History

In the body of mathematics, the notion of degree stands as a beautiful achievement of topology and one of the main contributions of the twentieth century, which has been called the century of topology. In Chapter I we try to outline how the ideas that led to this fundamental notion of degree were sparked and came to light. It is only natural that such a task is biased by our personal opinions and preferences. Thus, it is likely that a specialist in, say, partial differential equations would present the tale in a somewhat different way. All in all, a choice must be made and ours is this:

§1. *Prehistory:* Gauss, Cauchy, Liouville, Sturm, Kronecker, Poincaré, Picard, Bohl (1799–1910).

§2. *Inception and formation:* Hadamard, Brouwer (1910–1912).

§3. *Accomplishment:* Hopf, Leray, Schauder (1925–1934).

§4. *Renaissance and reformation:* Nagumo, de Rham, Heinz (1950–1970).

§5. *Axiomatization:* Führer, Amann, Weiss (1970–1972).

§6. *Further developments:* Equivariant theory, infinite dimensions.

The presentation of these topics is mainly discursive and descriptive, rigorous proofs being deferred to Chapters II through V where there will be complete arguments for all the most classical results presented here.

1. Prehistory

Rougly speaking, degree theory can be defined as the study of those techniques that give information *on the existence of solutions of an equation of the form* $y = f(x)$, where x and y dwell in suitable spaces and f is a map from one to the other. The theory also gives clues for *the number of solutions and their nature*. An important particular case is that of an equation $x = f(x)$, where f is a map from a domain D of a linear space into D itself: this is the so-called *Fixed Point Problem.*

By its very nature, it is clear that the origins of degree theory should be traced back to the first attempts to solve algebraic equations such as

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0,$$

where the coefficients a_i are complex numbers, $a_n \neq 0$. That such an equation always has some solution is the *Fundamental Theorem of Algebra.* This result was most beloved by KARL-FRIEDRICH GAUSS (1777–1855), who found at least four different proofs, in 1799, 1815, 1816, and 1849. It is precisely in the first and fourth proofs where we find what can be properly considered the first ideas of topological degree. By some properties of algebraic curves (which were formalized only in 1933 by ALEXANDER OSTROWSKI (1893–1986)), Gauss was able to prove that inside a circle of big enough radius, the algebraic curve corresponding to the real part of the polynomial shares some point with the algebraic curve corresponding to the imaginary part. In this way the following two lines of research were born:

Problem I. Find the common solutions of the equations

$$\begin{cases} f(x,y) = 0, \\ F(x,y) = 0 \end{cases}$$

inside a given closed planar domain, on whose border the two functions $f(x,y)$ and $F(x,y)$ do not vanish simultaneously.

Problem II. Find the number of real roots of a polynomial in one variable with real coefficients, in a given closed interval $[a,b]$ of the real line.

* * *

The first contributions to **Problem I** are due to AUGUSTIN LOUIS CAUCHY (1789–1857). In a memoir presented before the Academy of Turin, on November 17, 1831, and in the paper [Cauchy 1837a], Cauchy introduces a new calculus that, in its own words, can be used to solve equations of any degree.

Some parts of Cauchy's arguments are not completely precise, and the way these parts were made rigorous by JACQUES CHARLES FRANÇOIS STURM (1803–1855) and JOSEPH LIOUVILLE (1809–1882) is quite relevant in the history of the analytic definition of the topological degree of a continuous mapping.

Let us describe this. The definition of the *index of a function* given by Cauchy in [Cauchy 1837a] is the following:

Let x be a real variable and $f(x)$ a function that becomes infinite at $x = a$. If the variable x increases through a, the function will either change from negative to positive or change from positive to negative or not change sign at all. We will say that the index of f at a *is -1 in the first case, $+1$ in the second, and 0 in the third. We define the* integral index of f *between two given limits $x = x_0$ and $x = x_1$, denoted by $J_{x_0}^{x_1}(f)$, as the sum of the indices of f corresponding to the values of x in the interval $[x_0, x_1]$ at which f becomes infinite. If f is a function in two variables, we define the* integral index of f *between the limits $x_0, x_1; y_0, y_1$ to be the number*

$$ {}_{x_0}^{x_1}J_{y_0}^{y_1}(f) = \tfrac{1}{2}\left[J_{x_0}^{x_1}(f(\cdot, y_1)) - J_{x_0}^{x_1}(f(\cdot, y_0)) - J_{y_0}^{y_1}(f(x_1, \cdot)) + J_{y_0}^{y_1}(f(x_0, \cdot))\right]. $$

In his 1831 memoir, Cauchy obtained the index of a function by integral techniques and residues and proved the following result:

Theorem. *Let Γ be a closed curve that is the contour of an area S, and let $Z(z) = X(x, y) + iY(x, y)$ be an entire function. Then*

$$ \frac{1}{2\pi i}\int_{\Gamma}\frac{Z'(z)}{Z(z)}dz = \tfrac{1}{2}\, J_{s=s'}^{s=s''}\left(X/Y\right) $$

is the number of zeros of $Z(z)$ in S; here s stands for the arc length along Γ, and $s'' - s'$ is the length of Γ.

Cauchy generalized this result in a memoir published June 16, 1833, in Turin. The generalization follows:

Theorem. *Let $F(x, y)$ and $f(x, y)$ be two functions of the variables x, y, continuous between the limits $x = x_0, x = x_1, y = y_0, y = y_1$. We denote by $\Phi(x, y), \phi(x, y)$ the derivatives of the functions with respect to x, and by $\Psi(x, y), \psi(x, y)$ their derivatives with respect to y. Finally, let N be the number of the different systems of values x, y, between the above limits, verifying simultaneously the equations $F(x, y) = 0, f(x, y) = 0$. Then*

$$ N = {}_{x_0}^{x_1}J_{y_0}^{y_1}(\Delta), $$

where

$$ \begin{aligned} \Delta(x, y) &= \frac{f(x, y)}{F(x, y)}\big(\Phi(x, y)\psi(x, y) - \Psi(x, y)\phi(x, y)\big) \\ &= \frac{f(x, y)}{F(x, y)}\left(\frac{\partial F(x, y)}{\partial x}\frac{\partial f(x, y)}{\partial y} - \frac{\partial F(x, y)}{\partial y}\frac{\partial f(x, y)}{\partial x}\right). \end{aligned} $$

An elementary "proof" of this theorem appears in [Cauchy 1837b]. However, Liouville and Sturm in [Liouville-Sturm 1837] give three examples showing that the second theorem above above can fail. The first example is

$$\begin{cases} F(x,y) = x^2 + y^2 - 1, \\ f(x,y) = y. \end{cases}$$

In this example

$$\Delta(x,y) = \frac{2xy}{x^2 + y^2 - 1},$$

and drawing around the origin a rectangle containing the circle $x^2 + y^2 = 1$, one sees that

$${}_{x_0}^{x_1}J_{y_0}^{y_1}(\Delta) = 0,$$

because Δ never becomes infinity on the sides of the rectangle. However, the system

$$\begin{cases} x^2 + y^2 - 1 = 0, \\ y = 0 \end{cases}$$

has the two solutions $(1,0), (-1,0)$ inside the rectangle. Liouville and Sturm conclude their note with the following remark:

There is a theorem that can replace Cauchy's. Let us consider a closed contour Γ on which $F(x,y)$ and $f(x,y)$ do not vanish simultaneously, and let us also assume that inside this contour the function

$$\begin{aligned} w &= \Phi(x,y)\psi(x,y) - \Psi(x,y)\phi(x,y) \\ &= \frac{\partial F(x,y)}{\partial x}\frac{\partial f(x,y)}{\partial y} - \frac{\partial F(x,y)}{\partial y}\frac{\partial f(x,y)}{\partial x} \end{aligned}$$

does not vanish at the values (x,y) at which $f(x,y)$ and $F(x,y)$ vanish.

In this situation, among the solutions (x,y) of the equations $F(x,y) = 0$, $f(x,y) = 0$, inside Γ, some correspond to positive values of w and others to negative values of w. We denote by μ_1 the number of solutions of the first kind, and by μ_2 the number of solutions of the second kind. With this notation we have

$$\frac{1}{2}\delta = \mu_1 - \mu_2,$$

where δ stands for how many more times the function $\frac{f(x,y)}{F(x,y)}$ changes from positive to negative than from negative to positive, at those points in the contour Γ at which that function becomes infinite, when the contour is traced in the positive direction.

