Clearly f as defined on D^* satisfies $(1'), (3'), (4')$ and $(5')$. If $(2')$ is not already satisfied, it can only stem from the presence of a component of $E^* \cup \Gamma^*$ whose image misses S^{n-1}; the preimage of such a component should simply be deleted from E and Γ. □

Proof of Mapping Theorem 7.7.6. Choose a sequence $\delta_0 > \delta_1 > \cdots$ of positive numbers such that

(i) $4\delta_i < \epsilon/2^i$ for $i \geq 0$ and

(ii) δ_{i+1}-loops in $\overline{W}$ bound singular δ_i-disks in $\overline{W}$.

Lemma 7.7.7 provides a Delta structure $\Delta_0 = (D_0, E_0, \Gamma_0)$, where $D_0 = I^2$, plus a continuous function $h : D_0^* \to \overline{W}$ satisfying the conditions below for $j = 0$, where $f_0 = f|D_0$:

(1_j) $h(D_j^* \smallsetminus \operatorname{Int} E_j^*) \subset W$,

(2_j) $h(E_j^*) \subset B(S^{n-1}; \delta_{j+1})$,

(3_j) $\rho(h \circ (^*)|D_j, f_j) < \delta_{j+1}$,

(4_j) $h \circ (^*)|\partial D_j = f_j|\partial D_j$, and

(5_j) $\operatorname{diam} h(P) < \delta_{j+2}$ for each component P of E_j^* or Γ_j^*.

Assume inductively that $\Delta_0 \to \cdots \to \Delta_{i-1}$, $f_j : D_j \to \overline{W}$ and

$$h|D_0^* \cup \cdots \cup D_{i-1}^* : D_0^* \cup \cdots \cup D_{i-1}^* \to \overline{W}$$

have been obtained satisfying (1_j)–(5_j) for each $j \in \{0, \ldots, i-1\}$. For each component γ of Γ_{i-1}^*, let $D(\gamma)$ be a disk with boundary γ. By (5_{i-1}), $\operatorname{diam} h(\gamma) < \delta_{i+1}$, so (ii) assures the existence of a continuous extension $f_i(\gamma) : D(\gamma) \to \overline{W}$ of $h|\gamma$, the image of which has diameter less than δ_i. Set $D_i = \cup_\gamma D(\gamma)$ and define f_i on D_i as $f_i = \cup_\gamma f_i(\gamma)$. By Lemma 7.7.7 there exist a Delta structure (+) $\Delta_i = (D_i, E_i, \Gamma_i)$ and a map $h|D_i^* : D_i^* \to \overline{W}$ satisfying (1_i)–(5_i). This completes the inductive construction of

$$\Delta : \Delta_0 \to \Delta_1 \to \Delta_2 \to \cdots \quad \text{and} \quad h : C(\Delta) \to S^n/R.$$

Conditions (1), (3) and (4) of the Mapping Theorem obviously hold here. For each component $P \cup Q$ of $D_i^* \cup E_{i-1}^*$ $(i \geq 1)$, $P \subset D_i^*$, $Q \subset E_{i-1}^*$, we have

$$\operatorname{diam} h(P) \leq 2\rho(h \circ (^*)|P, f_i|P) + \operatorname{diam} f_i(P) < 3\delta_i$$

and $\operatorname{diam} h(Q) < \delta_{i+1}$; thus, $\operatorname{diam} h(P \cup Q) < 4\delta_i < \epsilon/2^i$ and (5) is satisfied. Also, for $i \geq 1$, $d(h(P), S^{n-1}) \leq \epsilon/2^i$, by (2_{i-1}), since $h(P) \cap h(E_{i-1}^*) = \emptyset$. But $\operatorname{diam} h(P) < \epsilon/2^i$, so $h(P) \subset B(S^{n-1}; \epsilon/2^{i-1})$, and (2) is satisfied. □

Theorem 7.7.8 (Štan′ko Complex Embedding)**.** *Let Δ, h and h' be as in the conclusion of Mapping Theorem 7.7.6. Then the ϵ-neighborhood of* Id : $S^n \to S^n$ *contains a cell-like embedding relation $R' : S^n \to S^n$ such that there is a PL embedding $h'' : C(\Delta) \to S^n$ with $(R')^{-1} \circ h'' = h'$.*

Proof. In view of Conditions (1) and (2) of the Mapping Theorem, the function $(h')^{-1}|h'(C(\Delta))$ is already continuous except possibly at points of $h'(\operatorname{Int}(E_0^* \cup E_1^* \cup \cdots))$. We will split S^n apart in stages, starting with $h'(\operatorname{Int} E_0^*)$, to provide enough room to isolate $h''(\operatorname{Int} E_0^*) = h'(\operatorname{Int} E_0^*)$ from $h''(C(\Delta) \setminus E_0^*)$. This will make $(h'')^{-1}$ continuous at points of $h''(\operatorname{Int} E_0^*)$. Iteration of the splitting will accomplish the same goal at images of $E_1^*, E_2^*, \ldots$ and will complete the proof of this Embedding Theorem.

The basic splitting move is the inverse of a simple collapsing map. Define $r : \hat{I}^2 \to [0,1]$ as $r(x) = (1/4) \cdot d(x, \partial \hat{I}^2) \in [0, 1/2]$ and set

$$\hat{I}^2 \times_r \hat{I}^{n-2} = \cup\{x \times [r(x) \cdot \hat{I}^{n-2}] \mid x \in \hat{I}^2\} \subset \hat{I}^2 \times \hat{I}^{n-2} \subset \hat{I}^n.$$

Note that $\hat{I}^2 \times_r \hat{I}^{n-2}$ is a closed neighborhood of $\operatorname{Int} \hat{I}^2 \times \mathbf{0}$ in $\hat{I}^n$. Let $\psi : \hat{I}^2 \times_r \hat{I}^{n-2} \to \hat{I}^2 \times \mathbf{0}$ denote projection to the first factor. Define a map

$\Psi : \hat{I}^n \to \hat{I}^n$ extending ψ, fixed on ∂I^n, sending each $x \times \hat{I}^{n-2}$ onto itself, and having as its nondegenerate point preimages precisely the nondegenerate point preimages of ψ. The relation Ψ^{-1} is called the *basic splitting relation.*

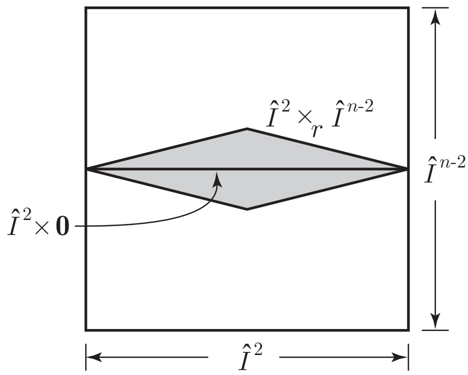

Figure 7.9. The basic splitting relation Ψ^{-1}

For each component E of $h'(E_0^*)$ there is a PL embedding $P_E : \hat{I}^2 \times \hat{I}^{n-2} \to S^n$ taking $\hat{I}^2 \times \mathbf{0}$ onto E and taking each fiber $x \times \hat{I}^{n-2}$ onto a very small set. The embeddings $\{P_E \mid E \subset h'(E_0^*)\}$ should be chosen with disjoint images. Define $R : S^n \to S^n$ splitting S^n along $h'(E_0^*)$ by the formula

$$R_0(x) = \begin{cases} P_E \Psi^{-1} P_E^{-1}(x) & \text{if } x \in P_E(\hat{I}^n) \\ \quad x & \text{if } x \notin \cup_E P_E(\hat{I}^n). \end{cases}$$

The restriction on $\hat{I}^{n-2}$ fiber size assures that R_0 lives in the ϵ-neighborhood (ϵ) of the relation $\mathrm{Id} : S^n \to S^n$. Define $h_0 : C(\Delta) \to S^n$ as

$$h_0(x) = \begin{cases} h'(x) & \text{if} \quad x \in E_0^* \\ R_0 \circ h'(x) & \text{if} \quad x \notin E_0^*. \end{cases}$$

Then h_0 is PL and injective, $R_0^{-1} \circ h_0 = h'$, and $h_0^{-1}|h_0(C(\Delta))$ is continuous at the points of $h_0(E_0^*)$. Choose a compact neighborhood N_0 of R_0 in (ϵ) slice-trivial in (ϵ), with $N_0^{-1} \circ N_1 \subset (1)$.

In the same manner choose $R_1 : S^n \to S^n$ splitting S^n along $h_0(E_1^*)$, fixing $R_0 \circ h'(D_0^*)$, and satisfying $R_1 \circ R_0 \subset \mathrm{Int}\, N_0$. Define $h_1 : C(\Delta) \to S^n$ by

$$h_1(x) = \begin{cases} h_0(x) & \text{if} \quad x \in E_1^* \\ R_1 \circ h_0(x) & \text{if} \quad x \notin E_1^*. \end{cases}$$

Choose a compact neighborhood N_1 of $R_1 \circ R_0$ in $\mathrm{Int}\, N_0$, slice-trivial in $\mathrm{Int}\, N_0$, with $N_1^{-1} \circ N_1 \subset (1/2)$.

In general, have R_i split S^n along $h_{i-1}(E_i^*)$, fixing $R_{i-1} \circ \cdots \circ R_0 \circ h'(D_{i-1}^*)$, and satisfying $R_i \circ R_{i-1} \circ \cdots \circ R_0 \subset \operatorname{Int} N_{i-1}$. Define $h_i : C(\Delta) \to S^n$ as

$$h_i(x) = \begin{cases} h_{i-1}(x) & \text{if} \quad x \in E_i^* \\ R_i \circ h_{i-1}(x) & \text{if} \quad x \notin E_i^*. \end{cases}$$

Again choose a compact neighborhood N_i of $R_i \circ \cdots \circ R_0$ in $\operatorname{Int} N_{i-1}$, slice-trivial in $\operatorname{Int} N_{i-1}$, with $N_i^{-1} \circ N_i \subset (1/(i+1))$.

Finally, define R' as $[R' = \cap_i N_i] : S^n \to S^n$. Just as in the proof of the 1-LCC Approximation Theorem, R' is a cell-like embedding relation. Define $h'' : C(\Delta) \to S^n$ as $h'' = \cup_i (h_i | D_i^*)$. That h'' is the embedding required in this Embedding Theorem is easily confirmed. □

Theorem 7.7.9 (Unknotting)**.** *Suppose $C(\Delta)$ is a Štan′ko complex PL embedded in a PL n-manifold M ($n \geq 5$) and Z is a compact subset of $C(\Delta)$. Then there exist a PL 3-cell Y_Z^3 and a PL embedding $\psi : Y_Z^3 \times I^{n-3} \to M$ such that $Z \subset \psi(Y_Z^3 \times \mathbf{0})$.*

Proof. All but the case $n = 5$ is covered by Lemma 5.5.8. By Lemma 5.5.7 Z is contained in some PL-embedded, collapsible, finite 2-complex in M. According to (Price, 1966), any two homotopic PL embeddings of a collapsible finite k-complex in M are ambient isotopic provided $n \geq 2k+1$, so the remaining $n = 5$ case follows like the others. □

Proof of Basic Lemma 7.7.5. Take Δ, h and h' from the conclusion of Štan′ko Complex Mapping Theorem 7.7.6, and then take the relation R' and the embedding h'' from the conclusion of the Embedding Theorem 7.7.8. Identify $C(\Delta)$ with $h''(C(\Delta))$ via the homeomorphism h''. Recall the combined identification map

$$(^*) : D_0 \sqcup D_1 \sqcup \cdots \to C(\Delta) = D_0^* \cup D_1^* \cup \cdots = h''(C(\Delta)) \subset S^n.$$

For $i \geq 0$ identify that D_i associated with the Delta structure $(+)\Delta_i = (\Delta_i, E_i, \Gamma_i)$ with the D_i from the template $(+)(A_i, B_i, C_i, D_i, e_i)$ in such a manner that $E_i \cup \Gamma_i \subset \operatorname{Int} B_i$ and $(D_i \cap e_i)^* = D_i^* \cap E_{i-1}^* \subset C(\Delta) \subset S^n$. Keep in mind that $\operatorname{Im}(R' \circ R) \cap C(\Delta) \subset \cup_i E_i^* \subset \cup_i B_i^*$.

By Unknotting Theorem 7.7.9 there exist a regular neighborhood N_i of $D_i^* \cup E_{i-1}^*$ in $C(\Delta)$, a PL 3-manifold Y_i and an embedded PL product $Y_i \times \hat{I}^{n-3} \subset S^n$ such that $N_i \subset Y_i = Y_i \times \mathbf{0} \subset Y_i \times \hat{I}^{n-3} \subset S^n$. For $i > 0$ we will use the sets $D_i^* \cup E_{i-1}^* \subset N_i$ and the product structure $\hat{I}^3 \times \hat{I}^{n-3}$ to construct an embedding $\alpha_i : \hat{I}^n = \mathscr{A}_i \cup \mathscr{B}_i \to S^n$ suitable for use in a basic Štan′ko move. The α_i's will be constructed in three steps, which proceed exactly like those of the second proof of Fundamental Lemma 5.5.2, only with X replaced throughout by $\operatorname{Im}(R' \circ R)$. Those steps are not reproduced here, but the properties of these α_i are listed below for easy reference later on in the verification steps:

(1) $\operatorname{Im}(\mathscr{A}_i \cup \mathscr{B}_i) \subset B(D_i^* \cup E_{i-1}^*; \epsilon/i)$ (component by component);

(2) of the sets in the list

$$[D_0^* \cap A_0^*], [B_0^*], [A_1^*, \operatorname{Im}\mathscr{A}_1], [B_1^*, \operatorname{Im}\mathscr{B}_1], [A_2^*, \operatorname{Im}\mathscr{A}_2], [B_2^*, \mathscr{B}_2], \dots,$$

only the ones in the same or adjacent square brackets can intersect,

(3) $X \cap \alpha_i(\mathscr{A}_i) \subset \alpha_i(\mathscr{C}_i)$;

(4) $\alpha_i(\mathscr{A}_i \cap \Phi_n(\mathscr{C}_i)) \subset S^n \smallsetminus C(\Delta)$.

(5) $\operatorname{Im}(R' \circ R) \cap \alpha_i(\mathscr{B}_i) \subset \alpha_{i+1}(\mathscr{C}_{i+1})$;

(6) $\alpha_i(\mathscr{B}_i) \cap \alpha_{i+1}(\mathscr{A}_{i+1} \cap \Phi_n(\mathscr{C}_{i+1})) = \emptyset$;

(7) $\alpha_i(\mathscr{B}_i \cap \Phi_n(\mathscr{C}_i)) \subset S^n \smallsetminus C(\Delta)$.

