Preface

This book springs from lectures on degree theory given by the authors during many years at the Departamento de Geometría y Topología at the Universidad Complutense de Madrid, and its definitive form corresponds to a three-month course given at the Dipartimento di Matematica at the Università di Pisa during the spring of 2006. Today mapping degree is a somewhat classical topic that appeals to geometers and topologists for its beauty and ample range of relevant applications. Our purpose here is to present both the history and the mathematics.

The notion of degree was discovered by the great mathematicians of the decades around 1900: Cauchy, Poincaré, Hadamard, Brouwer, Hopf, etc. It was then brought to maturity in the 1930s by Hopf and also by Leray and Schauder. The theory was fully burnished between 1950 and 1970. This process is described in Chapter I. As a complement, at the end of the book there is included an index of names of the mathematicians who played their part in the development of mapping degree theory, many of whom stand tallest in the history of mathematics. After the first historical chapter, Chapters II, III, IV, and V are devoted to a more formal proposition-proof discourse to define and study the concept of degree and its applications. Chapter II gives a quick presentation of manifolds, with special emphasis on aspects relevant to degree theory, namely regular values of differentiable mappings, tubular neighborhoods, approximation, and orientation. Although this chapter is primarily intended to provide a review for the reader, it includes some not so standard details, for instance concerning tubular neighborhoods. The main topic, degree theory, is presented in Chapters III and IV. In a simplified manner we can distinguish two approaches to the theory: the Brouwer-Kronecker degree and the Euclidean degree. The first is developed in Chapter III by differential means, with a quick diversion into the de Rham computation in cohomological terms. We cannot help this diversion: cohomology is too appealing to skip. Among other applications, we obtain in this chapter a differential version of the Jordan Separation Theorem. Then, we construct the Euclidean degree in

Chapter IV. This is mainly analytic and astonishingly simple, especially in view of its extraordinary power. We hope this partisan claim will be acknowledged readily, once we obtain quite freely two very deep theorems: the Invariance of Domain Theorem and the Jordan Separation Theorem, the latter in its utmost topological generality. Finally, Chapter V is devoted to some of those special results in mathematics that justify a theory by their depth and perfection: the Hopf and the Poincaré-Hopf Theorems, with their accompaniment of consequences and comments. We state and prove these theorems, which gives us the perfect occasion to take a glance at tangent vector fields.

We have included an assorted collection of some 180 problems and exercises distributed among the sections of Chapters II to V, none for Chapter I due to its nature. Those problems and exercises, of various difficulty, fall into three different classes: (i) suitable examples that help to seize the ideas behind the theory, (ii) complements to that theory, such as variations for different settings, additional applications, or unexpected connections with different topics, and (iii) guides for the reader to produce complete proofs of the classical results presented in Chapter I, once the proper machinery is developed.

We have tried to make internal cross-references clearer by adding the Roman chapter number to the reference, either the current chapter number or that of a different chapter. For example, III.6.4 refers to Proposition 6.4 in Chapter III; similarly, the reference IV.2 means Section 2 in Chapter IV. We have also added the page number of the reference in most cases.

One essential goal of ours must be noted here: we attempt the simplest possible presentation at the lowest technical cost. This means we restrict ourselves to elementary methods, whatever meaning is accepted for elementary. More explicitly, we only assume the reader is acquainted with basic ideas of differential topology, such as can be found in any text on the calculus on manifolds.

We only hope that this book succeeds in presenting degree theory as it deserves to be presented: we view the theory as a genuine masterpiece, joining brilliant invention with deep understanding, all in the most accomplished attire of clarity. We have tried to share that view of ours with the reader.

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