We see that the function w is the *Jacobian of the mapping* (F,f) (Liouville and Sturm always consider entire functions). Consequently, we find displayed here for the first time the importance of the sign of the functional determinant

$$w = \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{vmatrix}$$

when dealing with the computation of the number of solutions of the system

$$\begin{cases} F(x,y) = 0, \\ f(x,y) = 0 \end{cases}$$

in a planar region.

Today, in the hypotheses of the Liouville-Sturm Theorem, the number $\mu_1 - \mu_2$ is called the *topological degree of the mapping* (F, f) *at the origin,* and this is the starting point for the analytic definition of degree. But this will not take full shape until 1951.

* * *

In the later paper [Cauchy 1855], Cauchy states the *Argument Principle,* which is another way to compute the indices he has defined earlier. These results, translated into more modern terminology, read as follows.

Winding number (or index) of a planar curve around a point. *Let* $\Gamma \subset \mathbb{C}$ *be a closed oriented curve with a* $\mathcal{C}^1$ *parametrization:*

$$z(t) = x(t) + iy(t) + a, \quad 0 \leq t \leq 1, \quad z(0) = z(1), \quad a \in \mathbb{C} \setminus \Gamma.$$

Then,

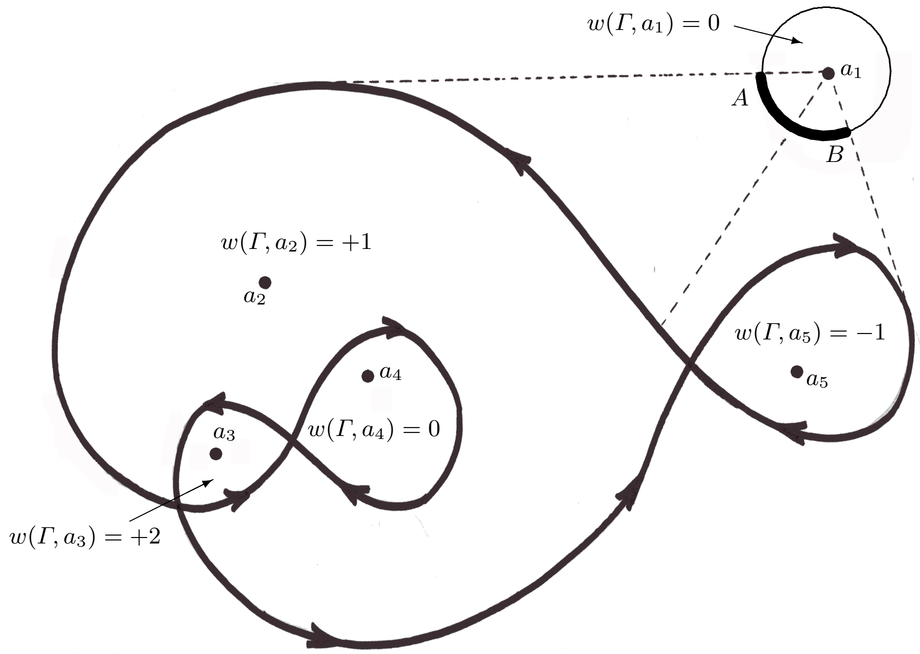
$$w(\Gamma, a) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-a} = \frac{1}{2\pi} \int_0^1 \frac{x(t)y'(t) - x'(t)y(t)}{x^2(t) + y^2(t)} dt$$

is an integer.

This integer is called the *winding number (or index) of* Γ *around* a.

Geometrically, the winding number tells us *how many times the curve wraps around the point.* In case Γ is only continuous, the winding number is defined through a $\mathcal{C}^1$ approximation Γ_1 of Γ, because $w(\Gamma_1, a)$ remains constant for Γ_1 close enough to Γ.

The following example illustrates this notion:



To proceed one step further, Cauchy considers a simply connected domain $G \subset \mathbb{C}$ (that is, G has no holes), a holomorphic function $f : G \to \mathbb{C}$, $\zeta = f(z)$, and a $\mathcal{C}^1$ closed curve $\varGamma \subset G$, on which f has no zeros. Then:

Argument Principle. *The following formula holds:*

$$w(f(\varGamma), 0) = \frac{1}{2\pi i} \int_{f(\varGamma)} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\varGamma} \frac{f'(z)}{f(z)} dz = \sum_k w(\varGamma, a_k)\alpha_k,$$

where the a_k's are the zeros of f in the domain D bounded by $\varGamma$ and the α_k's are their respective multiplicities.

Suppose next that $\varGamma$ has no self-intersection and that it has the positive (counterclockwise) orientation. Then D is a connected domain (this is the *Jordan Separation Theorem,* which we will discuss later), and $w(\varGamma, a) = +1$ for all $a \in D$, so that the last formula becomes

$$w(f(\varGamma), 0) = \sum_k \alpha_k,$$

that is:

Theorem. *The total number of zeros (counted with multiplicities) that f has in D is the winding number of the curve $f(\varGamma)$ around the origin.*

In general, the winding number can be negative, but we can still say that f has at least $|w(f(\Gamma), 0)|$ zeros in the domain bounded by Γ.

* * *

Let us now turn to **Problem II.** The first full solution is due to Sturm. In 1829 and 1835 he gave an algorithm to find the exact number of distinct real roots of a polynomial. The theorem was later generalized by Carl Gustav Jacob Jacobi (1804–1851), Charles Hermite (1822–1901), and James Joseph Sylvester (1814–1897).

Exploring the topological content of Sylvester's article [Sylvester 1853], Leopold Kronecker (1823–1891) introduces in his papers [Kronecker 1869a] and [Kronecker 1869b] a method that extends Sturm's. Indeed, at the end of his work Kronecker writes:

In my research developed in this article, I started from a theorem by Sturm. A generalization of that result was found by Hermite some time ago, but I have been able to extend the continued fraction algorithm developed by Sylvester to further widen Sturm's theorem.

Let us describe Kronecker's contribution. He starts with the following definition:

Regular function systems. A *regular function system* consists of $n+1$ real functions $F_0, F_1, \ldots, F_n$ in n real variables $x_1, \ldots, x_n$, such that

(a) $F_0, F_1, \ldots, F_n$ are continuous and have no common zeros. They admit partial derivatives with respect to all n variables, and those derivatives take finite values.

(b) The functions $F_0, F_1, \ldots, F_n$ take positive and negative values. Moreover, each function takes positive (resp., negative) values infinitely often.

(c) The domains $\{F_i < 0\}$, $i = 0, \ldots, n$, represent n-dimensional varieties that only contain finite values of the variables $x_1, \ldots, x_n$.

(d) No functional determinant

$$\left|\frac{\partial F_i}{\partial x_j}\right|_{\substack{k \neq i = 0,1,\ldots,n \\ j = 1,\ldots,n}}, \quad k = 0, 1, \ldots, n,$$

vanishes at any zero of the system $F_k \neq 0, F_0 = F_1 = \cdots = F_n = 0$.

(e) The common zero set of any chosen $n-1$ functions among $F_0, F_1, \dots, F_n$ is a $\mathcal{C}^1$ curve.

Then Kronecker looks at the orientations of the $\mathcal{C}^1$ curve involved in this definition (condition (e) above). He considers this part basic in his research on systems of functions in several variables:

Orientation Principle. Kronecker chooses for every pair (k, ℓ), $k < \ell$, an orientation of the $\mathcal{C}^1$ curve (recall (e) above)

$$F(k,\ell) = \{x \in \mathbb{R}^n : F_i(x) = 0 \ \text{ for } i \neq k, \ell\}.$$

This orientation is denoted by $|k\ell|$; he then puts $|\ell k| = -|k\ell|$.

Next, he defines:

(a) A point $e \in F(k,\ell) \cap \{F_k = 0\}$ is called an *incoming (eingang)* point of $F(k,\ell)$ (in $\{x \in \mathbb{R}^n : F_k(x) \cdot F_\ell(x) < 0\}$) if the following condition holds true: *walking the curve $F(k,\ell)$ as oriented by $|k\ell|$, we leave the set $\{x \in \mathbb{R}^n : F_k(x) \cdot F_\ell(x) > 0\}$ at the point e and enter $\{x \in \mathbb{R}^n : F_k(x) \cdot F_\ell(x) < 0\}$.*

The set of all these incoming points e is denoted by $E(k,\ell)$.