With the embeddings α_i all in place, we define the infinite Štan′ko move that leads to the desired cell-like embedding relation and establishes the Basic Lemma. Set

$$R''(x) = \begin{cases} \alpha_i \circ \Phi_n(x) \circ \alpha_i^{-1}(x) & \text{if } x \in \alpha_i(\mathscr{C}_i) \cap \operatorname{Im}(R' \circ R) \\ x & \text{otherwise,} \end{cases}$$

and note that R'' is a function. The cell-like embedding relation $R''' : S^{n-1} \to S^n$ whose existence is posited in the statement of the Basic Lemma is given by $R''' = R'' \circ R' \circ R$, and the continuous function $F^* : I^2 \to S^n \smallsetminus \operatorname{Im} R'''$ named there is given by $F^* = (*)|(I^2 = D_0)$.

Verification that R''' and F^* have the desired properties is a lengthy process. Each of the verification items labelled (Vi) below opens with a statement of what it confirms. The first of them is a technical calculation ultimately used for showing that $R''' \subset L$, that R''' is proper and that R''' is cell-like.

(V1) If $\langle x, y\rangle \in R' \circ R$ and $\langle x, R''y\rangle \in R''' \smallsetminus \cup_{i<N}[S^{n-1} \times \operatorname{Int}\alpha_i(\mathscr{A}_i \cup \mathscr{B}_i)]$ then

$$\langle x, R''y\rangle \in (\epsilon/N) \circ [R' \circ \pi^{-1} \circ (\epsilon/N) \circ \pi \circ (R')^{-1}] \circ (\epsilon/N) \circ [R' \circ R].$$

It suffices to check the case where $R''(y) \neq y$, and that is done by finding points $y_1, y_2 \in S^n$ such that

$$\langle x, y\rangle \in R' \circ R,\ y_1 \in (\epsilon/N)(y),$$

$$y_2 \in R' \circ \pi^{-1} \circ (\epsilon/N) \circ \pi \circ (R')^{-1}(y_1), \text{ and } R''(y) \in (\epsilon/N)(y_2).$$

By construction of R'' there exist an integer $j \geq N$ and a component P of $\alpha_j(\mathscr{A}_j \cup \mathscr{B}_j)$ containing both y and $R''(y)$. Let Q be the component of $D_j^* \cup E_{j-1}^*$ intersecting P. By (1), $P \subset (\epsilon/j)(Q)$. Thus, there exist points $y_1, y_2 \in Q$ satisfying $y_1 \in (\epsilon/j)(y)$ and $R''(y) \subset (\epsilon/j)(y_2) \subset (\epsilon/N)(y_2)$.

Since $h(Q) = \pi \circ (R')^{-1}(Q)$ has diameter less than ϵ/j by Conclusion (5) of 7.7.6, $y_2 \in R'' \circ \pi^{-1} \circ (\epsilon/N) \circ \pi \circ (R')^{-1}(y_1)$.

(V2) $R''' \subset L$. By (V1) with $N = 1$,

$$R''' \subset \epsilon \circ [R' \circ \pi^{-1} \circ \epsilon \circ \pi \circ (R')^{-1}] \circ \epsilon \circ [R' \circ R]$$

and the latter set lies in L_0, by Condition (S†) of the setting for the Basic Lemma. Since $L_0|S^{n-1} = L, R''' \subset L$.

(V3) R''' is proper. It suffices to show that R''' is compact. Let $\langle x_1, R''y_1\rangle$, $\langle x_2, R''y_2\rangle, \ldots$ be a sequence in R'''. Passing to a subsequence, if necessary, we see that it suffices to address two cases.

Case 1. The points $y_1, y_2, \ldots$ all lie in $\alpha_i\mathscr{A}_i$ for some fixed index i. Since $R''|\ [\alpha_i\mathscr{A}_i \cap \operatorname{Im}(R' \circ R)]$ is continuous, $\langle x_1, R''y_1\rangle, \langle x_2, R''y_2\rangle, \ldots$ all belong to the compact set

$$[(R' \circ R)^{-1}, R''][\alpha_i\mathscr{A}_i \cap \operatorname{Im}(R' \circ R)] \subset [(R' \circ R)^{-1}, R''] \operatorname{Im}(R' \circ R) \subset R'''.$$

Hence, the points cluster in R'''.

Case 2. For each integer $N > 0$, only finitely many of the points $y_1, y_2, \ldots$ lie in $\cup_{i<N} \operatorname{Int} \alpha_i\mathscr{A}_i$. Then the intersection of the sequence $\{\langle x_i, R''y_i\rangle\}_{i=N}^{\infty}$ with the set $Z_N = R''' - \cup_{i<N}[S^{n-1} \times \operatorname{Int} \alpha_i(\mathscr{A}_i \cup \mathscr{B}_i)]$ is contained in the portion of

$$(\epsilon/N) \circ [R' \circ \pi^{-1} \circ (\epsilon/N) \circ \pi \circ (R')^{-1}] \circ (\epsilon/N) \circ [R' \circ R|$$

outside $\cup_{i<N}[S^{n-1} \times \operatorname{Int} \alpha_i(\mathscr{A}_i \cup \mathscr{B}_i)]$ and is nonempty for each such N. The intersection of the Z_n (over all $N \geq 1$) equals

$$R' \circ R \smallsetminus \cup_{i=1}^{\infty}[S^{n-1} \times \operatorname{Int} \alpha_i(\mathscr{A}_i \cup \mathscr{B}_i)],$$

which is a compact subset of R'''. It follows easily that the points cluster at a point of R'''. We conclude that R''' is compact and, thus, proper.

(V4) R''' is injective. Since $R' \circ R$ is injective, it suffices to show that R'' is injective. We have

$$\operatorname{Im}(R' \circ R) = [\operatorname{Im}(R' \circ R) \smallsetminus \cup_i \operatorname{Im} \alpha_i] \cup [\operatorname{Im}(R' \circ R) \cap (\alpha_1\mathscr{C}_1 \cup \alpha_2\mathscr{C}_2 \cup \cdots)]$$

by conditions (3) and (5). The set $R''[\operatorname{Im}(R' \circ R) \cap \alpha_i\mathscr{C}_i]$ lies in $\alpha_i\Phi_n(\mathscr{C}_i)$. The sets $\alpha_i\Phi_n(\mathscr{C}_i)$ and $\alpha_j\Phi_n(\mathscr{C}_j)$ miss one another if $j \neq i-1, i, i+1$, by (2). Intersections of the form $\alpha_i\Phi_n(\mathscr{C}_i) \cap \alpha_{i+1}\Phi_n(\mathscr{C}_{i+1})$ lie in

$$\alpha_i(\mathscr{B}_i \cap \Phi_n\mathscr{C}_i) \cap \alpha_{i+1}(\mathscr{A}_{i+1} \cap \Phi_n\mathscr{C}_{i+1})$$

by (2) and the latter intersection is empty by (6). Thus R''' is injective.

(V5) R''' has nonempty point images. This follows immediately, since R'' is a function and $R' \circ R$ has nonempty point images.

(V6) $F' \subset O$. We have $F^* = (^*)|I^2 = h''(^*)|(I^2 = D_0)$, and

$$h''|D_0^* \subset R' \circ \pi^{-1} \circ h \subset R' \circ \pi^{-1} \circ \epsilon \circ f^* \subset \epsilon \circ \pi^{-1} \circ \epsilon \circ f^* \subset O$$

by definition of h'', limits on the distance between h and f^*, limits on the motion of R', and Condition (S‡) of the Setting.

(V7) $\operatorname{Im} F^* \cap \operatorname{Im} R'' = \emptyset$. Here we have

$$\operatorname{Im} F^* \cap \operatorname{Im}(R' \circ R) \subset E_0^* \subset \alpha_1 \mathscr{C}_1;$$

$$C(\Delta) \cap R''(\alpha_1 \mathscr{C}_1 \cap \operatorname{Im}(R' \circ R)) = \emptyset; \text{ and}$$

$$R''(\alpha_i \mathscr{C}_i \cap \operatorname{Im}(R' \circ R)) \cap \operatorname{Im} F^* = \emptyset \text{ for } i > 1,$$

since $\alpha_i(\mathscr{A}_i \cup \mathscr{B}_i) \cap \operatorname{Im} F^* = \emptyset$ for $i > 1$. Disjointness of the two images follows from these three observations.

(V8) R''' is cell-like. Consider any $x \in S^{n-1}$ and $X = R' \circ R(x)$. If X meets only finitely many of the sets $\alpha_i \mathscr{A}_i$, then $R''|X$ is a re-embedding of X, so $R''(X)$ is also cell-like. Assume then that X meets infinitely many of the $\alpha_i \mathscr{A}_i$. Let U_i denote the union of the components of $\alpha_i \mathscr{A}_i$ intersecting X, V_i the union of the components of $\alpha_i(\mathscr{A}_i \cup \mathscr{B}_i)$ intersecting U_i, and $X_i = X \cap \alpha_i \mathscr{A}_i$. There clearly exist homotopies $H_i : U_i \times [0,1] \to S^n$ starting at the inclusion, fixing FrU_i and having images in V_i such that $H_i(-,1)|X_i = R''|X_i$. By the same calculation performed in (V1), $V_N \cup V_{N+1} \cup \cdots$ is contained in

$$(\epsilon/N) \circ R' \circ \pi^{-1} \circ (\epsilon/N) \circ \pi \circ (R')^{-1} \circ (\epsilon/N) \circ R' \circ R(x) \smallsetminus \cup_{i<N} [S^{n-1} \times \operatorname{Int} \alpha_i(\mathscr{A}_i \cup \mathscr{B}_i)].$$

It follows from the Composition Theorem that, as $N \to \infty$, the various sets named immediately above converge uniformly to the compact set

$$X \smallsetminus \cup_{i=1}^{\infty} \operatorname{Int} \alpha_i(\mathscr{A}_i \cup \mathscr{B}_i) \subset R''(X).$$

Thus $X, U_1, U_2, \ldots, H_1, H_2 \ldots$ satisfy the hypotheses of Lemma 7.7.10 below, application of which will establish that $R''(X)$ is cell-like and will complete the proof of the Basic Lemma. □

Suppose given the following: a cell-like continuum X in S^n; pairwise disjoint compact subsets $U_1, U_2, \ldots$ in S^n; $X^- = X \smallsetminus \cup_{i=1}^{\infty} \operatorname{Int} U_i$; a sequence $\{\epsilon_i\}$ of positive numbers, with $\epsilon_i \to 0$; and a collection of homotopies $H_i : U_i \times [0,1] \to S^n$ starting at the identity, moving only points of $\operatorname{Int} U_i$ and having image in $B(X^-; \epsilon_i)$. Define a function $f : X \to S^n$ as $f|X^- = \mathrm{incl}_{X^-}$ and $f|X \cap U_i = H_i(-,1)$.

Lemma 7.7.10. *If f is injective, then $f(X)$ is cell-like.*

Remark. Even if it is injective, f need not be a homeomorphism.

Proof. Consider $x_1, x_2, \ldots \in X$. If infinitely many x_i belong to one of the sets $A_i = (X \cap U_i) \cup X^-$, then the sequence $f(x_1), f(x_2), \ldots$ clusters at some point of $f(A_i)$, since $f|A_i$ is continuous. On the other hand, if none of the A_i contains infinitely many points of $\{x_1, x_2, \ldots\}$, then we can assume that $x_i \in \operatorname{Int} U_{j(i)}$, where $j(1) < j(2) < \cdots$. Hence $f(x_i) \in B(X^-; \epsilon_{j(i)})$ for each

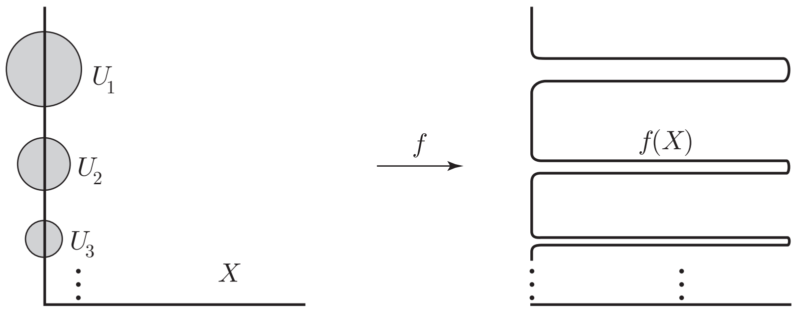

Figure 7.10. An injective f which is not a homeomorphism

i, and so $f(x_1), f(x_2), \ldots$ clusters at some point of $X^- \subset f(X)$. This yields that $f(X)$ is compact.

As an aid to showing $f(X)$ is cell-like, for each positive integer N define a map $\mu_N : S^n \to S^n$ as the inclusion on $S^n \smallsetminus \cup_{i=1}^N \operatorname{Int} U_i$ and as $H_i(-,1)$ on $\operatorname{Int} U_i$, $i = 1, \ldots, N$. Similarly, define another collection of maps $\hat{\mu}_N : f(X) \to S^n$ as inclusion on $f(X) \smallsetminus f(\cup_{i=1}^N \operatorname{Int} U_i)$ and as $[H_i(-,1)|X \cap U_i]^{-1}$ on $f(X \cap U_i)$. The injectivity of f assures that each $\hat{\mu}_N$ is well-defined and continuous.

Given an arbitrary neighborhood U of $f(X)$, choose an integer N so large that $i > N$ implies $H_i(U_i \times [0,1]) \subset U$. As a consequence,

$$\mu_N(X \cap U_i) = H(X \cap U_i, 0) \subset H_i(U_i, 0) \subset U \text{ for all } i > N,$$

and

$$\mu_N(X \cap U_i) = H(X \cap U_i) = f(X \cap U_i) \subset U \text{ for } i = 1, 2, \ldots, N.$$

Hence, $\mu_N(X) \subset U$. The map $(\mathrm{Id}, \mu_N) : S^n \to S^n \times S^n$ sends X into $S^n \times U$. Since $(\mathrm{Id}, \mu_N)(X)$ is cell-like, being homeomorphic to X, there is a neighborhood V of $(\mathrm{Id}, \mu_N)(X)$ that is contractible in $S^n \times U$. In turn, there is another neighborhood W of X in S^n such that $(\mathrm{Id}, \mu_N)(W) \subset V$.

Choose an integer $M > N$ so large that $i > M$ implies $H_i(U_i \times [0,1]) \subset W$. As above, $\hat{\mu}_M(f(X)) \subset W$.

