Preface

Classical differential geometry is the approach to geometry that takes full advantage of the introduction of numerical coordinates into a geometric space. This use of coordinates in geometry was the essential insight of René Descartes that allowed the invention of analytic geometry and paved the way for modern differential geometry. The basic object in differential geometry (and differential topology) is the smooth manifold. This is a topological space on which a sufficiently nice family of coordinate systems or “charts” is defined. The charts consist of locally defined n-tuples of functions. These functions should be sufficiently independent of each other so as to allow each point in their common domain to be specified by the values of these functions. One may start with a topological space and add charts which are compatible with the topology or the charts themselves can generate the topology. We take the latter approach. The charts must also be compatible with each other so that changes of coordinates are always smooth maps. Depending on what type of geometry is to be studied, extra structure is assumed such as a distinguished group of symmetries, a distinguished “tensor” such as a metric tensor or symplectic form or the very basic geometric object known as a *connection*. Often we find an interplay among many such elements of structure.

Modern differential geometers have learned to present much of the subject without constant direct reference to locally defined objects that depend on a choice of coordinates. This is called the “invariant” or “coordinate free” approach to differential geometry. The only way to really see exactly what this all means is by diving in and learning the subject.

The relationship between geometry and the physical world is fundamental on many levels. Geometry (especially differential geometry) clarifies, codifies and then generalizes ideas arising from our intuitions about certain aspects of our world. Some of these aspects are those that we think of as forming the spatiotemporal background of our activities, while other aspects derive from our experience with objects that have "smooth" surfaces. The Earth is both a surface and a "lived-in space", and so the prefix "geo" in the word geometry is doubly appropriate. Differential geometry is also an appropriate mathematical setting for the study of what we classically conceive of as continuous physical phenomena such as fluids and electromagnetic fields.

Manifolds have dimension. The surface of the Earth is two-dimensional, while the configuration space of a mechanical system is a manifold which may easily have a very high dimension. Stretching the imagination further we can conceive of each possible field configuration for some classical field as being an abstract point in an infinite-dimensional manifold.

The physicists are interested in geometry because they want to understand the way the physical world is in "actuality". But there is also a discovered "logical world" of pure geometry that is in some sense a part of reality too. This is the reality which Roger Penrose calls the Platonic world.[1] Thus the mathematicians are interested in the way worlds *could be in principle* and geometers are interested in what might be called "possible geometric worlds". Since the inspiration for what we find interesting has its roots in our experience, even the abstract geometries that we study retain a certain physicality. From this point of view, the intuition that guides the pure geometer is fruitfully enhanced by an explicit familiarity with the way geometry plays a role in physical theory.

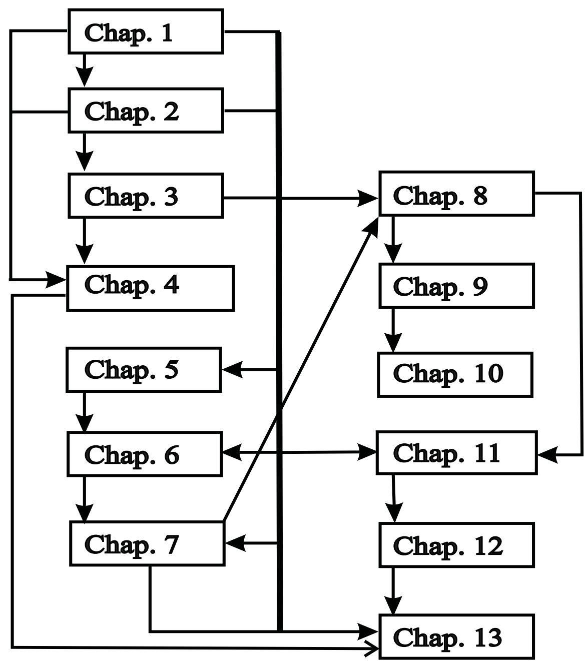
Knowledge of differential geometry is common among physicists thanks to the success of Einstein's highly geometric theory of gravitation and also because of the discovery of the differential geometric underpinnings of modern gauge theory[2] and string theory. It is interesting to note that the gauge field concept was introduced into physics within just a few years of the time that the notion of a connection on a fiber bundle (of which a gauge field is a special case) was making its appearance in mathematics. Perhaps the most exciting, as well as challenging, piece of mathematical physics to come along in a while is string theory mentioned above.

The usefulness of differential geometric ideas for physics is also apparent in the conceptual payoff enjoyed when classical mechanics is reformulated in the language of differential geometry. Mathematically, we are led to the subjects of symplectic geometry and Poisson geometry. The applicability of differential geometry is not limited to physics. Differential geometry is

[1] Penrose seems to take this Platonic world rather literally giving it a great deal of ontological weight as it were.

[2] The notion of a connection on a fiber bundle and the notion of a gauge field are essentially identical concepts discovered independently by mathematicians and physicists.

also of use in engineering. For example, there is the increasingly popular differential geometric approach to control theory.



There is a bit more material in this book than can be comfortably covered in a two semester course. A course on manifold theory would include Chapters 1, 2, 3, and then a selection of material from Chapters 5, 7, 8, 9, 10, and 11. A course in Riemannian geometry would review material from the first three chapters and then cover at least Chapters 8 and 13. A more leisurely course would also include Chapter 4 before getting into Chapter 13. The book need not be read in a strictly linear manner. We included here a flow chart showing approximate chapter dependence. There are exercises throughout the text and problems at the end of each chapter. The reader should at least read and think about every exercise. Some exercises are rather easy and only serve to keep the reader alert. Other exercises take a bit more thought.

Differential geometry is a huge field, and even if we had restricted our attention to just manifold theory or Riemannian geometry, only a small fragment of what might be addressed at this level could possibly be included. In choosing what to include in this book, I was guided by personal interest and, more importantly, by the limitations of my own understanding. While preparing this book I used too many books and papers to list here but a few that stand out as having been especially useful include [**A-M-R**], [**Hicks**], [**L1**], [**Lee, John**], [**ON1**], [**ON2**], and [**Poor**].

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