(b) A point $a \in F(k,\ell) \cap \{F_k = 0\}$ is called an *outgoing (ausgang)* point of $F(k,\ell)$ (off $\{x \in \mathbb{R}^n : F_k(x) \cdot F_\ell(x) < 0\}$) if the following condition holds true: *walking the curve $F(k,\ell)$ as oriented by $|k\ell|$, we leave the set $\{x \in \mathbb{R}^n : F_k(x) \cdot F_\ell(x) < 0\}$ at the point a and enter $\{x \in \mathbb{R}^n : F_k(x) \cdot F_\ell(x) > 0\}$.*

The set of all these outgoing points a is denoted by $A(k,\ell)$.

After the preceding preparation, Kronecker shows that the number

$$\#E(k,\ell) - \#A(k,\ell)$$

is *even* and does not depend on the indices k, ℓ, and then he defines:

Kronecker characteristic. The *characteristic of the regular function system* $F_0, F_1, \dots, F_n$ is the integer

$$\chi(F_0, F_1, \dots, F_n) = \tfrac{1}{2}\big(\#E(k,\ell) - \#A(k,\ell)\big).$$

It is convenient to stress that in the course of his proof of this fact

Kronecker obtains the following very modern description of his invariant:

$$\chi(F_0, F_1, \ldots, F_n) = (-1)^j \cdot \sum_{\substack{(F_0,\ldots,F_n)(x)=0 \\ F_j(x)<0}} \operatorname{sign} J_x(F_0, \ldots, \widehat{F}_j, \ldots, F_n)$$

$$= (-1)^j \cdot \sum_{\substack{(F_0,\ldots,F_n)(x)=0 \\ F_j(x)>0}} \operatorname{sign} J_x(F_0, \ldots, \widehat{F}_j, \ldots, F_n)$$

(here J stands for the Jacobian determinant) for any $j = 0, \ldots, n$.

Once this invariant is defined, Kronecker shows how it detects solutions of the given regular system:

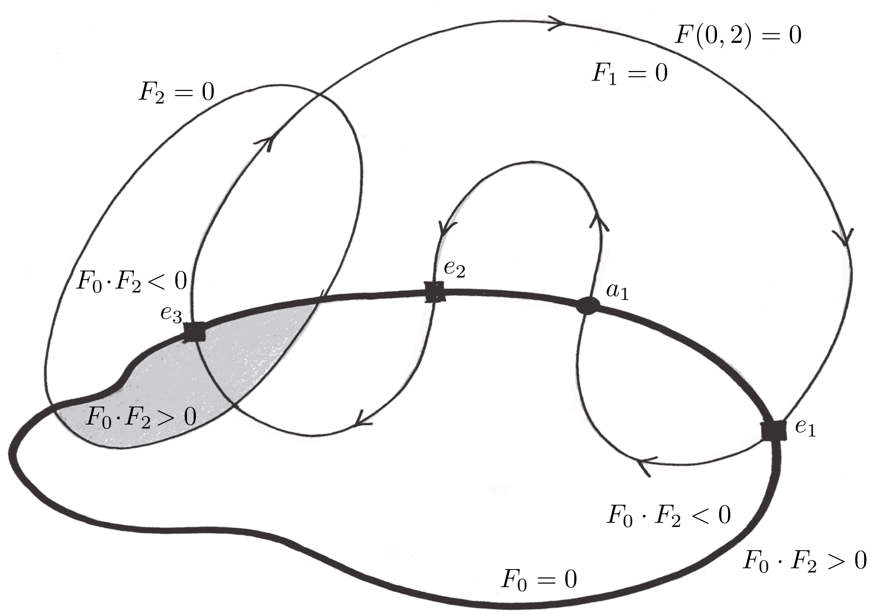
Kronecker Existence Theorem. *Let $F_0, F_1, \ldots, F_n$ be a regular function system. If $\chi(F_0, F_1, \ldots, F_n) \neq 0$, then for every $i = 0, 1, \ldots, n$, the system*

$$F_0(x) = 0, \ldots, F_{i-1}(x) = 0,\ F_i(x) < 0,\ F_{i+1}(x) = 0, \ldots, F_n(x) = 0$$

has some solution $x \in \mathbb{R}^n$.

This result and the above remark extend to a system of n functions with n unknowns what Liouville and Sturm had done thirty years earlier, as has already been mentioned.

We illustrate all of this with a simple example:



Here, we depict a regular system consisting of three functions $F_0, F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}$. Each set

$$F(k,\ell) = \{(x_1,x_2) \in \mathbb{R}^2 : F_i(x_1,x_2) = 0 \text{ for } i \neq k,\ell\} = \{F_i = 0\}, i \neq k,\ell,$$

is a $\mathcal{C}^1$ curve, and we have the incoming points $E(0,2) = \{e_1, e_2, e_3\}$ and the outgoing points $A(0,2) = \{a_1\}$. We get

$$\chi(F_0,F_1,F_2) = \frac{1}{2}\big(\#E(0,2) - \#A(0,2)\big) = \frac{1}{2}(3-1) = 1 \neq 0,$$

and we see immediately in the picture that each of the three systems

$$\begin{array}{ll} (0) & F_0 < 0,\ F_1 = 0,\ F_2 = 0, \\ (1) & F_0 = 0,\ F_1 < 0,\ F_2 = 0, \\ (2) & F_0 = 0,\ F_1 = 0,\ F_2 < 0 \end{array}$$

indeed has solutions.

In 1877, influenced by some discussions with his friend Karl Theodor Wilhelm Weierstrass (1815–1897) on the matter of complex analysis and potential theory, Kronecker gave a representation of his characteristic by means of an integral, today known as the *Kronecker integral*. Indeed, he shows this:

Kronecker Integral Theorem. *Let $F_0, F_1, \ldots, F_n$ be a regular system. Then for every $j = 0, 1, \ldots, n$,*

$$\chi(F_0,F_1,\ldots,F_n) = -\frac{1}{\operatorname{vol}(\mathbb{S}^{n-1})}\int_{S_j} \frac{\Delta}{\Phi^n}\, du$$

where

$$\Delta = \begin{vmatrix} F_0(w) & F_1(w) & \ldots & \widehat{F_j(w)} & \ldots & F_n(w) \\ \frac{\partial F_0}{\partial x_1} & \frac{\partial F_1}{\partial x_1} & \ldots & \widehat{\frac{\partial F_j}{\partial x_1}} & \ldots & \frac{\partial F_n}{\partial x_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial F_0}{\partial x_{n-1}} & \frac{\partial F_1}{\partial x_{n-1}} & \ldots & \widehat{\frac{\partial F_j}{\partial x_{n-1}}} & \ldots & \frac{\partial F_n}{\partial x_{n-1}} \end{vmatrix}$$

and $S_j = \{x \in \mathbb{R}^n : F_j(x) = 0\}$, $\Phi = \sqrt{F_0^2 + F_1^2 + \cdots + \widehat{F_j^2} + \cdots + F_n^2}$.

Note that in case $n = 2$, and supposing that F_1 and F_2 are the components of a holomorphic function $f(z)$ on $F_0 \leq 0$, the above integral for $j = 0$ becomes

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)} dz.$$

Thus, the Kronecker Existence Theorem is in fact a generalization of Cauchy's results.

There are antecedents to the Kronecker integral in dimension > 2. Even Kronecker remarks in [Kronecker 1869a] that a special case of his integral can be found in a previous paper by Gauss on potential theory [Gauss 1813]. In this case $n = 3$,

$$F_1 = 4\pi(x_1 - x_1^0),\ F_2 = 4\pi(x_2 - x_2^0),\ F_3 = 4\pi(x_3 - x_3^0),$$

and the Kronecker integral gives the flow through the surface $F_0 = 0$ of the electric field

$$V(z) = \frac{x - x^0}{\|x - x^0\|^3}$$

created by the unit charge placed at $x^0 = (x_1^0, x_2^0, x_3^0)$. Gauss establishes in this paper that the flow is equal to 4π or 0, according to whether the point x^0 is interior or exterior to the surface. In modern terms, the flow through the surface $F_0 = 0$ is expressed by

$$\int_{F_0=0} \langle V, \nu \rangle dS,$$

where ν is the outward normal vector field on the surface and dS is the area element.