Using $\pi_2 : S^n \times S^n \to S^n$ to denote projection to the second factor, consider the continuous function

$$\mu_N \circ \hat{\mu}_M = \pi_2 \circ (\mathrm{Id}, \mu_N) \circ \hat{\mu}_M : f(X) \to S^n.$$

It is null homotopic in U, since $(\mathrm{Id}, \mu_N) \circ \hat{\mu}_M(f(X)) \subset (\mathrm{Id}, \mu_N)(W) \subset V$ and V is null homotopic in $\pi_2^{-1}(U)$. But $\mathrm{incl}_{f(X)}$ and $\mu_N \circ \hat{\mu}_M$ are homotopic in U via the homotopy that fixes $f(X \cap U_i)$ for $i \notin \{N+1, \ldots, M\}$ and that moves $f(X \cap U_i)$ to $X \cap U_i$ by the reverse of the homotopy $H_i(-,t)|(X \cap U_i)$

in the other finite set of cases. Thus, $f(X)$ is cell-like, as it is null homotopic in an arbitrary neighborhood U. $\square$

Theorem 7.7.11. *Let $R : S^{n-1} \to S^n$ be a cell-like embedding relation and $\pi : S^n \to S^n/R$ the associated decomposition map. Let C denote the closure of one of the components of $S^n/R \smallsetminus \pi \circ R(S^{n-1})$, $T : C \to S^n$ a cell-like embedding relation, and L a neighborhood of T. Then L contains a 1-LCC cell-like embedding relation $T''' : C \to S^n$.*

Proof. The argument is a fairly straightforward repetition of the one just given for Theorem 7.7.2, except that in this setting the focus rests on loops lying toward only one specific side of the sequence of cell-like embedding relations. Here $\pi^{-1} : S^n/R \to S^n$ restricts to a cell-like embedding relation $\hat{R} : C \to S^n$. One needs an analog of Basic Lemma 7.7.5 for $\hat{R}$-disks that meet $\operatorname{Im} \hat{R}$ only at points of $\hat{R}(\operatorname{Bd} C)$. Given a countable dense collection $\{f_i : S^1 \to S^n\}$ of embedded loops in S^n, as in the proof that 7.7.5 implies 7.7.2, the adapted Basic Lemma gives rise to a sequence of cell-like embedding relations $\hat{R}_i : C \to S^n$, neighborhoods L_i of $\hat{R}_i$ and controlled mappings $F_i : I^2 \to S^n$ such that $\operatorname{Im} \hat{R}_i \cap \operatorname{Im} F_i = \emptyset$ as before, unless $f_i(S^1) \cap \hat{R}_{i-1}(\operatorname{Int} C) \neq \emptyset$, in which case $\hat{R}_i = \hat{R}_{i-1}$ and F_i, which is essentially irrelevant, is chosen arbitrarily. $\square$

Corollary 7.7.12. *Let $R : S^{n-1} \to S^n$ be a cell-like embedding relation and let $\pi : S^n \to S^n/R$ be the associated decomposition map. Suppose C is the closure of one of the components of $S^n/R \smallsetminus \pi \circ R(S^{n-1})$ and $\pi \circ R(S^{n-1})$ is 1-LCC in C. Then C is an n-cell.*

Proof. Let $T''' : C \to S^n$ denote the 1-LCC cell-like embedding relation promised by Theorem 7.7.11, and use π''' to denote the quotient map for the decomposition of S^n whose elements are $\{T'''(x) \mid x \in C\}$ and the singletons from $S^n \smallsetminus \operatorname{Im} T'''$. Here the frontier of $\pi''' T'''(C)$ in S^n/T''' is the 1-LCC embedded $(n-1)$-sphere $\pi''' T'''(\operatorname{Bd} C)$, so Proposition 7.4.13 assures that S^n/T''' is the n-sphere. Furthermore, being 1-LCC, $\pi''' T'''(\operatorname{Bd} C)$ is flatly embedded, so $\pi''' T'''(C) \cong C$ is an n-cell. $\square$

Corollary 7.7.13. *If $\lambda : S^{n-1} \to S^n$ is an embedding, $n \geq 5$, and C is the closure of one of the components of $S^n \smallsetminus \lambda(S^{n-1})$ such that $\lambda(S^{n-1})$ is 1-LCC in C, then C is an n-cell.*

Theorem 7.7.14. *Let $\lambda : S^{n-1} \to S^n$ be an embedding, $n \geq 5$, C the closure of one of the components of $S^n \smallsetminus \lambda(S^{n-1})$, and $\epsilon > 0$. Then there exists an embedding $\lambda' : C \to S^n$ such that $\rho(\lambda', \operatorname{incl}_C) < \epsilon$ and $\operatorname{Cl}(S^n \smallsetminus \lambda'(C))$ is an n-cell.*

Proof. Apply Theorem 7.7.11 to obtain a 1-LCC embedding relation $T''' : C \to S^n$ in the ϵ-neighborhood of $\operatorname{incl}_C : C \to S^n$. Form the associated decomposition $\mathscr{G}$ into the sets $\{T'''(x) \mid x \in C\}$ and the singletons from $S^n \smallsetminus \operatorname{Im} T'''$, and let C' denote the closure of the component of $S^n/\mathscr{G}$ not containing the image of $T'''(\operatorname{Int} C)$. Corollary 7.7.12 indicates that C' is an n-cell. Proposition 7.4.14 yields that $S^n/\mathscr{G}$ is topologically S^n. Hence, $S^n \cong S^n/\mathscr{G}$ is expressed as the union of a copy of C and the n-cell C', where $C \cap C' = \partial C'$. Obtaining this reembedded copy of C pointwise close to C itself comes about by a controlled shrinking of $\mathscr{G}$, as in the proof of Corollary 7.7.3. □

The Locally Flat Approximation Theorem (7.7.1) combines with Corollary 7.6.11 to give

Corollary 7.7.15. *A locally homotopically unknotted $(n-2)$-sphere Σ in S^n $(n \geq 5)$ is flat if and only if Σ bounds an $(n-1)$-cell $E \subset S^n$.*

As another consequence of 7.7.1, local issues about codimension-one embeddings involving manifolds reduce to problems about embeddings of $(n-1)$-spheres in S^n.

Theorem 7.7.16. *Suppose $n \geq 5$, Q is an $(n-1)$-manifold embedded in an n-manifold M as a closed subset, and $q \in Q$. Then there exist an $(n-1)$-sphere Σ in S^n, a neighborhood N_q of q in M, and an embedding $e : N_q \to S^n$ such that $e(N_q \cap Q) \subset \Sigma$.*

Proof. Exploiting coordinate charts in M, we can simplify the setting so that $q \in Q \subset \mathbb{R}^n \subset S^n$. Fix an $(n-1)$-cell B with $q \in \operatorname{Int} B \subset B \subset Q$, and choose a neighborhood U of q in S^n for which $Q \cap \overline{U} \subset \operatorname{Int} B$. According to Theorem 7.7.1 we can assume that Q is locally flat at each point of $Q \smallsetminus \overline{U}$. Here ∂B is flat (Corollary 7.7.15). Thus we can assume that ∂B is the standard $(n-2)$-sphere in S^n and that a collar C on ∂B in B lies in the standard copy of $S^{n-1} \subset S^n$.

Consider the universal cover $p : E \to S^n \smallsetminus \partial B$; E is homeomorphic to $\operatorname{Int} B^{n-1} \times \mathbb{R}$ in such a way that $p(\operatorname{Int} B \times \{0\}) \subset S^{n-1}$. Lift $\operatorname{Int} B$ to the universal cover E via $\lambda : \operatorname{Int} B \to E$ with the collar C on ∂B mentioned earlier going into, say, $\operatorname{Int} B^{n-1} \times \{0\}$, and extend λ to a lift $\lambda' : N \to E$ defined on some neighborhood N of $\operatorname{Int} B$ for which $N \cap Q = \operatorname{Int} B$. Find an embedding h of the universal cover into a neighborhood of $p(\operatorname{Int} B^{n-1} \times \{0\})$ such that $p\lambda(x) = p\lambda'(x) = x$ for all $x \in C$ and $h(E) \cap S^{n-1} = p(\operatorname{Int} B \times \{0\})$. Then

$$\Sigma = h\lambda'(B) \cup (S^{n-1} \smallsetminus p(\operatorname{Int} B \times \{0\}))$$

is an $(n-1)$-sphere in S^n and $e = h\lambda'$ embeds the neighborhood N of q in the required way. □

Appendix on Embedding Relations

Standard results about continuous relations that carry over from the function setting are:

(1) If $R : X \to Y$ and $S : Y \to Z$ are continuous, then $S \circ R : X \to Z$ is continuous.

(2) If $R : X \to Y$ is continuous with compact point images and $C \subset X$ is compact, then $R(C)$ is compact.

(3) If $R : X \to Y$ is continuous with nonempty, connected point images and $C \subset X$ is connected, then $R(C)$ is connected.

For simplicity we assume all spaces under consideration to be locally compact, separable metric spaces. The relations most useful in geometric topology are the proper relations—recall that a relation $R : X \to Y$ is *proper* if both R and R^{-1} are continuous with compact point images. The equivalent but asymmetric defining property usually ascribed is the following:

(4) A relation $R : X \to Y$ is proper provided R is continuous with compact point images and the inverse of each compact subset of Y is compact.

The basic results on proper relations are:

(5) A proper relation $R : X \to Y$ is a closed subset of $X \times Y$. Each neighborhood of R in $X \times Y$ contains a proper neighborhood $N : X \to Y$ of R in $X \times Y$. Each closed subset of a proper relation is a proper relation.

(6) Composition Theorem. Suppose $R : X \to Y$ and $S : Y \to Z$ are relations where R^{-1} and S both are continuous with compact point images and U is a neighborhood of $S \circ R$ in $X \times Z$. Then there exist neighborhoods V of R in $X \times Y$ and W of S in $Y \times Z$ such that $W \circ V \subset U$.

Proof. Fix $y \in Y$ and observe that the (possibly empty) compact set $R^{-1}(y) \times S(y)$ lies in U, since

$$U \supset S \circ R = S \circ \mathrm{Id}_Y \circ R = (R^{-1} \times S)(\mathrm{Id}_Y).$$

There exist open neighborhoods B_y of $R^{-1}(y)$ in X and C_y of $S(y)$ in Z such that $B_y \times C_y \subset U$. Continuity of R^{-1} and S gives an open neighborhood A_y of y such that $R^{-1}(A_y) \subset B_y$ and $S(A_y) \subset C_y$.

The paracompactness of Y assures the existence of a precise, locally finite, open refinement $\{A'_y \mid y \in Y\}$ of the open cover $\{A_y \mid y \in A\}$ that covers Y and satisfies $\mathrm{Cl}(A'_y) \subset A_y$ for each $y \in Y$. Define $V : X \to Y$ and $W : Y \to Z$ by the formulae $V^{-1}(y) = \cap\{B_{y(0)} \mid y \in \mathrm{Cl}(A'_{y(0)})\}$ and $W(y) = \cap\{C_{y(0)} \mid y \in \mathrm{Cl}(A'_{y(0)})\}$. $\square$

(7) Corollary. If $R : X \to Y$ and $S : Y \to Z$ are continuous with compact point images and U is a neighborhood of $S \circ R$ in $X \times Z$, then there is a neighborhood V of R in $X \times Y$ such that $S \circ V \subset U$.

Proof. Choose compact sets $X_1, X_2, \ldots$ whose interiors cover X. The relations $R|X_i : X_i \to Y$ are proper by (4). Accordingly, by the Composition Theorem, there exist neighborhoods V_i of $R|X_i$ in $X_i \times Y$ with $S \circ V_i \subset U$, and $V = \cup_i V_i$ satisfies the requirements of the Corollary. □

The characteristic feature of cell-like embedding relations is that, by and large, they can be approximated by continuous functions.

(8) Continuous Approximation Theorem. Suppose $R : X \to Y$ is a continuous cell-like embedding relation from a finite-dimensional space X to an ANR Y. Then each neighborhood of R in $X \times Y$ contains a continuous function from X to Y.

See (Cannon, 1975) for a proof. The Continuous Approximation Theorem is not used in this book.

(9) Slice Triviality Theorem. Suppose $R : X \to Y$ is a continuous relation with cell-like point images and $L'' : X \to Y$ is a neighborhood of R. Then there exists another neighborhood L' of R such that $L' \subset L''$ and $x \in X$ implies $L'(x)$ contracts in $L''(x)$.

Proof. For each $x \in X$ find an open set $W \supset R(x)$ in Y and a neighborhood V of x such that $V \times W \subset L''$. Cell-likeness of $R(x)$ leads to another $W_x \supset R(x)$ that contracts in W. Moreover, x has a neighborhood $U_x \subset V$, such that $R(x') \subset W_x$ for all $x' \in U_x$. In case X is compact, corresponding to the open cover $\mathcal{U} = \{U_x \times W_x \mid x \in X\}$ of R is a $\delta > 0$, a kind of Lebesgue number for $\mathcal{U}$, such that for the δ-neighborhood L' of $R \subset X \times Y$ and for arbitrary $x \in X$, some $U_z \times W_z \in \mathcal{U}$ contains $\{x\} \times L'(x)$. Hence, $L'(x) \subset W_z$ contracts in $L''(x)$. The general case (X locally compact) is left to the reader. □

Historical Notes. There are other approaches to the Locally Flat Approximation Theorem. Cannon, Bryant and Lacher (1979) performed multiple grope replacements in M, changing it to ANR homology n-manifold Y equipped with a cell-like map $p : Y \to M$ and a 1-LCC embedding $\lambda : Q \to Y$ of the given $(n-1)$-manifold Q such that $Y \smallsetminus \lambda(S)$ is an n-manifold and $p\lambda = \mathrm{incl}_Q$. Either work of S. Ferry (1979) or application of Quinn's Index Theorem (see §8.5) yields that Y actually is an n-manifold. Upon approximating p by a homeomorphism (Corollary 7.4.2), one obtains a 1-LCC approximation $h\lambda(Q)$ to Q.

Ferry (1992) provided another argument for the Locally Flat Approximation Theorem which combines surgery below the middle dimension with an application of his α-Approximation Theorem, a topic treated briefly in Chapter 8.

As an alternative, in case Q of Theorem 7.7.1 is two-sided, let $C+, C-$ denote the closures of the components of $M \setminus Q$. Form a new space M^* from $C + \sqcup (Q \times [-1,1]) \sqcup C-$ by identifying each $\langle q, -1\rangle$ with $q \in C-$ and, similarly, identifying each $\langle q, +1\rangle$ with $q \in C+$. By (Quinn, 1987) M^* is the cell-like image of an n-manifold, and quite obviously it satisfies the Disjoint Disks Property, so M^* itself is an n-manifold. Moreover, there is an evident cell-like map $p : M^* \to M$ that sends each arc $q \times [-1,1]$ to q; approximating p by a homeomorphism, one notices that the image of $Q \times \{0\} \subset Q \times [-1,1] \subset M^*$ is a bicollared approximation to the original Q.

L. L. Lininger (1965) and N. Hosay (1963) independently proved the 3-dimensional version of Theorem 7.7.11 for embeddings $T : C \to S^n$; Daverman (1977), (1987) did the same in higher dimensions.

Exercises

7.7.1. Suppose $\Sigma^{n-2} \subset S^n$ is a locally flat $(n-2)$-sphere that bounds a topologically embedded $(n-1)$-cell $B \subset S^n$, $n \geq 5$. Then Σ^{n-2} is flat.

7.7.2. If $R : X \to S^n$ is a 1-LCC cell-like embedding relation defined on the compact, k-dimensional space X, where $k \leq n-3$ and $n \geq 5$, then the decomposition of S^n into points of $S^n \setminus \operatorname{Im} R$ and the sets $\{R(x) \mid x \in X\}$ is shrinkable.

7.8. Kirby-Siebenmann obstruction theory

The PL Structure Theorem 6.8.2 of Kirby and Siebenmann has crucially important consequences in codimension one, just as it does in codimension two.