* * *

After Kronecker's pioneering work on the characteristic and the integral of a regular system, there were a number of papers on the subject, with important applications to geometric and topological questions. Following the chronological development, me must mention first the contributions by Jules Henri Poincaré (1854–1912), who used the Kronecker integral in the qualitative theory of autonomous (independent of time) ordinary differential equations. Already in 1881, Poincaré uses the index of a planar curve to study these differential equations, but in 1883 he uses Kronecker's theorem in a note in Comptes Rendus [Poincaré 1883]. There he writes:

Mr. Kronecker has presented to the Berlin Academy, in 1869, a memoir on functions of several variables, including an important theorem from which the following result follows easily:

Let $\xi_1, \xi_2, \dots, \xi_n$ *be* n *continuous functions in the* n *variables* $x_1, x_2, \dots, x_n$*, the variable* x_i *restricted to range among the limits* $-a_i$ *and* $+a_i$*. Let us suppose that for* $x_i = a_i$ *the function* ξ_i *is always positive, and for*

$x_i = -a_i$ the function ξ_i is always negative. Then I say that there is a system of values for the x at which all the ξ vanish. This result can be applied to the three-body problem to prove it has infinitely many special solutions with important properties that we are to present.

This is an *Intermediate Value Theorem in arbitrary dimension.* Three years later Poincaré published the paper [Poincaré 1886] in which he studies the curves defined by differential equations of the second order. There he looks for the singular points of those equations and discusses their distribution using the Kronecker integral. This contains the argument for the invariance of the characteristic under continuous deformations, which is used in the proof of the result stated above. Kronecker had himself considered this invariance, and actually he could prove it in some particular cases.

* * *

Another important question implicit in Kronecker's work is the determination of the *exact* number of solutions of a system of equations. But it is Charles Emile Picard (1856–1941) who in 1891 published a note in Comptes Rendus [Picard 1891] with the precise formulation of the problem whose resolution was to be his main contribution to degree theory. Picard states:

Let us consider n equations $f_i(x_1, x_2, \ldots, x_n) = 0$, $i = 1, 2, \ldots, n$, where we suppose the f_i's represent continuous functions in n real variables $x_1, x_2, \ldots, x_n$ defining a point in some domain D. The question of finding the number of common roots of these equations in that domain has held the attention of geometers for a long time. A formula has been given in this sense by Mr. Kronecker, in his famous investigations of the characteristic of function systems. Unfortunately, the Kronecker integral, a multiple integral of order $n - 1$ on the surface of the domain D, does not give the number of roots we are looking for. The functional determinant of the system plays a fundamental role in this theory, and one only obtains the difference of the numbers of roots at which the determinant is positive and the number of roots at which the determinant is negative.

I will show here, in a few lines, that it is possible to represent the exact number of roots by a suitable integral.

Picard's method consists of attaching new equations to get a new system that has the same roots and whose functional determinant is always positive and applying Kronecker's theory afterwards. Picard presented his results in full in classes at La Sorbonne, Paris, and finally published them in a two-volume treatise on analysis [Picard 1891/1905]. After this, Kronecker's theory became a classic.

In 1904 Piers Bohl (1865–1921) published a paper [Bohl 1904] where he proves the following result:

Theorem. *Let* (G) *be the domain defined by* $-a_i \leq x_i \leq a_i$ $(a_i > 0, i = 1, 2, \ldots, n)$. *There do not exist continuous real-valued functions* $F_1, \ldots, F_n$, *without common zeros in* (G), *such that* $F_i = x_i$ $(i = 1, 2, \ldots, n)$ *on the boundary of* (G).

For the proof, Bohl uses the Kronecker integral and *Stokes' Theorem* (proving along the way the Kronecker Existence Theorem). From the above proposition, and by continuous deformation techniques, Bohl deduces the following two statements:

Proposition. (a) *Let* (G) *be the domain defined by* $-a_i \leq x_i \leq a_i$ $(a_i > 0,\ i = 1, 2, \ldots, n)$, *and let* $F_1, \ldots, F_n$ *be continuous real-valued functions such that* $F_i = x_i$ $(i = 1, 2, \ldots, n)$ *on the boundary of* (G). *Then, for every point* $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ *in* (G) *there is another* $(x_1, x_2, \ldots, x_n)$ *such that* $F_i(x_1, x_2, \ldots, x_n) = \gamma_i$ $(i = 1, 2, \ldots, n)$.

(b) *Let* (G) *be the domain defined by* $-a_i \leq x_i \leq a_i$ $(a_i > 0,\ i = 1, 2, \ldots, n)$, *and let* $f_1, \ldots, f_n$ *be continuous real-valued functions that do not vanish simultaneously in* (G). *Then there are a constant* $N < 0$ *and a point* $(u_1, \ldots, u_n)$ *in the boundary of* (G) *such that* $f_i(u_1, u_2, \ldots, u_n) = N u_i$ $(i = 1, 2, \ldots, n)$.

It is possible that these results were already in the Ph.D. thesis defended by Bohl in 1900 at Dorpat University. Note here that (b) implies what is today the well-known fact that the boundary of (G) is not a retract of (G), which in turn implies a fixed point result. Thus we can say that Bohl was really close to discovering the *Brouwer Fixed Point Theorem.*

On the other hand, in 1895 Poincaré published his famous memoir *Analysis Situs* [Poincaré 1895b], where, starting from the attempts by Georg Friedrich Bernhard Riemann (1826–1866) and Enrico Betti (1823–1892), he starts the development of what will later be called *combinatorial topology* and *simplicial homology.* Poincaré refined these ideas in two complements to his *Analysis Situs,* [Poincaré 1899, Poincaré 1900], where we find the first notion of *(abstract) polyhedron,* which will later be called *triangulation of a compact manifold* V *of dimension* n. This was a finite set $\mathcal{T}$ of disjoint cells of different dimensions $\leq n$ such that:

(1) Every cell of dimension d, or d-cell, is the image in V of an open ball $B \subset \mathbb{R}^d$ via a diffeomorphism from an open neighborhood of the closure $\overline{B}$ into a submanifold W of V of dimension d.

(2) The boundary in W of such a d-cell is a union of cells of dimensions $\leq d-1$.

(3) V is a union of cells of $\mathcal{T}$.

We will come back to this, when ten years later the notion of manifold comes to full development.

2. Inception and formation

In 1910 Jacques Salomon Hadamard (1865–1963) published a remarkable paper [Hadamard 1910], which is in fact an appendix to the book [Tannery 1910]. This paper marks the transition from the origins of degree theory to the construction of a more elaborate and systematic theory.

Hadamard says:

The proof given by Mr. Ames for the Jordan Theorem in the plane *is based on the index (or variation of the argument) of a planar curve with respect to a point (the Cauchy concept). The generalization of this concept to higher dimensions is provided by the Kronecker index. This concept is now classic, mainly after the* Traité d'Analyse *by Mr. Picard (vol. I, p. 123; vol. II, p. 193) of 1891 and 1893, and after several contemporary papers that give new applications of that index.*

Later Hadamard mentions Poincaré, Bohl, and Luitzen Egbertus Jan Brouwer (1881–1966).

Let us now describe things more explicitly. The result is the following well-known statement:

Jordan Separation Theorem. *Let Γ be a Jordan curve in $\mathbb{R}^2$, that is, a curve homeomorphic to a circle. Then $\mathbb{R}^2 \setminus \Gamma$ has exactly two connected components whose common boundary is Γ.*

The argument that Hadamard refers to in his comment runs as follows:

(1) First, by some geometric and topological means, one sees that $\mathbb{R}^2 \setminus \Gamma$ has at most two connected components and only one unbounded.

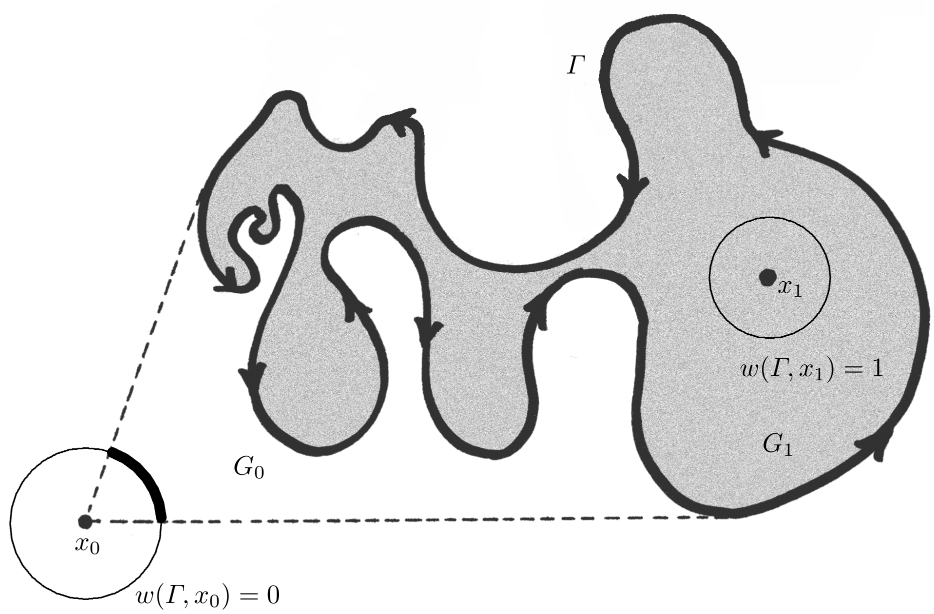
This reduces the result to its most essential part, that Γ *disconnects the plane.*

(2) Second, one computes the index with respect to Γ. This gives:

   (a) Points at which the index vanishes. These belong to the unbounded component of $\mathbb{R}^2 \setminus \Gamma$, which is called the *exterior of* Γ.

   (b) Points at which the index is ± 1. These are in a second (bounded) component, the *interior of* Γ.

The figure below illustrates the second (and most relevant) part of this proof.



Camille Jordan (1838–1922) stated and tried to prove this theorem in [Jordan 1893]. As mentioned above, the essential content of the result is that *a Jordan curve divides the plane,* which jointly with the fact that *a simple arc does not divide the plane* is the oldest proposition of *set topology* in Euclidean spaces. The first complete proof of the Jordan Theorem was given by Oswald Veblen (1880–1960) in [Veblen 1905].

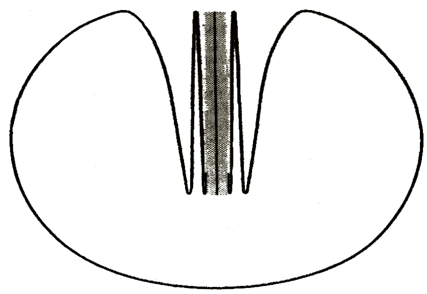
In [Schönflies 1902], Arthur Moritz Schönflies (1853–1928) gives the following additional information concerning this theorem:

Schönflies Theorem. *Let Γ be a Jordan curve in $\mathbb{R}^2$ and G_1 its interior. Then, for all $x \in G_1$ and all $a \in \Gamma$, there is a simple arc from x to a whose points other than a are all in G_1.*

We say that a is *accessible from* G_1. This finally led to the following characterization of Jordan curves:

If a compact set in $\mathbb{R}^2$ has two complementary domains, from which every point of the set is accessible, then the set is a Jordan curve.

Here we see the typical topological beast which is not a Jordan curve:



* * *

Let us now take a closer look at Hadamard's paper. In the first paragraph, Hadamard analyzes the proof above of the Jordan Separation Theorem, focusing on the part involving the order (= index) of a point with respect to the curve. He thus presents clearly what he wants to generalize. To do that, he defines, in the second paragraph, what he means by *a surface in the Euclidean space*. His notions are based on ideas introduced by Poincaré, at the beginning of the century, to treat polyhedra. We will not go into detail here, but in a sketchy way, we can think of an $(n-1)$-*surface in* $\mathbb{R}^n$ as a subspace that decomposes into pieces homeomorphic to the standard $(n-1)$-simplex, which glue in a suitable way along their faces. Hadamard concludes this paragraph by stating *Green's Theorem,* which reduces a volume (triple) integral to a surface (double) one.

In the third paragraph, Hadamard defines the order of a point with respect to a closed oriented surface in $\mathbb{R}^n$. Let us suppose we are given a hypersurface S, with coordinates $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with respect to parameters $(u_1, \ldots, u_{n-1})$, such that the x_i's have continuous partial derivatives with respect to the u_j's (later, this condition is weakened to the mere existence of partial derivatives); suppose also that the origin is not a point of S. Then, Hadamard shows this key fact:

Hadamard Integral Theorem. *The following integral is an integer:*

$$w = -\frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})} \int_S \frac{\Delta}{\gamma^n}\, du_1 \cdots du_{n-1}$$

where

$$\Delta = \begin{vmatrix} x_1 & \dots & x_n \\ \frac{\partial x_1}{\partial u_1} & \dots & \frac{\partial x_n}{\partial u_1} \\ \dots & \dots & \\ \frac{\partial x_n}{\partial u_{n-1}} & \dots & \frac{\partial x_n}{\partial u_{n-1}} \end{vmatrix} \quad \textit{and} \quad \gamma = \sqrt{x_1^2 + \cdots + x_n^2}\,.$$

Hadamard calls this integer w, the *order of the origin with respect to* S. The order with respect to an arbitrary point $a = (a_1, \ldots, a_n) \notin S$ is obtained by translation, that is, by replacing x_i with $x_i - a_i$ everywhere in the formula above.

The proof of this theorem proceeds by induction, using Green's Theorem, the starting step being the order (= index) of a point with respect to a planar curve. It is a long proof: seven pages in the paper! Along the way, Hadamard obtains several important byproducts. For instance:

Proposition. *Let S be a hypersurface, and consider a point $a \notin S$. If there exists a half-line starting at a and not meeting S, then the order of S with respect to a is zero.*

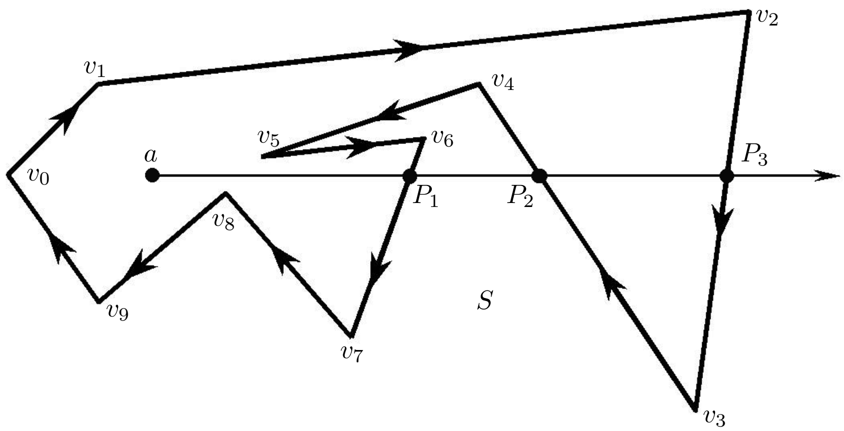
This of course corresponds to the unbounded component of $\mathbb{R}^n \setminus S$ (if we advance the Jordan Separation Theorem in dimension n). But among other things, Hadamard explains how the order can be computed:

Proposition. *Let S be a hypersurface, which is a closed polyhedron. Let w stand for the order with respect to S of a point $a \notin S$. Then*

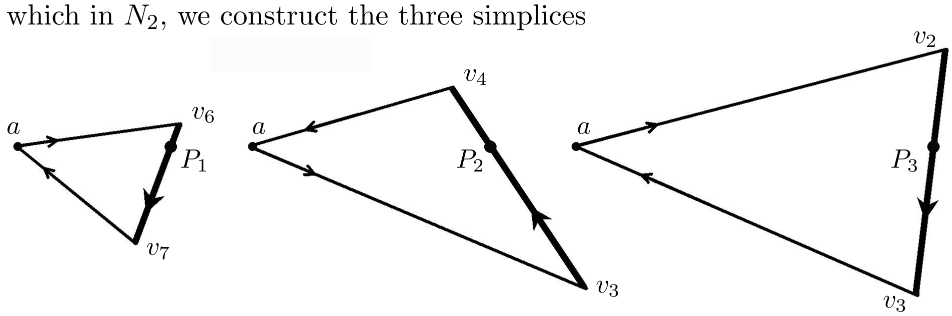
$$w = N_1 - N_2,$$

where N_1, N_2 are computed as follows. Pick a half-line ℓ starting at a and meeting S only at points lying in $(n-1)$-simplices (not in faces). Then $N_1 + N_2$ is the number of points in $\ell \cap S$, and such a point is counted in N_1 (resp., N_2) if the n vertices of the simplex that contains it, ordered in the corresponding orientation and preceded by the point a, determine an n-simplex oriented according to the orientation of the coordinate system.