Say that an embedding $\varphi : Q^{n-1} \to M^n$ of one PL manifold in another is *PL locally flat* if φ is PL and if all link pairs in $(M^n, \varphi(Q^{n-1}))$ are PL standard pairs.

Corollary 7.8.1 (Codimension-One Taming). *Suppose Q^{n-1} is a PL $(n-1)$-manifold such that $H^3(Q^{n-1}; \mathbb{Z}_2) = 0$, $h : Q^{n-1} \to M^n$ is a locally flat topological embedding of Q^{n-1} into a PL n-manifold M^n, $n \geq 5$, and $\epsilon > 0$. Then h is ambient isotopic to a PL locally flat embedding via an ϵ-isotopy of M^n.*

The proof coincides with that for the codimension-two taming result (Corollary 6.8.3).

Corollary 7.8.2. *If Q^{n-1} is a PL $(n-1)$-manifold, $n \geq 5$, such that $H^3(Q^{n-1};\mathbb{Z}_2) \neq 0$, then there exist a PL n-manifold M^n and a locally flat embedding $h : Q^{n-1} \to M^n$ that is not ambient isotopic to a PL locally flat embedding.*

Proof. The Product Structure Theorem promises a PL structure on $M^n = Q^{n-1} \times \mathbb{R}$ not compatible with the obvious product structure on $Q^{n-1} \times \mathbb{R}$. If there were a PL locally flat embedding $Q^{n-1} \to M^n$ isotopic to $Q^{n-1} \to Q^{n-1} \times \{0\}$, then an open PL bicollar on the image could be expanded to produce a PL homeomorphism $Q^{n-1} \times \mathbb{R} \to M^n$. □

Locally Flat Approximation Theorem 7.7.1 then leads to a PL Approximation Theorem for the PL manifolds with trivial $\mathbb{Z}_2$-cohomology in dimension 3.

Corollary 7.8.3 (Codimension-One PL Approximation)**.** *Suppose Q^{n-1} is a PL $(n-1)$-manifold such that $H^3(Q^{n-1};\mathbb{Z}_2) = 0$, and suppose $h : Q^{n-1} \to M^n$ is a topological embedding of Q^{n-1} into a PL n-manifold M^n, $n \geq 5$. Then h can be approximated, arbitrarily closely, by PL locally flat embeddings.*

7.9. Detecting 1-LCC embeddings

§4.6 presents conditions for detecting 1-LCC embeddings of codimension-three compacta. This section does the same for embeddings of codimension-one manifolds. Among the conditions covered are local flatness modulo certain twice-flat subsets (Theorem 7.9.2), singular regular neighborhoods (Theorem 7.9.8), and a local spanning property (Theorem 7.9.15).

Here is a slight strengthening of an observation appearing earlier in the proof that 7.7.5 implies 7.7.2.

Lemma 7.9.1. *Let β denote an $(n-1)$-cell in an n-manifold M^n, $f : I^2 \to M^n$ a map such that $f(\partial I^2) \cap \beta = \emptyset$, and Z the component of $I^2 \setminus f^{-1}(\beta)$ containing ∂I^2. Then there exists a map $g : I^2 \to M^n$ such that $g|Z = f|Z$ and $g(I^2 \setminus Z) \subset \beta$; moreover, g can be obtained so that $g(I^2 \setminus Z) \cap \partial\beta \subset f(I^2) \cap \partial\beta$.*

Proof. This follows using the Tietze Extension Theorem to extend

$$f|\overline{Z} \cap (I^2 \setminus Z) : \overline{Z} \cap (I^2 \setminus Z) \to \beta$$

to a map $I^2 \setminus Z \to \beta$. For the strengthened conclusion, simply apply the result using an $(n-1)$-cell $\beta' \subset \beta$ such that $\beta' \supset f(I^2) \cap \beta$ and $\beta' \cap \partial\beta = f(I^2) \cap \partial\beta$. □

Theorem 7.9.2. *Suppose Σ^{n-1} is an $(n-1)$-manifold embedded as a closed subset of an n-manifold M^n and X is a closed subset of Σ^{n-1} such that Σ^{n-1} is 1-LCC in M^n at each point of $\Sigma^{n-1} \setminus X$ and X is LCC^1 in both Σ^{n-1} and M^n. Then Σ^{n-1} is 1-LCC in M^n.*

Proof. Start with $s \in \Sigma^{n-1}$ and $\epsilon > 0$. Identify an $(n-1)$-cell β in Σ with $s \in \operatorname{Int}\beta \subset \beta \subset B(s;\epsilon)$, and then choose a simply connected neighborhood $W_s \subset B(s;\epsilon)$ with $W_s \cap \Sigma^{n-1} \subset \beta$. Given any loop $f : \partial I^2 \to W_s \setminus \Sigma$, extend f to $F : I^2 \to W_s$. Since X is LCC^1 in M^n, Lemma 3.3.3 assures that F can be approximated by a map $F^* : I^2 \to W_s \setminus X$ such that $F^*|\partial I^2 = F|\partial I^2 = f$. Let H denote the component of $I^2 \setminus (F^*)^{-1}(\Sigma)$ containing ∂I^2, and note that $F^*(\overline{H}) \cap X = \emptyset$. Apply Lemma 7.9.1 to obtain a map $g : I^2 \to F^*(H) \cup \beta$ such that $g|H = F^*|H$ and $g(I^2 \setminus H) \subset \beta$. Since X is also LCC^1 in Σ^{n-1}, a mild extension of Lemma 3.3.3 assures that g can be approximated by a map $g^* : I^2 \to F^*(H) \cup \beta$ such that $g^*|H = g|H = F^*|H$ and $g^*(I^2 \setminus H) \subset \beta \setminus X$. Note that $g^*(I^2) \subset (F^*(H) \cup \beta) \setminus X \subset B(s;\epsilon)$. Now g^* can be approximated by a map $g : I^2 \to B(S;\epsilon) \setminus \Sigma^{n-1}$ with $g|\partial I^2 = g^*|\partial I^2 = f$, since Σ^{n-1} is 1-LCC at points of $\Sigma^{n-1} \cap g^*(I^2)$. □

Corollary 7.9.3. *If the $(n-1)$-sphere $\Sigma^{n-1} \subset S^n$, $n \geq 5$, is locally flat modulo a finite set, then Σ^{n-1} is flat.*

Corollary 7.9.4. *If the $(n-1)$-sphere $\Sigma^{n-1} \subset S^n \geq 5$, is locally flat modulo a twice-flat k-cell or k-sphere, $k \leq n-4$, then Σ^{n-1} is flat.*

Remark. Suspensions of examples like the Fox-Artin 2-sphere in S^3 (§2.8.3) indicate that Corollary 7.9.4 fails when $k = n-3$.

Definition. Let Σ^{n-1} denote an $(n-1)$-sphere topologically embedded in S^n and $C \subset \Sigma^{n-1}$ a Cantor set. Extending a previous definition for cells and spheres in Σ^{n-1}, we say that C is *twice flat* if it is flat as a subset of both Σ^{n-1} and S^n.

Corollary 7.9.5. *If the $(n-1)$-sphere $\Sigma^{n-1} \subset S^n$, $n \geq 5$, is locally flat modulo a twice-flat Cantor set, then Σ^{n-1} is flat.*

Definition. Let Σ denote a connected $(n-1)$-manifold topologically embedded in an n-manifold M as a closed, separating subset, and let U denote a component of $M \setminus \Sigma$. We say that Σ can be *homeomorphically approximated from U* if for each $\epsilon > 0$ there exists an embedding $\lambda_\epsilon : \Sigma \to U \subset M$ such that $\rho(\lambda_\epsilon, \mathrm{incl}_\Sigma) < \epsilon$.

Theorem 7.9.6. *Let Σ denote a connected $(n-1)$-manifold topologically embedded in an n-manifold M as a closed, separating subset, and let U denote a component of $M \setminus \Sigma$ such that Σ can be homeomorphically approximated from U. Then Σ is 1-LCC embedded in $\overline{U}$.*

The proof is an exercise. Here there is no need to restrict n.

Corollary 7.9.7. *Let Σ denote a connected $(n-1)$-manifold topologically embedded in an n-manifold M as a closed, separating subset. Then Σ is locally flat in M if and only if it can be homeomorphically approximated from each component of $M \smallsetminus \Sigma$.*

Definition. Let Q denote a connected $(n-1)$-manifold topologically embedded in an n-manifold M as a closed, separating subset, and let U denote a component of $M \smallsetminus Q$. We say that Q is *singularly collared in $\overline{U}$* if there exists a map $\mu : Q \times [0,1] \to \overline{U}$ such that $\mu_0 = \mathrm{incl}_Q$ and $\mu(Q \times (0,1]) \subset U$.

Theorem 7.9.8. *Suppose Q is a connected $(n-1)$-manifold topologically embedded in a connected n-manifold M as a closed, separating subset, and suppose U is a component of $M \smallsetminus Q$ such that Q is singularly collared in $\overline{U}$. Then Q is 1-LCC in $\overline{U}$.*

The argument for 7.9.8 hangs on the notion of degree for maps between (orientable) manifolds, on a result about degree being locally determined relative to the target, and on another result that degree-one maps induce epimorphisms of fundamental groups.

Definitions. An *orientation* of a connected, orientable n-manifold U is a choice of generator of $H^n_c(U;\mathbb{Z})$. Given connected n-manifolds U, V equipped with orientations γ_U, γ_V, respectively, the *degree* of a proper map $f : U \to V$ is the integer d such that $f^*(\gamma_V) = d \cdot \gamma_U$. Functorial properties assure that the degree of a composite (of proper mappings) is the product of degrees.

Lemma 7.9.9. *Suppose $f : M \to N$ is a proper map between connected, oriented n-manifolds and V is a connected open subset of N such that $U = f^{-1}(V)$ is connected. Then the degree of f equals the degree of $f|U$, provided U, V are oriented with the orientations obtained by restriction from M, N, respectively.*

Proof. Let $\rho_U : H^n_c(U;\mathbb{Z}) \to H^n_c(M;\mathbb{Z})$ and $\rho_V : H^n_c(V;\mathbb{Z}) \to H^n_c(N;\mathbb{Z})$ denote the isomorphisms induced by extension. Equality of the degrees follows immediately from the commutativity of

$$\begin{tikzcd} H^n_c(U;\mathbb{Z}) \arrow[d, "\rho_U", "\cong"'] & H^n_c(V;\mathbb{Z}) \arrow[l, "(f|U)^*"'] \arrow[d, "\rho_V", "\cong"'] \\ H^n_c(M;\mathbb{Z}) & H^n_c(N;\mathbb{Z}), \arrow[l, "f^*"'] \end{tikzcd}$$

together with the prescription that the vertical isomorphisms preserve preferred orientations. $\square$

Corollary 7.9.10. *Every proper, non-surjective mapping $f : M \to N$ between orientable n-manifolds has degree* 0.

Proof. Properness implies that $f(M)$ is closed in N. Applying the analysis of 7.9.9 to a connected open subset V of $N \setminus f(M)$, observe that f^* factors through the trivial group $H^n_c(\emptyset;\mathbb{Z})$. $\square$

Lemma 7.9.11. *Suppose $f : M \to N$ is a proper map between connected, oriented n-manifolds, V is a connected open subset of N, $U_1, U_2 \ldots$ are the components of $f^{-1}(V)$, and $f|U_i : U_i \to V$ has degree d_i $(i = 1, 2, \ldots)$. Then $d_i = 0$ for all but finitely many values and the degree of f equals $\Sigma_i d_i$.*

Proof. We re-employ the notation from the preceding lemma and examine the diagram:

$$\begin{tikzcd} H^n_c(f^{-1}(V);\mathbb{Z}) \arrow[d, "\rho"] & H^n_c(V;\mathbb{Z}) \arrow[l, "(f|)^*"'] \arrow[d, "\rho_V", "\cong"'] \\ H^n_c(M;\mathbb{Z}) & H^n_c(N;\mathbb{Z}). \arrow[l, "f^*"'] \end{tikzcd}$$

As above, choices of orientations for M, N give rise to preferred generators γ_V of $H^n_c(V;\mathbb{Z})$ and γ_i for $H^n_c(U_i;\mathbb{Z})$. Here, by hypothesis $(f|U_i)^*$ sends γ_V to $d_i \cdot \gamma_i \in H^n_c(U_i;\mathbb{Z})$. For $v \in V$, at most finitely many components of $f^{-1}(V)$ meet $f^{-1}(v)$, by properness, and Corollary 7.9.10 attests that $d_i = 0$ for those U_i that do not surject to V. Moreover, $H^n_c(f^{-1}(V);\mathbb{Z}) \cong H^n_c(\cup U_i;\mathbb{Z}) \cong \oplus_i H^n_c(U_i;\mathbb{Z})$, so the image of γ_V under the homomorphism in the upper row is $\xi = \Sigma_i(d_i \cdot \gamma_i)$. Under the extension $\rho : H^n_c(\cup_i U_i;\mathbb{Z}) \to H^n_c(M)$, we see that $\rho(\xi) = (\Sigma_i d_i) \cdot \gamma_M$, since each γ_i is sent to the preferred orientation class $\gamma_M \in H^n_c(M;\mathbb{Z})$ via the extension ρ. $\square$

Lemma 7.9.12. *Let p denote a positive integer. If $\theta : \widetilde{M} \to M$ is a p-fold covering map between oriented PL n-manifolds, then the degree of θ is $\pm p$.*

Proof. This widely known result typically is based on other definitions of orientability. In the context at hand, the usual diagram

$$\begin{tikzcd} H^n_c(V;\mathbb{Z}) \arrow[d, "\rho_V", "\cong"'] & H^n_c(V;\mathbb{Z}) \arrow[l, "(h|V)^*"'] \arrow[d, "\rho_V", "\cong"'] \\ H^n_c(M;\mathbb{Z}) & H^n_c(M;\mathbb{Z}) \arrow[l, "h^*=\mathrm{Id}^*"'] \end{tikzcd}$$

reveals that, given a homeomorphism $h : M \to M$ properly homotopic to Id_M and open subset V such that $h(V) = V$, $h|V$ must be orientation preserving. Let $V \subset M$ denote the interior of a PL n-cell evenly covered by θ, $U_1, \ldots, U_p$ the components of $\theta^{-1}(V)$, and ϵ_i the degree of $\theta|U_i : U_i \to V$. Since $\theta|U_i$ is a homeomorphism, $\epsilon_i = \pm 1$. Should there be a pair $\{\epsilon_i, \epsilon_j\}$ with $\epsilon_i = -\epsilon_j$, then just as in (Rourke and Sanderson, 1972, p. 44ff) an isotopy $\widetilde{M} \to \widetilde{M}$ carrying U_i to U_j could be used to produce a homeomorphism $h : M \to M$ properly isotopic to Id_M whose restriction to V reverses orientations. Consequently, Lemma 7.9.11 implies that θ has degree $p \cdot \epsilon_1 = \pm p$. $\square$

Lemma 7.9.13. *Any proper map $f : M \to N$ between connected, oriented, PL n-manifolds of degree ± 1 induces an epimorphism $f_* : \pi_1(M) \to \pi_1(N)$.*

Proof. Suppose to the contrary that $f_*(\pi_1(M)) \neq \pi_1(N)$. Construct the covering space $\theta : \widetilde{N} \to N$ corresponding to $f_*(\pi_1(M))$. Then f lifts to a map $\widetilde{f} : M \to \widetilde{N}$ such that $f = \theta\widetilde{f}$. Localization as in Lemma 7.9.9 assures that f is surjective, for otherwise it would have degree 0.