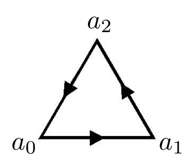
This is much simpler than it reads. Consider the following polygonal curve in the plane:



We have three points P_1, P_2, P_3 in $\ell \cap S$, and to decide which is in N_1 and which in N_2, we construct the three simplices



Comparing with the standard orientation of the coordinate system



we immediately see that $P_1 \in N_2$, $P_2 \in N_1$, and $P_3 \in N_2$. Hence, $w(S,a) = 1 - 2 = -1$.

Furthermore, Hadamard analyzes the variation of w as a moves and concludes:

Proposition. (a) *The order is constant if the point varies without crossing the surface.*

(b) *The order is* ± 1 *if* S *is a convex polyhedron and the point is an interior point.*

In the fourth paragraph, Hadamard defines the Kronecker index of a function system for functions defined on a closed hypersurface:

Definition. *Let S be a hypersurface of the Euclidean space $\mathbb{R}^n$, and let $f_1, \dots, f_n$ be a system of n continuous functions defined on S, which do not vanish simultaneously at any point of S. The* index of the system *is the order of the origin with respect to the hypersurface $(f_1, \dots, f_n)(S)$ generated by the functions.*

Note that this definition has meaning by *the non-vanishing condition,* which guarantees that the origin does not belong to the hypersurface $(f_1, \dots, f_n)(S)$.

The main property of this index is given in the following theorem:

Boundary Theorem. *Let S be the boundary of a domain $V \subset \mathbb{R}^n$, and let $f_1, \dots, f_n$ be defined and continuous on the whole domain V. If $f_1, \dots, f_n$ do not vanish simultaneously at any point of V, then the index w of the system $f_1, \dots, f_n$ on S is zero.*

This can be trivially rewritten as a solutions existence statement:

Corollary. *Let S be the boundary of a domain $V \subset \mathbb{R}^n$, and let $f_1, \dots, f_n$ be defined and continuous on S, such that they do not vanish simultaneously at any point of S. If the index of $f_1, \dots, f_n$ on S is not zero, then the system $f_1 = 0, \dots, f_n = 0$ has some solution in V.*

On the other hand, the computation of the index of a function system on a surface depends on the following important result:

Poincaré-Bohl Theorem. *Let us consider, on the same closed hypersurface $S \subset \mathbb{R}^n$, two function systems $f_1, \dots, f_n$ and $g_1, \dots, g_n$, both satisfying the non-vanishing condition on S. Then:*

(i) *If the systems have different indices, then there is at least one point x in S such that*
$$\frac{f_1(x)}{g_1(x)} = \cdots = \frac{f_n(x)}{g_n(x)} < 0.$$

(ii) *If the indices of the two systems are not in the ratio $(-1)^n$, then there is at least one point x in S such that*
$$\frac{f_1(x)}{g_1(x)} = \cdots = \frac{f_n(x)}{g_n(x)} > 0.$$

Using this, Hadamard obtains:

Schönflies Theorem. *Consider the closed disc*

$$\overline{D}^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

and its boundary $\mathbb{S}^1$. *Let* $f, g : \overline{D}^2 \to \mathbb{R}$ *be two continuous functions such that the mapping* $(f,g) : \overline{D}^2 \to \mathbb{R}^2$ *is injective, and let* C *be the closed curve defined by the parametrization* $(f|_{\mathbb{S}^1}, g|_{\mathbb{S}^1})$. *Then, for every point* (X,Y) *interior to* C, *there is a point* $(x,y) \in \overline{D}^2$ *such that* $(f(x,y), g(x,y)) = (X,Y)$.

Note how the Jordan Separation Theorem for plane curves is used here. Moreover, Hadamard states the Jordan Separation Theorem *in* $\mathbb{R}^n$ without proof and then deduces the Schönflies theorem *in* $\mathbb{R}^n$:

Schönflies Theorem in $\mathbb{R}^n$**.** *Let* V *be a domain with boundary a surface* S *in one piece (this hypothesis is not essential), and let* $f_1, \ldots, f_n$ *be continuous functions from* V *into* $\mathbb{R}$ *such that the mapping* $f = (f_1, \ldots, f_n) : V \to \mathbb{R}^n$ *is injective. Consider the surface* S' *parametrized by* $f|_S = (f_1|_S, \ldots, f_n|_S)$. *Then for every point* $(X_1, \ldots, X_n)$ *interior to* S', *there is a point* $(x_1, \ldots, x_n)$ *in* V *such that* $f(x_1, \ldots, x_n) = (X_1, \ldots, X_n)$.

As a consequence, Hadamard gets:

Proposition. *If* $f : V \to V'$ *is a homeomorphism between two domains, the interior points of* V *map to interior points of* V', *and the boundary points of* V *map to boundary points of* V'.

This is a particular case of the *Invariance of Domain Theorem.* Concerning the Jordan Separation Theorem in $\mathbb{R}^n$, it is very likely, as we will see later, that Hadamard had known it from some private communication with Brouwer.

Another very important result that Hadamard proves follows:

Brouwer Fixed Point Theorem. *Every continuous mapping from the closed ball* $\overline{D}^n \subset \mathbb{R}^n$ *into itself has at least one fixed point.*

Proof. The argument runs as follows. Let $f : \overline{D}^n \to \overline{D}^n$ be a continuous mapping. We can suppose $f(x) \neq x$ for all $x \in \partial\overline{D}^n = \mathbb{S}^{n-1}$ (otherwise we are done). Then, by the Poincaré-Bohl Theorem $\mathrm{Id}_{\mathbb{S}^{n-1}} - f|_{\mathbb{S}^{n-1}}$ and $\mathrm{Id}_{\mathbb{S}^{n-1}}$ have the same index: were it

$$\frac{x_1 - f_1(x)}{x_1} = \cdots = \frac{x_n - f_n(x)}{x_n} = \mu < 0,$$

then $f(x) = (1-\mu)x \notin \overline{D}^n$. Consequently, the index of $\mathrm{Id}_{\mathbb{S}^{n-1}} - f|_{\mathbb{S}^{n-1}}$ is 1, and by the basic properties of indices, there is an $x_0 \in \overline{D}^n$ such that $x_0 - f(x_0) = 0$, that is, $f(x_0) = x_0$. □

Another remarkable statement (without proof) by Hadamard is the famous *Poincaré-Hopf Index Theorem,* relating the Poincaré index (characteristic) of a domain to the zeros of a vector field. In particular, Hadamard proves:

Vanishing of tangent fields. *Let V be a domain that admits a tangent space at every point, depending continuously on the point. If the index σ of V is not zero, then it is not possible to attach continuously a tangent line to V at every point (that is, the tangent line must be undefined somewhere).*

This is the case, for instance, for the sphere in $\mathbb{R}^3$. The fact that Hadamard formulates this theorem seems to result from a misunderstanding with Brouwer, who had proved it for spheres.

To conclude, Hadamard states (again without proof) the following theorem due to Brouwer:

Proposition. *Every continuous mapping from a sphere into itself that preserves orientations has at least one fixed point.*

The result is false if the mapping reverses orientation (think of the antipodal isometry)

* * *

One should mention here a footnote (p. 476 of [Hadamard 1910]) where Hadamard explains that the method for the proof of the previous result was communicated by Brouwer. Thus, it becomes more and more evident that there was a quite fluent exchange of ideas between Hadamard and Brouwer. What follows is a letter from Brouwer to Hadamard that spares any further comment on the matter. We translate from [Brouwer 1976]:

> Paris, January 4, 1910
> 6 Rue de l'Abbé de l'Epée
>
> Dear Sir,
>
> I can now communicate some extensions of the fixed point theorem for bijective continuous transformations of the sphere. They are reduced to arbitrary continuous transformations of the sphere. To such a transformation one can attach a finite number n as its *degree*. From a degree n transformation one can construct by continuous variations any other degree n transformation, but no more than those. In particular, one can always construct in this way a degree n rational transformation of the complex sphere.

To determine this degree, we use homogeneous coordinates (in the double sense), write x, y, z for the initial sphere, and ξ, η, ζ for the image, split the sphere into a finite number of regions, and consider firstly the transformations defined by relations

$$\xi : \eta : \zeta = f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z),$$

where f_1, f_2, f_3 are polynomials, which on the other hand can vary for the different regions of the sphere. We call this transformation *a polynomial transformation*. We choose an orientation on the sphere: then every point P in the image, *in general position,* will occur r_P times with the positive orientation, and s_P times with the negative orientation. In this situation, one can prove that $r_P - s_P$ is a constant: it is the degree of the polynomial transformation.