It can be easily shown that $\widetilde{f}$ is a proper mapping. When θ is a p-fold covering, $p < \infty$, both θ and $\widetilde{f}$ are proper. By Lemma 7.9.12 $|\mathrm{degree}(\widetilde{f})| = [\pi_1(N) : f_*(\pi_1(M))] > 1$. But this is impossible, as it would yield $1 = |\mathrm{degree}(f)| = |\mathrm{degree}(\theta)| \cdot |\mathrm{degree}(\widetilde{f})| > 1$.

We conclude by explaining why θ must be a finite-sheeted cover. Adjusting f via a proper homotopy, we can assume that for some small open set V in N, $f|f^{-1}(V)$ is PL. Let V' denote the interior of an n-simplex σ in V whose preimage under f consists of finitely many n-simplices, each mapped homeomorphically to σ. If θ were infinite-sheeted, $\widetilde{f}$ would have degree 0, as it could not be onto. Let $U_1, \ldots, U_k$ denote the components of $f^{-1}(V')$ and ϵ_i the degree of $f|U_i : U_i \to V'$. The collection $\{U_1, \ldots, U_k\}$ is partitioned into finitely many subcollections corresponding to the preimages under $\widetilde{f}$ of the various components V^* of $\theta^{-1}(V')$. Over each subcollection the various degrees of the restricted $\widetilde{f}$ sum to 0, by Lemma 7.9.11. Upon composing θ and $\widetilde{f}$, we obtain $\epsilon_1 + \cdots + \epsilon_k = 0$, yielding $\mathrm{degree}(f) = 0$, contrary to hypothesis. $\square$

Proof of Theorem 7.9.8. Focus on a point $q \in Q$. In light of Corollary 7.7.16, we can transfer to the setting in which there are an embedding $\lambda : N_q \to S^n$ defined on some neighborhood N_q of q in M and an $(n-1)$-sphere Σ in S^n with $\lambda(N_q \cap Q) \subset \Sigma$. Let U' denote the component of $S^n \smallsetminus \Sigma$ containing points of $\lambda(N_q \cap U)$ arbitrarily close to $s = \lambda(q)$. It will suffice to prove that U' is 1-LC at s.

Fix $\epsilon > 0$. As Q is singularly collared in $\overline{U}$, there exist a small $(n-1)$-cell D on Σ with $s \in \mathrm{Int}\, D$ and a map $\mu : D \times [0,1] \to B(s;\epsilon) \cap \overline{U'}$ such that $\mu_0 = \mathrm{incl}_D$ and $\mu(D \times (0,1]) \subset U'$.

Lemma 7.9.14. *In the setting above, $\overline{U'}$ contains an open subset W such that* $\operatorname{Int} D \subset W \subset f(D \times [0,1])$.

Proof. Use μ to produce a map h of $\Sigma \times [0,1]$ to $\Sigma \cup \mu(D \times [0,1]) \subset S^n$ such that $h(z,0) = z$ for all $z \in \Sigma$, $h(z',t) = z'$ for all $z' \in \Sigma \smallsetminus \operatorname{Int} D$, and $h(\operatorname{Int} D \times \{t\}) \cap \Sigma = \emptyset$ for all $t > 0$. Define W as the intersection of $\overline{U'}$ and the component of $S^n \smallsetminus h_1(\Sigma)$ that contains $\operatorname{Int} D$. We show that $W \subset \mu(D \times [0,1])$.

Suppose to the contrary that there exists

$$w \in W \smallsetminus \mu(D \times [0,1]) \subset W \smallsetminus h(\Sigma \times [0,1]).$$

Choose $z_0 \in N_s \smallsetminus \overline{U'}$ and regard $S^n \smallsetminus \{z_0\}$ as $\mathbb{R}^n$. As in (Hurewicz and Wallman, 1948), for $x \neq z_0$ let π_x denote the radial map of $\mathbb{R}^n \smallsetminus \{x\} = S^n \smallsetminus \{x, z_0\}$ onto the unit $(n-1)$-sphere centered at x. Then $\pi_w h_0$ and $\pi_w h_1$ are homotopic maps of Σ to S^{n-1}. For any $y \in S^n \smallsetminus \overline{U'}$ we see that w and y belong to different components of $S^n \smallsetminus h(\Sigma \times \{0\})$ and belong to the same component of $S^n \smallsetminus h(\Sigma \times \{1\})$. This leads to the desired contradiction, because according to Theorem VI.10 of (Hurewicz and Wallman, 1948), $\pi_w h_0$ is an essential map and $\pi_w h_1$ is an inessential map. $\square$

Continuing with the proof of Theorem 7.9.8, apply Lemma 7.9.14 to obtain the promised connected open subset W of $\overline{U'}$ such that

$$\operatorname{Int} D \subset W \subset \mu(D \times [0,1]) \subset B(s;\epsilon).$$

Identify the component Y of $\mu^{-1}(W)$ containing $\operatorname{Int} D \times \{0\}$. Set $W_U = W \cap U'$, $Y_U = Y \cap \mu^{-1}(W_U)$, and $\mu_U = \mu | Y_U$. It follows immediately that $\mu_U : Y_U \to W_U$ is a proper map between (orientable) n-manifolds, which implies that the degree of μ_U is defined. We shall prove that μ_U has degree ± 1.

By Theorem 7.7.14 we can assume that U' is embedded in S^n with an n-cell as its complement. Hence, μ_U extends to a map

$$\widetilde{\mu} : Y_U \cup (\operatorname{Int} D \times (-1,0]) \to S^n$$

with $\widetilde{\mu}| \operatorname{Int} D \times (-1,0]$ an embedding into $S^n \smallsetminus U'$ such that

$$\widetilde{\mu}(\operatorname{Int} D \times (-1,0]) \cap \Sigma = \widetilde{\mu}(\operatorname{Int} D \times \{0\}).$$

It follows that $\widetilde{\mu}$ is a proper mapping between orientable n-manifolds. Since the image obviously contains the connected open subset $\widetilde{\mu}(\operatorname{Int} D \times (-1,0))$ over which $\widetilde{\mu}$ is a homeomorphism, Lemma 7.9.9 assures that $\widetilde{\mu}$ and μ_U have degree ± 1.

As a result, μ_U induces an epimorphism at the fundamental group level (Lemma 7.9.13). To each loop α in W_U corresponds a loop α' in Y_U with $(\mu_U)_*([\alpha']) = [\alpha]$. Since α' is contractible in $\operatorname{Int} D \times (0,1]$, α is contractible in $\mu(\operatorname{Int} D \times (0,1]) \subset B(s;\epsilon) \cap U'$. Hence, U' is 1-LC at s, and Q is 1-LCC in $\overline{U}$. □

Definition. Suppose Q is an $(n-1)$-manifold topologically embedded in an n-manifold M as a closed subset, and suppose U is a component of $M \smallsetminus Q$. We say that Q can be *locally spanned from* U if, for each neighborhood N_q of each $q \in Q$, there exist $(n-1)$-cells $B \subset Q$ and $D \subset \overline{U}$ such that $s \in \operatorname{Int} B$, $D \cap Q = \partial B$, and $B \cup D \subset N_q$.

Theorem 7.9.15. *Suppose Σ is a connected $(n-1)$-manifold topologically embedded in a connected n-manifold M as a closed, separating subset, and suppose U is a component of $M \smallsetminus \Sigma$ such that Σ can be locally spanned from U. Then Σ is 1-LCC in $\overline{U}$.*

Proof. Consider a map f of ∂I^2 into a small subset of U. Extend f to a map F of I^2 into a small subset of $\overline{U}$. In view of the locally spanned condition, we determine a finite collection of very small $(n-1)$-cells $B_1, ..., B_k$ in Σ whose interiors cover $\Sigma \cap F(I^2)$, and corresponding $(n-1)$-cells $D_1, ..., D_k$ in $\overline{U}$ such that $D_i \cap \Sigma = \partial B_i$ and $D_i \cap f(\partial I^2) = \emptyset$ for each i. Application of Lemma 7.9.1 yields a new map $F_1 : I^2 \to U$ with small image contained in $F(I^2) \cup D_1 \smallsetminus \operatorname{Int} B_1$ and with $F_1|\partial I^2 = F|\partial I^2 = f$, where in particular $F_1(I^2) \cap \partial B_1 \subset F(I^2) \cap \partial B_1$. Consequently, $F_1(I^2) \cap \Sigma \subset F(I^2) \cap (\Sigma \smallsetminus \operatorname{Int} B_1)$. Another $k-1$ applications of Lemma 7.9.1 yield a new map $F_k : I^2 \to U$ with small image in $(F(I^2) \cup (\cup_i D_i)) \smallsetminus \Sigma$ and $F_k|\partial I^2 = F|\partial I^2 = f$. The key is that, once a point of Σ is removed from the image of F_i, it reappears in none of the succeeding images. □

Corollary 7.9.16. *An $(n-1)$-sphere Σ in S^n is 1-LCC if and only if Σ is locally spanned from each component of $S^n \smallsetminus \Sigma$.*

Corollary 7.9.17. *An $(n-1)$-sphere Σ in S^n, $n \geq 5$, is flat if and only if Σ is locally spanned from each component of $S^n \smallsetminus \Sigma$.*

Historical Notes. The distinction between the cases $n = 3$ and $n > 3$ of codimension-one embeddings was first displayed by J. C. Cantrell's doctoral dissertation, ultimately refined into the result about $(n-1)$-spheres in S^n, $n > 3$, necessarily being flat if they are locally flat modulo a single point. Key ideas appeared in the treatment of Theorem 2.9.3. Corollary 7.9.3 is a mild generalization. Kirby (1968a) provided an elegant geometric construction to establish Corollary 7.9.5 for all $n > 3$.

For embeddings in 3-manifolds flatness results emerged in an order rather opposite to those in high dimensions. Bing's original flattening theorem (1959b) was the low-dimensional version of Corollary 7.9.7. Later he used it to show that a surface in a 3-manifold is locally flat if it is 1-LCC embedded (Bing, 1961b).

J. Hempel (1964) showed that a compact 2-manifold in S^3 which is singularly collared on both sides is 1-LCC embedded; Daverman (1976) developed the argument used here for Theorem 7.9.8 that works in all ambient dimensions; Bryant and Lacher (1975) independently obtained the same result and generalizations to other codimensions as well.

P. Olum (1953) proved that maps of degree 1 between connected oriented n-manifolds induce epimorphisms at the π_1 level. D. B. A. Epstein (1966) introduced a valuable geometric notion of degree; he showed that a proper map : $M \to N$ between connected, oriented n-manifolds has degree p if and only if p is the minimum integer such that, for some map g properly homotopic to f and small open n-cell $V \subset N$, $g^{-1}(V)$ has exactly p components, on each of which g restricts to a homeomorphism.

A singular regular neighborhood can be regarded as the image of a homotopy of a codimension-one manifold S that instantly deforms S into its complement. Instead of homotopies, one might consider a sequence of approximations to the inclusion; a codimension-one sphere Σ in S^n is said to be *free* if, for each $\epsilon > 0$ and each component U of $S^n \smallsetminus \Sigma$, there is a map $f_\epsilon : \Sigma \to U$ that moves points less than ϵ. As of this writing, whether free $(n-1)$-spheres in S^n, $n \geq 3$, must be 1-LCC embedded is still unknown; a partial result about the implications of freeness toward flatness appears in §7.11.

C. E. Burgess (1965) introduced the locally spanned concept and proved Theorem 7.9.15 in the 3-dimensional setting; his proof immediately applies in all dimensions.

Exercises

7.9.1. Suppose $\Sigma^{n-1} \subset S^n$, $n \geq 5$, is a wildly embedded $(n-1)$-sphere with complementary domains U, V for which there is a homeomorphism $\psi : \overline{U} \to \overline{V}$ with $\psi|\Sigma = \text{incl}_\Sigma$. Also suppose Σ is locally flat modulo a k-cell, $k \leq n-3$, or a Cantor set C. Show that C is wildly embedded in S^n.

7.9.2. Prove Theorem 7.9.6.

7.9.3. Show that most embeddings of S^{n-1} in S^n, $n \geq 5$, are locally flat (i.e., show that the locally flat embeddings form a dense, G_δ-subset of $\text{Emb}(S^{n-1}, S^n)$).

7.10. Sewings of crumpled n-cubes

Definitions. Let Σ be an $(n-1)$-sphere topologically embedded in S^n and U one of the components of $S^n \smallsetminus \Sigma$. Then $\overline{U}$ is a *crumpled n-cube*, and the sphere Σ is called the *boundary* of $\overline{U}$, denoted $\mathrm{Bd}\,\overline{U}$. A crumpled n-cube C in S^n is a *closed n-cell complement* if $\mathrm{Cl}(S^n \smallsetminus C)$ is an n-cell.

Restated in this terminology, Theorem 7.7.14 certifies that every crumpled n-cube ($n \geq 5$) admits an embedding in S^n as a closed n-cell complement. This signals that wildness on one side of an embedded codimension-one manifold forces no corresponding wildness on the other side. Our attention next turns to a complementary issue: what sorts of wildness on one side can be matched with wildness on the other side?

Definitions. A *sewing* of crumpled n-cubes C_1 and C_2 is a homeomorphism $h : \mathrm{Bd}\,C_1 \to \mathrm{Bd}\,C_2$. Associated with any such sewing h is the *sewing space* $C_1 \cup_h C_2$, namely, the quotient space obtained from the disjoint union $C_1 \sqcup C_2$ after identification of each $x \in \mathrm{Bd}\,C_1$ with $h(x) \in \mathrm{Bd}\,C_2$.

Sewings of crumpled cubes enhance the proliferation of wildness. The crucial question about a sewing of crumpled n-cubes is whether the sewing space is an n-manifold. When it is, the manifold necessarily is S^n (see Corollary 7.10.3), and then S^n contains a separating $(n-1)$-sphere Σ bounding copies of the two crumpled cubes, which are matched up along Σ exactly as prescribed by the sewing.

In light of Theorem 7.7.14, the sewing will always yield S^n when no wild point in one crumpled cube is matched with a wild point in the other. This section presents several examples indicating that an arbitrary sewing of crumpled cubes need not have a manifold as its sewing space and it probes conditions under which a sewing space is S^n, despite possible overlapping of the wildness. It also introduces an inflation technique for producing wildness, which it exploits to fabricate wild spheres in $\mathbb{R}^n$ that are locally flat modulo subspheres flatly embedded in $\mathbb{R}^n$. The most elaborate example—Example 7.10.14, which relies upon the construction of ramified wild Cantor sets from §4.8—is a crumpled cube C such that $C \cup_{\mathrm{Id}} C$ does not yield S^n. This C contains two embedded loops such that any singular disk in C bounded by the first meets every singular disk in S^n bounded by the second. That is precisely the feature necessitating the elaborate blow-up procedure of §7.7 used to establish the 1-LCC Approximation Theorem.