Let us come back to an arbitrary continuous transformation. It can be approximated by a sequence of polynomial transformations: one proves then that the latter have all the same degree: it is again the degree of the limit transformation.

The degree is always a finite integer, either positive or negative. The degree of a bijective transformation is $+1$, if the orientation is preserved, and -1, if the orientation is reversed.

Now the generalized fixed point theorem becomes what follows: Every continuous transformation of the sphere, whose degree is not -1, has at least one fixed point.

Moreover, I have extended this theorem to m-dimensional spheres. It reads then in the following way: Every continuous transformation of the m-dimensional sphere has at least one fixed point, except a) when m is odd and the degree n is $+1$, b) when m is even and the degree n is -1.

In particular, if the transformation is *bij*ective [in margin, strenger formuleeren], there is at least one fixed point a) if m is odd and the orientation is reversed, b) if m is even and the orientation is preserved.

For the volume of an m-dimensional sphere [sic] in the space of dimension $m+1$ (if we include there the sphere itself) I was able just recently to establish a still more general theorem, namely: Every (possibly not bijective) continuous transformation of the volume of the m-dimensional sphere has at least one fixed point.

Concerning general continuous tangent distributions on the sphere, two articles of mine will appear soon, where I study certain questions that refer to the Dirichlet principle and to the decomposition of a field into a "quellenfrei" part and a "wirbelfrei" part. To that end, I determine first the most general form that tangent curves (characteristic curves after Poincaré) can have. As the main result in the first article, one should take the property that a characteristic curve that does not tend to a singular point must be a spiral, whose two limit cycles are also characteristic. The property that there is at least one singular point is necessary; it is not in the end an accessory corollary, on which I have insisted because it was the first easy to formulate result and because there seemed to be some close relation between this theorem and that of the fixed point in the sphere, a relation that has been clarified only through your correspondence. In the second article I have included your beautiful, direct, and more complete proof of the existence of at least one singular point.

My address will be in Paris till January 15. Maybe there will be the occasion for us to meet?

Yours sincerely,

L.E.J. Brouwer

* * *

Now we turn to Brouwer's fundamental work. We can agree that the greatest contribution due to Brouwer is the definition of the degree of a continuous mapping of n-manifolds solely by geometric and topological means. The manifolds he considers come from the ideas concerning polyhedra that, as was already mentioned, Poincaré introduced at the end of the nineteenth century and the beginning of the twentieth century. These ideas were the germ of what we know today as *combinatorial topology.* It is remarkable that Brouwer never mentions Poincaré's writtings on the topic.

The notion of a manifold used by Brouwer coincides with what nowadays we call a *pseudomanifold*. A manifold of dimension n, or n-manifold, is built up from simplices of dimension d, or d-simplices, of smaller dimensions d. Skipping strict formalisms, we can put it as follows:

Definition. *A subspace M of a Euclidean space is an n-*manifold *when it is homeomorphic to the polyhedron $|K|$ associated to a* simplicial complex K of dimension n. *That is, $|K|$ is a subspace of a Euclidean space consisting of a union of disjoint simplices of dimension $\leq n$ (points, open segments, open triangles, open tetrahedrons, etc.) such that:*

(i) Every simplex of $|K|$ is a face of some n-simplex of $|K|$.

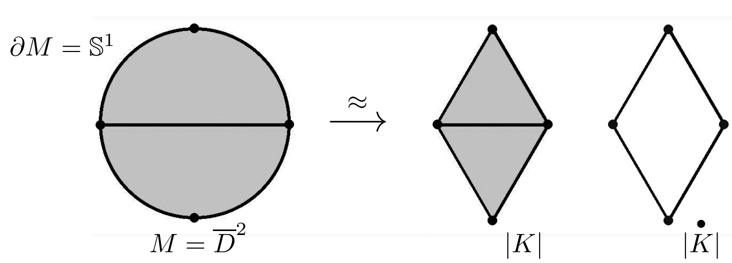
(ii) Every $(n-1)$-simplex of $|K|$ is a face of at most two n-simplices of $|K|$.

(iii) For every pair of n-simplices S and S' of $|K|$, there is a finite sequence $S = S_1, S_2, \ldots, S_m = S'$ of n-simplices of $|K|$ such that S_i and S_{i+1} have a common $(n-1)$-face for $i = 1, \ldots, m-1$.

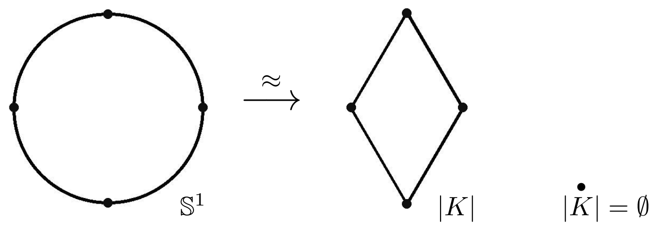
The boundary of $|K|$, *which will be denoted by $|\overset{\bullet}{K}|$, is the union of all $(n-1)$-simplices that are faces of exactly one n-simplex. Hence, if $|\overset{\bullet}{K}| = \emptyset$, every $(n-1)$-simplex is a face of exactly two n-simplices of $|K|$.*

The following examples are very simple illustrations of these notions:

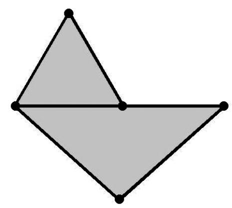
(a) *A 2-manifold with boundary.*



(b) *A* 1-*manifold without boundary.*



(c) *A space which is not a polyhedron.*



Once the notion of a manifold is fixed in this form, Brouwer constructs the topological degree of a continuous mapping of manifolds in [Brouwer 1912a], article dated in Amsterdam, July 1910. In a footnote in the first page we read:

> *While this paper was in print, the note by J. Hadamard,* Sur quelques applications de l'indice de Kronecker, *has appeared in the second volume of J. Tannery's* Introduction à la théorie des fonctions d'une variable. *In that note some aspects of the theory we present here are anticipatedly developed.*

This once again confirms the mutual influence between the two mathematicians.

Let now M and N be two n-manifolds, which we assume to be connected, compact, boundaryless, and oriented. For the definition of degree, Brouwer proceeds in two steps:

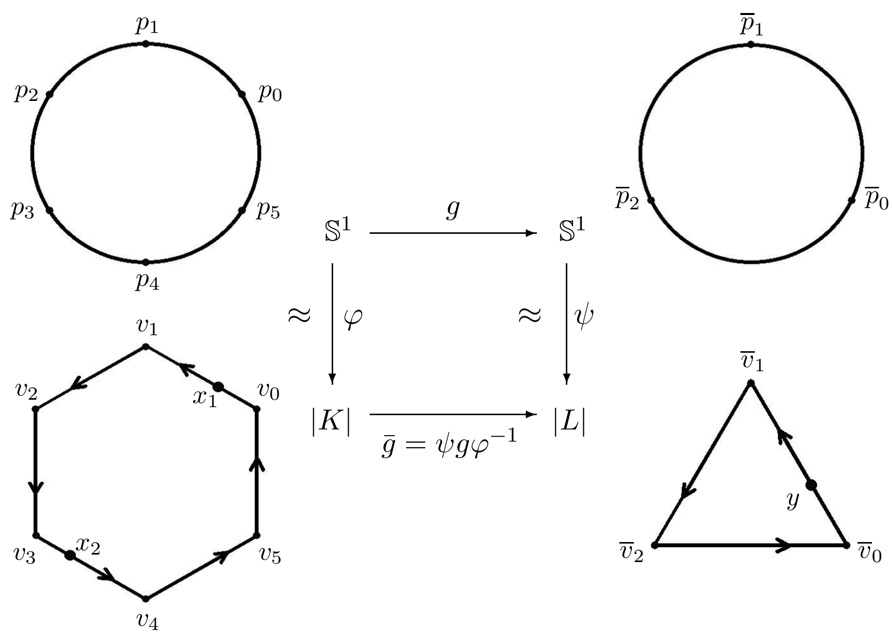
Step I: The simplicial case. Consider a *simplicial continuous mapping* $g : M \to N$. This means that g is continuous, and there are triangulations $\varphi : |K| \equiv M$ and $\psi : |L| \equiv N$ of M and N such that the *localization*

$$\bar{g} = \psi \circ g \circ \varphi^{-1} : |K| \to |L|$$

restricts to an affine map on each simplex of $|K|$. For such a g, Brouwer shows that there is a dense connected open set Ω in $|L|$ such that if $y \in \Omega$ is not in the union S of the images by $\bar{g}$ of the $(n-1)$-simplices of $|K|$, then $g^{-1}(y)$ is a finite set $\{x_1, \dots, x_r\}$ each of whose points belongs to an (open) n-simplex of $|K|$. These simplices σ are disjoint, and the restriction of $\bar{g}$ to every σ is a homeomorphism onto $\bar{g}(\sigma)$. Let p (resp., q) be the number of simplices on which the restriction $\bar{g}|_\sigma$ preserves (resp., reverses) the orientation, and consider the difference $p - q$. Then Brouwer shows that the number $p - q$ is the same for all $y \in \Omega$ off S. To that end, he joins any two such y's by a polygonal contained in Ω and moves y along: p and q increase or decrease simultaneously by the same number when the polygonal crosses S; hence $p-q$ does not change. This happens because each $(n-1)$-simplex is a face of exactly two n-simplices of $|K|$. Consequently, the integer $d = p - q$ is well defined and is called the *degree* $d(g)$ *of* g.