Here is an elementary consequence of Theorem 7.7.14.

Proposition 7.10.1. *If C is any crumpled n-cube in S^n, $n \geq 5$, and $h : \mathrm{Bd}\,C \to \partial B^n$ any sewing to the boundary of an n-cell, then $C \cup_h B^n \cong S^n$.*

Proposition 7.10.2. *For any sewing $h : \mathrm{Bd}\,C_1 \to \mathrm{Bd}\,C_2$ of crumpled n-cubes, $n \geq 5$, there exists a cell-like mapping $S^n \to C_1 \cup_h C_2$.*

Proof. Apply Theorem 7.4.14 or the preceding proposition to regard C_2 as embedded in S^n so that $\mathrm{Cl}(S^n \smallsetminus C_2)$ is an n-cell B. Specify a collar

$\lambda : \partial B \times [0,1] \to B$ on $\partial B = \operatorname{Bd} C_2$ (with $\lambda_0 = \mathrm{incl}_{\partial B}$), set

$$B' = C_2 \cup \lambda(\partial B \times [0,1]),$$

and note that, by Corollary 2.4.12 to the Generalized Schönflies Theorem, B' is also an n-cell (see Figure 7.11). Define a sewing $h' : \operatorname{Bd} C_1 \to \partial B'$ as $h'(x) = \lambda(h(x), 1)$. Then $C_1 \cup_{h'} B' \cong S^n$, by Proposition 7.10.1, and the decomposition of $S^n = C_1 \cup_{h'} B'$ into points and the arcs

$$\{\lambda(\{b\} \times [0,1]) \subset B' \mid b \in \partial B\}$$

gives rise to a cell-like mapping $S^n \to C_1 \cup_h C_2$. $\square$

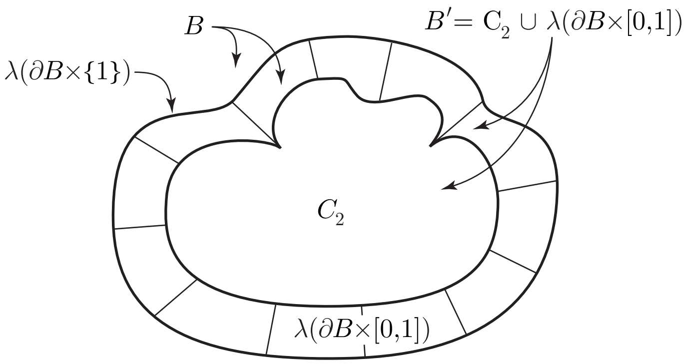

Figure 7.11. The domain of the cell-like map $S^n \to C_1 \cup_h C_2$

With an application of Corollary 7.4.2 we obtain:

Corollary 7.10.3. *If a sewing $h : \operatorname{Bd} C_1 \to \operatorname{Bd} C_2$ of crumpled n-cubes ($n \geq 5$) yields a manifold, then $C_1 \cup_h C_2 \approx S^n$.*

Proposition 7.10.4. *Let C_1 and C_2 be closed n-cell complements in S^n, $n \geq 5$, and $h : \operatorname{Bd} C_1 \to \operatorname{Bd} C_2$ a sewing. Then a necessary condition for $C_1 \cup_h C_2$ to be S^n is that any two maps $f_i : I^2 \to C_i$, $i \in \{1,2\}$, can be approximated, arbitrarily closely, by maps $F_i : I^2 \to S^n$ such that*

$$F_2(I^2) \cap h(\operatorname{Bd} C_1 \cap F_1(I^2)) = \emptyset.$$

Proof. Let $\lambda_i : C_i \to S^n, i \in \{1,2\}$, be embeddings such that

$$\lambda_1(C_1) \cap \lambda_2(C_2) = \lambda_1(\operatorname{Bd} C_1) = \lambda_2(\operatorname{Bd} C_2),$$

$\lambda_1(C_1) \cup \lambda_2(C_2) = S^n$, and $\lambda_1 | \operatorname{Bd} C_1 = \lambda_2 h$. Find a small neighborhood W_i of $\lambda_i(C_i)$ and a retraction $R_i : W_i \to \lambda_i(C_i)$ close to the identity on W_i, with $R_i(W_i \smallsetminus \lambda_i(C_i)) \subset \lambda_i(\operatorname{Bd} C_i)$. Approximate the maps $\lambda_i f_i : I^2 \to \lambda_i(C_i)$ by maps $g_i : I^2 \to W_i$ with disjoint images and with $\lambda_i^{-1} R_i g_i$ close to f_i. Set $U_i = g_i^{-1}(W_i \smallsetminus \lambda_i(C_i))$, observe that $\lambda_i^{-1} R_i g_i(U_i) \subset \operatorname{Bd} C_i$, and use the fact that each C_i is a closed n-cell complement in S^n to adjust each $\lambda_i^{-1} R_i g_i$ to a new map $F_i : I^2 \to S^n$ such that $F_i|I^2 \smallsetminus U_i = \lambda_i^{-1} R_i g_i | I^2 \smallsetminus U_i = \lambda_i^{-1} g_i | I^2 \smallsetminus U_i$ and $F_i(U_i) \subset S^n \smallsetminus C_i$. Set $Y_i = (F_i)^{-1}(\mathrm{Bd}\, C_i)$. It follows that

$$\begin{aligned} F_2(I^2) \cap h(\mathrm{Bd}\, C_1 \cap F_1(I^2)) &\subset \lambda_2^{-1} g_2(Y_2) \cap h\lambda_1^{-1} g_1(Y_1) \\ &\subset \lambda_2^{-1}(g_2(Y_2) \cap g_1(Y_1)) = \emptyset. \end{aligned} \qquad \square$$

Theorem 7.10.5. *Let C_1 and C_2 be crumpled n-cubes ($n \geq 5$) satisfying the Disjoint Disks Property and let $h : \mathrm{Bd}\, C_1 \to \mathrm{Bd}\, C_2$ be a sewing such that any two maps $f_i : I^2 \to C_i$, $i \in \{1, 2\}$, can be approximated, arbitrarily closely, by maps $F_i : I^2 \to C_i$ such that*

$$F_2(I^2) \cap h(\mathrm{Bd}\, C_1 \cap F_1(I^2)) = \emptyset.$$

Then $C_1 \cup_h C_2$ is topologically S^n.

Proof. This is mainly a rephrasing of Proposition 7.4.12. Its hypotheses hold by Proposition 7.10.2 and Corollary 7.4.8. $\square$

Remark. Regardless of whether C_1 and C_2 satisfy the Disjoint Disks Property, a sewing $h : \mathrm{Bd}\, C_1 \to \mathrm{Bd}\, C_2$ yields S^n if h satisfies the mismatch property of Theorem 7.10.5 (Cannon and Daverman, 1981).

Theorem 7.10.6. *For any crumpled n-cube C, $n \geq 5$, $C \cup_{\mathrm{Id}} C \cong S^n$ if and only if C satisfies the Disjoint Disks Property.*

Proof. If C satisfies the Disjoint Disks Property, then the identity sewing satisfies the mismatch property of Theorem 7.10.5 and $C \cup_{\mathrm{Id}} C \cong S^n$.

For the other implication, in case $C \cup_{\mathrm{Id}} C \cong S^n$, there exists a retraction $r : S^n \to C$ that is 1-1 over $\mathrm{Bd}\, C$; r simply folds one of the copies of C over onto the other. As in 7.10.4, let $\lambda_i : C \to S^n$, $i \in \{1, 2\}$, be embeddings such that $\lambda_1(C) \cup \lambda_2(C) = S^n$, $\lambda_1(C) \cap \lambda_2(C) = \lambda_1(\mathrm{Bd}\, C) = \lambda_2(\mathrm{Bd}\, C)$ and $r\lambda_1 = r\lambda_2 = \mathrm{Id}_C$. Given maps $\mu_1, \mu_2 : I^2 \to C$, approximate $\lambda_1\mu_1, \lambda_2\mu_2$ by maps $\mu_1', \mu_2' : I^2 \to S^n$ with disjoint images. Then $r\mu_1', r\mu_2'$ are maps of I^2 to C such that

$$r\mu_1'(I^2) \cap r\mu_2'(I^2) \cap \mathrm{Bd}\, C = \emptyset.$$

Make a further (general position) approximation over the n-manifold $\mathrm{Int}\, C$ to obtain maps $\mu_1'', \mu_2'' : I^2 \to C$ with disjoint images. $\square$

Corollary 7.10.7. *If C is a crumpled n-cube, $n \geq 5$, satisfying the Disjoint Disks Property, then there exists an involution $u : S^n \to S^n$ with orbit space homeomorphic to C. In particular, the fixed point set of u is a wild $(n-1)$-sphere (provided C is not an n-cell).*

Definition. The *orbit space* of an involution $u : S^n \to S^n$ is the quotient of S^n obtained by identifying each $x \in S^n$ with $u(x)$.

To obtain quick applications of Theorem 7.10.5 it is advantageous to develop conditions under which a crumpled cube C has the Disjoint Disks Property.

Lemma 7.10.8. *If the boundary of a crumpled n-cube $C \subset S^n$ is locally flat modulo a Cantor set, k-cell or k-sphere Z that is flat in $\mathrm{Bd}\, C$, where $n \geq 5$ and $1 \leq k \leq n-4$, then C has the Disjoint Disks Property.*

Proof. As before, identify a neighborhood W of C in S^n and retraction $R : W \to C$ for which $R(W \setminus C) \subset \mathrm{Bd}\, C$. Given maps $f_1, f_2 : I^2 \to C$, produce approximations $f_1', f_2' : I^2 \to W \subset S^n$ such that $f_1'(I^2) \cap f_2'(I^2) = \emptyset$. Restrict W to assure that each Rf_i' is a close approximation to f_i. Let $U_i = (f_i')^{-1}(W \setminus C)$. Invoke the hypothesis about Z being flat in $\mathrm{Bd}\, C$ to determine maps $F_1, F_2 : I^2 \to C$ such that $F_i | I^2 \setminus U_i = Rf_i' | I^2 \setminus U_i = f_i' | I^2 \setminus U_i$, $F_i(U_i) \subset \mathrm{Bd}\, C \setminus Z$, and F_i is close to Rf_i'. Since $\mathrm{Bd}\, C$ is locally collared in C at points of $F_i(U_i)$, the maps F_i can be further adjusted, fixing $I^2 \setminus U_i$ while pushing points of $F_i(U_i)$ away from $\mathrm{Bd}\, C$, thereby yielding maps $F_i' : I^2 \to C$ such that

$$F_1'(I^2) \cap F_2'(I^2) \cap \mathrm{Bd}\, C \subset f_1'(I^2 \setminus U_1) \cap f_2'(I^2 \setminus U_2) \cap \mathrm{Bd}\, C = \emptyset.$$

Finally, they can be adjusted once more over $\mathrm{Int}\, C$ so as to have disjoint images. □

Example 7.10.9. *There exists a wild $(n-1)$-sphere Σ in $\mathbb{R}^n$ that is locally flat modulo a k-sphere flatly embedded in $\mathbb{R}^n$ $(1 \leq k \leq n-2)$.*

Such an example arises by inflating any crumpled $(n-1)$-cube C in $\mathbb{R}^{n-1}$ whose double $C \cup_{\mathrm{Id}} C$ is S^{n-1} (or, equivalently, C has the Disjoint Disks Property). Think of C as a subset of $\mathbb{R}^{n-1}$—for definiteness, assume C to be collared in $\mathbb{R}^{n-1} \setminus \mathrm{Int}\, C$—and let $\nu : C \to [0,1]$ be a map such that $\nu^{-1}(0) = \mathrm{Bd}\, C$. By an *inflation of* C we mean

$$\mathrm{Infl}(C, \nu) = \{\langle x, t\rangle \in \mathbb{R}^{n-1} \times \mathbb{R}^1 = \mathbb{R}^n \mid x \in C \text{ and } |t| \leq \nu(x)\}.$$

The frontier Σ of $\mathrm{Infl}(C, \nu)$ is the union of the two copies of C, the graphs of $\pm\nu$, sewn together via the Identity map along their boundaries and so, by hypothesis, is an $(n-1)$-sphere. Clearly, the topological type of $\mathrm{Infl}(C, \nu)$ does not depend on the choice of map ν, so from here on out we shall refer to such a construction as an *inflation of* C, denoted $\mathrm{Infl}(C)$, without reference to any specific ν.

The $(n-1)$-sphere $\Sigma = \mathrm{Bd}\,\mathrm{Infl}(C)$ is locally flat modulo the $(n-2)$-sphere $\Sigma \cap (\mathbb{R}^{n-1} \times \{0\}) \cong \mathrm{Bd}\, C$, which is flat in $\mathbb{R}^n$ (see Exercise 6.3.2 and Theorem 6.3.6). Moreover, Σ is wildly embedded (assuming C is not a cell) because $C \times \{0\}$ is a strong deformation retract of $\mathrm{Infl}(C)$ via a deformation that moves points vertically—in the $\mathbb{R}^1$ direction—and preserves interiors;

hence, the interior of $\mathrm{Infl}(C)$ is 1-LCC at $\langle x, 0\rangle \in \mathrm{Bd}\, C \times \{0\}$ if and only if $\mathrm{Int}\, C$ is 1-LCC at x.

Take C to be a crumpled $(n-1)$-cube in $\mathbb{R}^{n-1}$, $n > 5$, whose frontier is locally flat modulo a Cantor set X standardly embedded in $\mathrm{Bd}\, C$. Theorem 7.10.6 and Lemma 7.10.8 assure that $C \cup_{\mathrm{Id}} C \cong S^{n-1}$. For $k = 1, \ldots, n-2$ let D_k denote a k-cell in $\mathrm{Bd}\, C$ containing X. Then Σ is locally flat modulo the k-cell $D_k \times \{0\} \subset \mathrm{Bd}\, C \times \{0\} \subset \Sigma$, which is flat in $\mathbb{R}^n$ by Corollary 4.6.10 when $k < n-2$ and by Theorem 6.3.6 when $k = n-2$. For that matter, Σ is locally flat modulo the Cantor set $X \times \{0\}$, which also is flat in $\mathbb{R}^n$.

Lemma 7.10.10. *If the boundary of a crumpled n-cube C is locally flat modulo a Cantor set, k-cell or k-sphere that is flat in S^n, where $n \geq 5$ and $1 \leq k \leq n-3$, then C has the Disjoint Disks Property.*

The proof, which is similar to that of 7.10.8, is left as an exercise.

Definition. A group G of homeomorphisms on a space X is said to act *semifreely* on X if there exists a subset Z of X such that for every $g \in G$, $g(z) = z$ for all $z \in Z$ and $g(x) \neq x$ for all $x \in X \setminus Z$ and all $g \neq \mathrm{Id}_X$.

Theorem 7.10.11. *Let C be a crumpled n-cube, $n \geq 5$, such that $C \cup_{\mathrm{Id}} C \cong S^n$. Then there exists a semifree S^1-action on S^{n+1} having an $(n-1)$-sphere as its fixed point set and having orbit space homeomorphic to C.*

Proof. There is a map $p : S^1 \times C \to \mathrm{Infl}(C) \cup_{\mathrm{Id}} \mathrm{Infl}(C)$ which is 1-1 on $S^1 \times \mathrm{Int}\, C$ and which behaves like projection to the second factor on $S^1 \times \mathrm{Bd}\, C$. The obvious free S^1-action on $S^1 \times C$ (trivial on the C factor) descends under p to a semifree action on $\mathrm{Infl}(C) \cup_{\mathrm{Id}} \mathrm{Infl}(C)$, and the latter is topologically S^n, since $\mathrm{Infl}(C)$ has the Disjoint Disks Property (another exercise). $\square$

Example 7.10.12. *There exist a crumpled n-cube C having the Disjoint Disks Property and a homeomorphism $h : \mathrm{Bd}\, C \to \mathrm{Bd}\, C$ such that $C \cup_h C$ fails to be a manifold.*

Proof. Start with a wild Cantor set X_n in S^n equipped with a special geometric defining sequence and an embedded loop $e_n(\partial I^2) \subset S^n \setminus X_n$ as in Lemma 4.8.5. Construct an n-cell B in $S^n \setminus e_n(\partial I^2)$ containing X_n as a standardly embedded Cantor set in ∂B, and set $C = S^n \setminus \mathrm{Int}\, B$. Apply Mixing Lemma 4.8.1 to obtain a homeomorphism $\tau : X_n \to X_n$ mixing the admissible subsets of X_n. Since X_n is flat in $\mathrm{Bd}\, C = \partial B$, τ extends to a homeomorphism $h : \mathrm{Bd}\, C \to \mathrm{Bd}\, C$. Now by Proposition 7.10.4 $C \cup_h C$ cannot be an n-manifold: there is a special Cantor set X, the image of X_n in the sewing space, and disjoint loops away from X, one in each copy of C, which ought to bound essentially disjoint disks in the n-manifold (at least for $n \geq 5$), but any two such singular disks must meet somewhere in X, by the mixing property. $\square$

Example 7.10.13. *There exist a crumpled n-cube $C^* \subset S^n$ and embedded loops $e_n(\partial I^2)$, $e'(\partial I^2)$ in $\operatorname{Int} C^*$ such that the image of every singular disk in C^* bounded by e_n intersects the image of every singular disk in S^n bounded by e'.*

For $k = n-1, n$, apply Lemma 4.8.7 to produce wild Cantor sets X_k in S^k equipped with compatible special geometric defining sequences, each with the strong interior inessential property. Identity loops $e_k(\partial I^2) \subset S^k \smallsetminus X_k$ such that every singular disk in S^k bounded by $e_k(\partial I^2)$ contains an admissible subset of X_k (Corollary 4.8.6). Build k-cells $B_k \subset S^k \smallsetminus e_k(\partial I^2)$ with X_k standardly embedded in ∂B_k and ∂B_k locally flat modulo X_k. Form the complementary crumpled k-cube $C_k = S^k \smallsetminus \operatorname{Int} B_k$.

Note that C_k has the Disjoint Disks Property by Lemma 7.10.8 (provided $k \geq 5$) . Inflate C_{n-1} to a crumpled n-cube $C' = \operatorname{Infl}(C_{n-1})$. There is a natural embedding $e' : \partial I^2 \to C'$ for which every singular disk $F'(I^2)$ in C' bounded by $e'(\partial I^2)$ contains an admissible subset of X_{n-1}. Construct an $(n-1)$-cell $\beta' \subset \operatorname{Bd} C'$ containing X_{n-1} in its boundary as a standardly embedded subset, with β' locally flatly embedded in $\operatorname{Bd} C'$ modulo X_{n-1}, and construct a similar $(n-1)$-cell $\beta \subset \operatorname{Bd} C_n$ with X_n standardly embedded in $\operatorname{Bd} \beta$. Produce a homeomorphism $h : \beta' \to \beta$ such that $h|X_{n-1} : X_{n-1} \to X_n$ mixes the admissible subsets of the Cantor sets.

Let $C^* = C' \sqcup_h C_n$ denote the object obtained from the disjoint union of C' and C_n by gluing $\beta' \subset \operatorname{Bd} C'$ to $\beta \subset \operatorname{Bd} C_n$ via h. Regard C' as embedded in S^n so $S^n \smallsetminus \operatorname{Int} C'$ is an n-cell B'. Thicken β' to an n-cell $D' \subset B'$ locally flat modulo $X_{n-1} \subset \partial D'$, where β' and X_{n-1} are flatly embedded in $\partial D'$. Then D' is flat in S^n, so $S^n \smallsetminus \operatorname{Int} D'$ is an n-cell $D \supset C'$. The homeomorphism $h : \beta' \to \beta$ extends to a homeomorphism $H : \partial D \to \operatorname{Bd} C_n$. Then $C^* = C' \cup_h C_n$ has an obvious embedding in $D \cup_H C_n \cong S^n$.

This object C^* contains two noteworthy loops: $e_n : \partial I^2 \to C_n \subset C^*$ and $e' : \partial I^2 \to C' \subset C^*$. Consider singular disks $F_n(I^2), F'(I^2)$ in S^n, C' bounded by e_n, e', respectively. Then $F_n(I^2)$ contains an admissible subset of $j(X_n) \subset j(C_n)$. To see that $F'(I^2)$ contains an admissible subset of $j'(X_{n-1}) \subset j'(C')$, one can modify F' using Lemma 7.9.1 to obtain another map $F^* : I^2 \to C'$ such that $F^*(I^2) \subset F'(I^2) \cup \beta^*$, where β^* denotes the image of $\beta = h(\beta')$ in C^*, and where $F^*(I^2) \cap \partial\beta^* = F'(I^2) \cap \partial\beta^*$. Hence, $F'(I^2) \cap j'(X_{n-1}) = F^*(I^2) \cap j(X_{n-1})$ contains the image under j' of an admissible subset of X_{n-1}. This implies that $F'(I^2)$ and $F_n(I^2)$ intersect.

Clearly C^* cannot have the Disjoint Disks Property. As a result, it also serves as:

Example 7.10.14. *There exists a crumpled n-cube C^* such that $C^* \cup_{\mathrm{Id}} C^*$ fails to be a manifold.*

Historical Notes. Bing was the first to produce periodic homeomorphisms of $\mathbb{R}^n$ and S^n having wild fixed point sets; in perhaps his most widely known example of this type, he demonstrated (Bing, 1952) that $AH \cup_{\mathrm{Id}} AH \cong S^3$, where AH denotes the crumpled 3-cube bounded by Alexander's horned sphere, so S^3 admits an involution fixing a wild 2-sphere. R. J. Daverman and W. T. Eaton (1969) proved that an arbitrary sewing $h : \mathrm{Bd}\, C_1 \to \mathrm{Bd}\, C_2$ of crumpled 3-cubes can be approximated by another sewing h' such that $C_1 \cup_{h'} C_2 \cong S^3$; nothing comparable is known about arbitrary sewings of crumpled n-cubes, $n \geq 5$. Eaton (1972) showed the mismatch property of Theorem 7.10.5 to be a necessary and sufficient condition for a sewing of two crumpled 3-cubes to yield S^3; Cannon and Daverman (1981) showed it be a sufficient condition for a sewing of crumpled n-cubes to yield S^n, $n \geq 4$. Daverman (see comments in (1981)) introduced the inflation process as a method of constructing wild codimension-one embeddings. He also (2007) provided various mismatch properties under which a sewing of crumpled cubes yields S^n.

Exercises

7.10.1. If the inflation $\mathrm{Infl}(C)$ of a crumpled $(n-1)$-cube C is bounded by a sphere Σ, $n \geq 5$, then Σ is collared from $\mathrm{Cl}(\mathbb{R}^n \smallsetminus \mathrm{Infl}(C))$.

7.10.2. Prove Lemma 7.10.10.

7.10.3. If the crumpled cube C satisfies the Disjoint Disks Property, then so does $\mathrm{Infl}(C)$.

7.10.4. For $n > 3$ the suspension of any crumpled n-cube C satisfies the Disjoint Disks Property.

7.10.5. If the crumpled cube C satisfies the Disjoint Disks Property, then each map $f : I^2 \to C$ can be approximated by an embedding $F : I^2 \to C$ such that $F(I^2) \cap \mathrm{Bd}\, C$ is 0-dimensional.

7.11. Wild examples and mapping cylinder neighborhoods

The presence of mapping cylinder neighborhoods imposes considerable regularity on an embedding, but not enough regularity to ensure local flatness. At the heart of §7.11 is a construction in Example 7.11.2 of a codimension-one sphere wildly embedded in S^n despite possessing a mapping cylinder neighborhood. Complementing the example is an initial result indicating that the combination of mapping cylinder neighborhood and freeness implies local flatness for codimension-one manifold embeddings.

The methods for producing Example 7.11.2 lead to other applications, including the construction here of a wild Cantor set whose embedding satisfies a strong homogeneity property.

Theorem 7.11.1. *Suppose Σ^{n-1} is a connected, two-sided $(n-1)$-manifold in an n-manifold M, $n \geq 5$, such that Σ^{n-1} has a mapping cylinder neighborhood and is free. Then Σ^{n-1} is bicollared.*

Proof. Specify a component U of $M \smallsetminus \Sigma^{n-1}$. It suffices to show that Σ^{n-1} is 1-LCC in $\overline{U}$. Apply the hypothesis to obtain a proper map $\psi : N^{n-1} \to \Sigma^{n-1}$ defined on an $(n-1)$-manifold N^{n-1} such that Σ^{n-1} has a closed neighborhood in $\overline{U}$ naturally homeomorphic to $\mathrm{Map}(\psi)$, the mapping cylinder of ψ.

Consider $s \in \Sigma^{n-1}$ and $\epsilon > 0$ such that $B(s;\epsilon)$ lies in a Euclidean patch in M. Identify a small $(n-1)$-cell $D \subset \Sigma^{n-1} \cap B(s;\epsilon)$ with $s \in \operatorname{Int} D$. Pushing down the mapping cylinder structure of $\mathrm{Map}(\psi)$, if necessary, we assume the part W of the mapping cylinder determined by $\psi^{-1}(D)$ lies in $B(s;\epsilon)$. Let D_1, D_2, D_3 be additional $(n-1)$-cells with

$$s \in \operatorname{Int} D_{i+1} \subset D_{i+1} \subset \operatorname{Int} D_i \subset D_i \subset \operatorname{Int} D \subset D = D_0,$$

and then let W_i denote the portion of $\mathrm{Map}(\psi)$ determined by $\psi^{-1}(\operatorname{Int} D_i)$ $(i = 0,1,2,3)$. Delete all points of N^{n-1} from W_i to form W_i^* $(i = 1,2,3)$.

We claim that any loop α in $W_3^* \smallsetminus \Sigma$ is null-homotopic in $B(s;\epsilon) \cap U$. Find $\gamma \in (0,1)$ so close to 1 that the image $W_3^- \subset W^*$ of $\psi^{-1}(\operatorname{Int} D_3) \times (0,\gamma)$ in $\mathrm{Map}(\psi)$ contains α. Use freeness of Σ^{n-1} in U to obtain a map $g : D \to U$ so close to $\mathrm{incl}_D : D \to \overline{U}$ in the ANR $\overline{U}$ to allow a homotopy $\mu : D \times [0,1/3] \to \overline{U}$ between $\mu_0 = \mathrm{incl}_D$ and $\mu_{1/3} = g$; do this so the image of μ lies in the portion of $\mathrm{Map}(\psi)$ corresponding to $N^{n-1} \times [\gamma, 1]$. The mapping cylinder structure offers the means to push the image of g out to the frontier $\operatorname{Fr}\mathrm{Map}(\psi)$ (relative to $\overline{U}$); push first through the image of $N^{n-1} \times [\gamma, 1)$ to the level corresponding to γ, and then through the image of $N^{n-1} \times [0,\gamma]$ out to the frontier. This gives an extension of μ to a map $\mu : D \times [0,1] \to \overline{U}$ satisfying

$\mu(D \times [1/3, 1]) \subset U$,

$\mu_1(D) \subset \operatorname{Fr}\mathrm{Map}(\psi)$ (relative to $\overline{U}$),

$\mu(D \times [1/3, 2/3]) \subset N^{n-1} \times [\gamma, 1) \subset \mathrm{Map}(\psi) \cap U$,

$\mu(D \times [2/3, 1]) \subset N^{n-1} \times [0,\gamma] \subset \mathrm{Map}(\psi) \cap U$, and

$\mu(D \times [0,1)) \cap \operatorname{Fr}\mathrm{Map}(\psi) = \emptyset$.

Furthermore, this can be arranged so that

$\mu(D \times [0,1)) \subset B(s;\epsilon)$,

$\mu(\partial D_0 \times [0,1]) \cap W_1 = \emptyset$,

$\mu((D_0 \smallsetminus \operatorname{Int} D_i) \times [0,1]) \cap W_{i+1} = \emptyset$ $(i = 1, 2)$, and
$\mu(D_i \times [0,1]) \subset W_{i-1}$ $(i = 1, 2, 3)$.

By the argument given for Lemma 7.9.14, $\mu(D_0 \times [0,1])$ contains points of U very close to each point of $\operatorname{Int} D_1$; in view of the connections along mapping cylinder lines and away from $\mu(\partial D_0 \times [0,1])$, the same argument gives that $\mu(D_0 \times [0,1]) \supset W_1$. In like fashion, $\mu(D_i \times [0,1]) \supset W_{i+1}$ $(i = 1, 2)$.

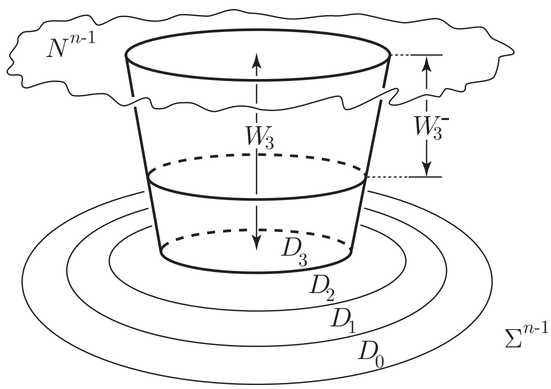

Figure 7.12. The neighborhood W_3 and other structures near s

Let Y_1^* denote the component of $\mu^{-1}(W_1^*)$ containing $\operatorname{Int} D_1 \times \{0\}$. As in the proof of Theorem 7.9.8, one can append a collar $\operatorname{Int} D_1 \times (-1, 0]$ to W_1^* and extend $\mu | Y_1^* : Y_1^* \to W_1^*$ to a map

$$\nu : Y_1^* \cup (\operatorname{Int} D_1 \times (-1, 0]) \to W_1^* \cup (\operatorname{Int} D_1 \times (-1, 0])$$

in the obvious way. Here ν has degree ± 1, since it is a homeomorphism over the appended collar (Lemma 7.9.9). Note that $Y_1^* \supset \operatorname{Int} D_2 \times (0, 1)$.

Form $Y_3^- = \mu^{-1}(W_3^-)$. Properties of μ force $Y_3^- \subset \operatorname{Int} D_2 \times (2/3, 1) \subset Y_1^*$. List the components $U_1, U_2, \ldots$ of Y_3^- and use d_i to denote the degree of $\mu | U_i : U_i \to W_3^-$. Then $\Sigma_i d_i = \pm 1$, by Lemma 7.9.11.

Let W_1^-, like W_3^-, denote the portion of W_1 corresponding to the image of $N^{n-1} \times (0, \gamma)$ and let Y_1^- denote the component of $\mu^{-1}(W_1^-)$ containing $\operatorname{Int} D_2 \times (2/3, 1)$. Now we have $Y_3^- \subset Y_1^- \subset D_0 \times (2/3, 1)$. Another appeal to Lemma 7.9.11 assures that $\mu | Y_1^- : Y_1^- \to W_1^-$ is a degree ± 1 map between connected manifolds. Thus, Lemma 7.9.13 promises a loop $\alpha' \subset Y_1^-$ such that $\mu(\alpha')$ is homotopic to α in W_1^-. As α' is null homotopic in $D \times (2/3, 1)$, α is null homotopic in $\mu(D \times (2/3, 1)) \subset B(s; \epsilon) \cap U$. $\square$

Example 7.11.2. *For $n \geq 6$, S^n contains a wildly embedded $(n-1)$-sphere Σ with a mapping cylinder neighborhood.*

This example introduces a remarkably useful and direct new method for producing wildness. It involves decompositions into acyclic sets. Typically the methodology brings about an associated decomposition space S that contains an object which obviously is "wild", in the sense of failing to be 1-LCC embedded; however, it can be far from obvious that S is a manifold. Although the given decomposition is only acyclic, not cell-like, in many instances there does exist a cell-like map from another manifold onto S, in which event the Cell-like Approximation Theorem possibly could be exploited to detect that S is a manifold.

For the specific issue at hand, Example 7.11.2, the decomposition space S associated with an acyclic decomposition of S^n contains an object Σ related to the $(n-1)$-sphere, and the latter obviously has a neighborhood P with the structure of a mapping cylinder. Moreover, the sphere-like subspace Σ has wildness features, by virtue of containing a 1-sphere that fails to be 1-LCC in S.

Construction of the Example. Assume $n \geq 7$; something similar can be done for $n = 6$, but we will ignore that special case. Fix a finite, acyclic 2-complex A that is PL embedded in S^{n-2} with contractible complement (see Example 0.10.3). Take a regular neighborhood $N(A)$ of the embedded A and then spin an $(n-2)$-ball B, $N(A) \subset \operatorname{Int} B \subset B \subset S^{n-2}$, to produce a PL embedding of $N(A) \times S^1$ in S^{n-1}. Treat S^{n-1} as an equatorial sphere in S^n. Form the decomposition of S^n having the sets $\{A \times \{s\} \mid s \in S^1\}$ as nondegenerate elements, and let $p : S^n \to S$ denote the map to the associated decomposition space S. We will show that both S and $p(S^{n-1})$ are spheres. The image under p of an annular neighborhood of S^{n-1} in S^n will be a mapping cylinder neighborhood of $p(S^{n-1})$. The 1-sphere $p(A \times S^1)$ will be wildly embedded in S (and in $p(S^{n-1})$ as well) because it fails to be 1-LCC.

Lemma 7.11.3. *The quotient space $N(A)/A$ is the cell-like image of a ∂-manifold W under a cell-like map that restricts to a homeomorphism on a neighborhood of ∂W.*

Proof. The complex A is embedded in S^{n-2} with contractible complement. Therefore, the closed complement C' of a collar on $\partial N(A)$ in $S^{n-2} \smallsetminus \operatorname{Int} N(A)$ is contractible, and $N(A)/A \cong (S^{n-2} \smallsetminus \operatorname{Int} N(A))/C'$, since each is a cone over $\partial N(A)$. $\square$

Lemma 7.11.4. *Suppose M is an m-manifold and $p : M \times S^1 \to X$ is a closed, surjective mapping for which there exist an $(m-2)$-dimensional, compact ANR Z in M and a closed subset C of S^1 such that the nondegenerate point preimages under p are the sets $\{Z \times \{s\} \mid s \in C\}$. Let D be a* *dense subset of C. Then each map $f : I^2 \to X$ can be approximated by a map $F : I^2 \to X$ such that $F(I^2) \cap p(Z \times C) \subset p(Z \times D)$.*

Proof. Given $f : I^2 \to X$, choose a triangulation T on I^2 so images of its simplices under f have small diameter. Let L denote the subcomplex of T containing the 1-skeleton plus all 2-simplices $\tau \in T$ with $f(\tau) \cap p(Z \times C) = \emptyset$. Here f can be approximately lifted to a map $g : L \to M$: for each 2-simplex $\tau \in L$ set $g|\tau = p^{-1} f|\tau$; for vertices and 1-simplices of L not contained in any 2-simplex there, the existence of such an approximate lift g follows readily from the dimension restriction on Z. The desired map F will coincide with pg on L. For each 2-simplex $\sigma \in T \setminus L$, $g|\partial\sigma$ is homotopic in $(U \setminus Z) \times J$ to a map sending $\partial\sigma$ into $U \times \{d\}$, where U is a small neighborhood of Z in M, J is a small subset of S^1, and $d \in D$. Thus, $F|\partial\sigma = pg|\partial\sigma$ extends over σ to a map $F : \sigma \to p((U \setminus Z) \times J) \cup (U \times \{d\})$, which gives the desired approximation to f. □

Corollary 7.11.5. *The space $(N(A)/A) \times S^1$ satisfies the DDP.*

Proof. Specify disjoint dense subsets D_1, D_2 of S^1. Let $p : N(A) \times S^1 \to X = (N(A)/A) \times S^1$ denote the decomposition map. Given two maps $f_i : I^2 \to (N(A)/A) \times S^1$, approximate by maps $F_i : I^2 \to (N(A)/A) \times S^1$ with $F(I^2) \cap p(A \times S^1) \subset p(A \times D_i)$ $(i = 1, 2)$. A general position adjustment near points of $F_1(I^2) \cap F_2(I^2)$ in the ∂-manifold $p((N(A) \setminus A) \times S^1)$ yields disjoint approximations. □

It follows from the Cell-like Approximation Theorem that $(N(A)/A) \times S^1$ is a ∂-manifold, and hence $p(S^{n-1})$ is a manifold. The latter is a sphere since it is a simply-connected homology sphere (by the Vietoris-Begle Theorem). Similarly, $S = p(S^n)$ is an n-sphere. Finally, $p(A \times S^1)$ is wildly embedded in S since it fails to be 1-LCC: it has a compact neighborhood P of the form $(c * \partial(N(A) \times [-1, 1])) \times S^1$ and its complement in P deformation retracts to $\partial(N(A) \times [-1, 1]) \times S^1$. □

Definition. A subset X of a space S is *strongly homogeneously embedded* in S if every homeomorphism $h : X \to X$ extends to a homeomorphism $H : S \to S$.

Example 7.11.6. *For $n \geq 6$, S^n contains a wild, strongly homogeneously embedded Cantor set.*

Proof. Again we use acyclic decompositions to build a space S containing a Cantor set X that is both strongly homogeneously embedded and wild, in the sense of failing to be 1-LCC embedded. The real work involves showing that S is a manifold.

As before, consider the acyclic 2-complex A PL embedded in S^{n-2} with contractible complement. Let C be a Cantor set,

$$C \subset \operatorname{Int} I = \operatorname{Int} I \times \{1/2\} \subset \operatorname{Int} I^2.$$

Consider $S^{n-2} \times I^2$ as a PL subset of S^n.

Let $p : S^n \to S$ be the decomposition map associated with the decomposition of S^n into points and the sets $\{A \times \{c\} \mid c \in C\}$. The Cantor set of interest is $X = p(A \times C)$. Obviously it is strongly homogeneously embedded in S because for any homeomorphism $h : X \to X$ there exists a homeomorphism $g : C \to C$ rendering the following diagram commutative:

$$\begin{tikzcd} A \times C \arrow[r, "\mathrm{Id}\times g"] \arrow[d, "p|_{A\times C}"] & A \times C \arrow[d, "p|_{A\times C}"] \\ X \arrow[r, "h"] & X \end{tikzcd}$$

This g extends to a homeomorphism $G : I^2 \to I^2$ that reduces to the identity on ∂I^2, and

$$\mathrm{Id} \times G : S^{n-2} \times I^2 \to S^{n-2} \times I^2$$

extends to $H : S^n \to S^n$ via the Identity off $S^{n-2} \times I^2$. Then H, in turn, induces a homeomorphism $\hat{H} : S \to S$ as $\hat{H} = pHp^{-1}$, and $\hat{H}|X = h$.

By Corollary 7.11.5, S has the DDP.

Finally, we explain why S is the cell-like image of a manifold. Let N_1 be a regular neighborhood of A in $\operatorname{Int} N(A)$ and let E_1 be the union of a pair of disjoint 2-cells in $\operatorname{Int} I^2$, with $\operatorname{Int} E_1 \supset C$. As $N(A) \times I^2 \subset S^n$ is a regular neighborhood of a copy of A, $\partial(N(A) \times I^2)$ bounds a compact, contractible n-manifold Q_0. We claim that $\operatorname{Int} Q_0$ contains a pair of disjoint copies of Q_0 whose union Q_1 satisfies $Q_0 \smallsetminus \operatorname{Int} Q_1 \cong N(A) \times (I^2 \smallsetminus \operatorname{Int}(N_1 \times E_1))$. Form a second compact contractible n-manifold Q' by removing $\operatorname{Int}(N_1 \times E_1)$ from $N(A) \times I^2$ and attaching in a copy of Q_0 to what remains along each component of $\partial(N_1 \times E_1)$. Remarkably, Q' is PL homeomorphic to Q_0, by the Relative h-Cobordism Theorem (Rourke and Sanderson, 1972, p. 87): the union of Q_0, Q' and a collar joining ∂Q_0 to $\partial Q'$ is an n-sphere, hence that union bounds an $(n+1)$-cell W, the resulting triple (W, Q_0, Q') is a relative h-cobordism, so $W \cong Q_0 \times [0,1]$ with Q_0 corresponding to $Q_0 \times \{0\}$ and Q' to $Q_0 \times \{1\}$.

Let $N_1 \supset N_2 \supset \cdots N_k \supset \cdots$ be regular neighborhoods of A in S^{n-2} such that $N_k \subset \operatorname{Int} N_{k-1}$ and $\cap_k N_k = A$. For $k \geq 2$ let E_k be a union of 2^k pairwise disjoint 2-cells in $\operatorname{Int} E_{k-1}$, where $\cap_k E_k = C$. Each component of $N_{k-1} \times E_{k-1} \smallsetminus \operatorname{Int}(N_k \times E_k)$ is homeomorphic to $N(A) \times I^2 \smallsetminus \operatorname{Int}(N_1 \times E_1)$. Now $S^{n-2} \times I^2$ contains a sequence $Q_0 \supset Q_1 \supset \cdots Q_{k-1} \supset Q_k \supset \cdots$, where for $k > 0$ Q_k consists of 2^k pairwise disjoint copies of Q_0, $Q_k \subset \operatorname{Int} Q_{k-1}$ and

$$Q_{k-1} \setminus \operatorname{Int} Q_k \cong N_{k-1} \times E_{k-1} \setminus \operatorname{Int}(N_k \times E_k).$$

As a result, there is a natural surjective map $q : Q_0 \to p(N(A) \times I^2) \subset S$ sending distinct components of $\cap_k Q_k$ to distinct points of X and being 1-1 on $Q_0 \setminus \cap_k Q_k$. Rather obviously, the components of $\cap Q_i$ are cell-like sets, so q is a cell-like mapping and therefore a near-homeomorphism. It follows that S is a manifold. □

Constructions like those of Example 7.11.2 or §2.6 give wild but strongly homogeneously embedded 1-spheres in $S^5 \times S^1$ and $S^3 \times S^1$, respectively. Exactly the same methods, with S^k in place of S^1, lead to wild but strongly homogeneously embedded k-spheres in codimensions 5 and 3. Whether there is a wild and strongly homogeneously embedded (or even just homogenously embedded) codimension-one manifold example remains an open question.

Historical Notes. Theorem 7.11.1 is due to Bryant and Lacher (1975), who did considerably more, showing there that the combination of mapping cylinder neighborhood and a generalized concept of freeness implies local flatness for embedded manifolds of all other codimensions. In low dimensions freeness is not a necessary ingredient: V. Nicholson (1969) proved that complexes in 3-manifolds with mapping cylinder neighborhoods are tame, and Lacher and A. Wright (1970) showed that 3-manifolds with mapping cylinder neighborhoods in 4-manifolds are locally flat.

S. Ferry and E. Pederson (1991) produced a catalogue of wildly embedded circles in S^n ($n \geq 7$) similar to the wild circles of Example 7.11.2. Theirs are indexed by Wall's finiteness obstruction.

M. A. Kervaire (1969) proved that every PL homology n-sphere, $n \geq 5$, bounds a compact, contractible, PL $(n+1)$-manifold; Kervaire derived the same result in the smooth category, provided one allows modification of the homology sphere by taking its connected sum with a (unique) smooth homotopy sphere.

Alternate ways of getting a cell-like mapping from a manifold onto spaces like these acyclic decomposition spaces are treated in Chapter 8.

The methods arising in the development of Example 7.11.2 are those used to settle the Double Suspension Problem, discussed later here in §8.10. The connection is exposed in Exercise 7.11.2 below.

The strongly homogeneous but wildly embedded Cantor set is due to Daverman (1979).

Exercises

7.11.1. Suppose Σ^{n-1} is a connected, two-sided $(n-1)$-manifold in a connected n-manifold M $(n \geq 5)$, $\psi : \Sigma^{n-1} \to \Sigma^{n-1}$ is a cell-like mapping, and U is a component of $M \smallsetminus \Sigma^{n-1}$ such that Σ^{n-1} has a closed neighborhood in $\overline{U}$ naturally homeomorphic to the mapping cylinder of ψ. Then Σ^{n-1} is collared from $\overline{U}$.

7.11.2. Show that the join of S^1 and $\partial N(A)$, where $N(A)$ is the acyclic, $(n-2)$-dimensional ∂-manifold described in Example 7.11.2, is topologically S^n.

7.11.3. Show that S^n contains a wild, homogenously embedded $(n-2)$-torus $(n > 2)$. [Hint: spin the Bing sling of Subsection 2.8.5.]