Step II: The general case. To deal with arbitrary non-simplicial mappings, Brouwer introduces two constructions that have become fundamental in combinatorial topology: *barycentric subdivision* of a simplicial complex (subdivision of each simplex by taking its barycenter as a new vertex) and *simplicial approximation* of continuous mappings. By means of these refined geometric techniques, he defines the degree $d(f)$ of an arbitrary continuous mapping $f : M \to N$ through a *good* simplicial approximation of f. In fact, he sees that any two close enough simplicial approximations of f have the same degree as defined in Step I. In fact, Brouwer shows that close approximations are homotopic by a piecewise linear homotopy, which implies the equality of degrees.

Let us make some pictures for this construction. Suppose we have the following simplicial mapping:



Here the localization $\bar{g}$ is affine on every 1-simplex of $|K|$; hence it is determined by the images of the vertices: in this case we suppose

$$\bar{g}(v_0) = \overline{v}_0, \quad \bar{g}(v_1) = \overline{v}_1, \quad \bar{g}(v_2) = \overline{v}_2,$$

$$\bar{g}(v_3) = \overline{v}_0, \quad \bar{g}(v_4) = \overline{v}_1, \quad \bar{g}(v_5) = \overline{v}_2.$$

Then take a point

$$y \in |L| \setminus \bar{g}(\{v_0, v_1, v_2, v_3, v_4, v_5\}),$$

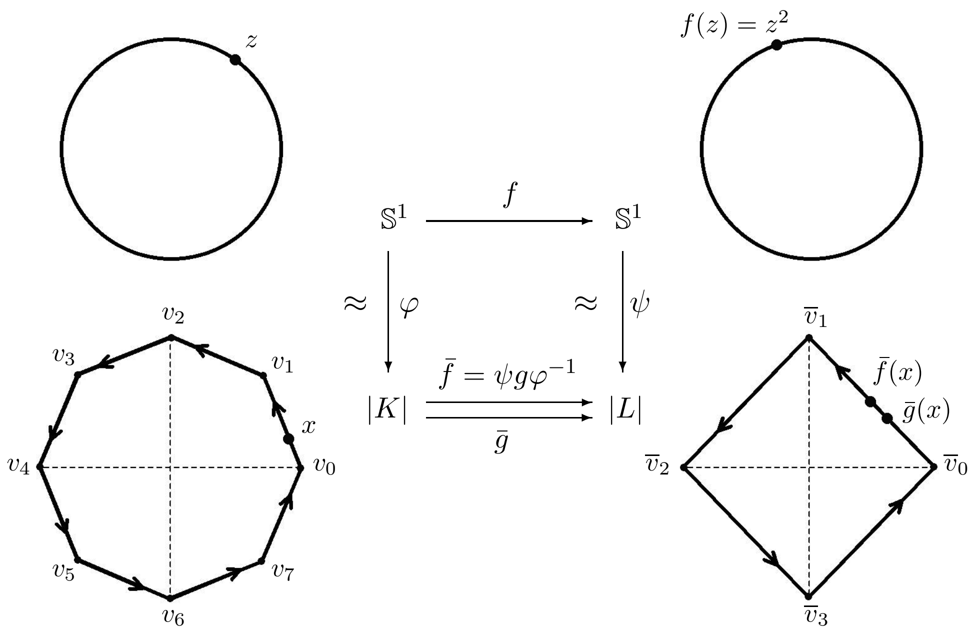
and let $\bar{g}^{-1}(y) = \{x_1, x_2\}$. In this situation, the restrictions

$$\bar{g}|_{(v_0,v_1)} : (v_0, v_1) \to (\overline{v}_0, \overline{v}_1) \text{ and } \bar{g}|_{(v_3,v_4)} : (v_3, v_4) \to (\overline{v}_0, \overline{v}_1)$$

are homeomorphisms that preserve the orientation; hence the degree of f is

$$d(f) = d(\bar{g}) = p - q = 2 - 0 = 2.$$

Next, we look at the general case: a non-affine map like $f(z) = z^2$. The picture is the following:



Here $\bar{g}$ is the simplicial approximation of $\bar{f}$, and we have

$$\bar{g}(v_0) = \overline{v}_0, \quad \bar{g}(v_1) = \overline{v}_1, \quad \bar{g}(v_2) = \overline{v}_2, \quad \bar{g}(v_3) = \overline{v}_3,$$

$$\bar{g}(v_4) = \overline{v}_0, \quad \bar{g}(v_5) = \overline{v}_1, \quad \bar{g}(v_6) = \overline{v}_2, \quad \bar{g}(v_7) = \overline{v}_3.$$

Computing as above, $d(\bar{g}) = 2$, and we conclude $d(f) = 2$.

For his notion of degree, Brouwer obtains the following essential properties:

Proposition. *The following statements hold:*

(a) $d(f) = 0$ *if* f *is not surjective.*

(b) $d(f) = \pm 1$ *if* f *is a homeomorphism.*

(c) $d(g \circ f) = d(g) \cdot d(f)$ *for the composition* $g \circ f$ *of two continuous mappings* $f : M \to N$, $g\ : N \to P$.

(d) $d(f_1) = d(f_2)$ *if* $f_1, f_2 : M \to N$ *are homotopic.*

Furthermore, Brouwer proves the following two important theorems:

Theorem. *A continuous vector field tangent to an* n*-sphere of even dimension always has singular points (= zeros).*

Theorem. *A continuous mapping without fixed points from an* n*-sphere into itself has degree* -1 *if* n *is even and degree* $+1$ *if* n *is odd.*

Proof. The proof of the second result follows. First, the homotopy

$$F(t, x) = \frac{tf(x) - (1-t)x}{\|tf(x) - (1-t)x\|}$$

is well defined because f has no fixed point. Thus, the antipodal mapping F_0 is homotopic to $f = F_1$, and thus F_0 and f have the same degree, which is ± 1 according to the dimension. □

From this, Brouwer deduces:

Corollary. (a) *Every non-surjective continuous mapping of an* n*-sphere must have some fixed point.*

(b) *Every continuous mapping of a sphere of even dimension that is homotopic to the identity must have some fixed point.*

(c) *Every continuous mapping of a sphere of odd dimension that is homotopic to the antipodal mapping must have some fixed point.*

Proof. For (a), a non-surjective mapping has degree $0 \neq \pm 1$, and the preceding theorem applies. For (b), the degree of the given mapping is that of the identity; hence it is $1 \neq -1$. As the dimension is even, again the preceding theorem gives the conclusion. Assertion (c) is proven in the same way, since the degree of the antipodal map in odd dimension is -1. □

Another very important result due to Brouwer (and as mentioned before, revisited by Hadamard) is the following:

Brouwer Fixed Point Theorem. *A continuous mapping from a closed n-ball $\overline{D}^n$ into itself must have some fixed point.*

Proof. The proof runs as follows. Given a continuous mapping $f : \overline{D}^n \to \overline{D}^n$, Brouwer identifies $\overline{D}^n$ with the upper hemisphere $\mathbb{S}^n_+$ of the n-sphere $\mathbb{S}^n$ and defines

$$g : \mathbb{S}^n \to \mathbb{S}^n : x \mapsto g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{S}^n_+, \\ f(x_1, \ldots, x_n, -x_{n+1}) & \text{if } x \notin \mathbb{S}^n_+. \end{cases}$$

This mapping is continuous, but not surjective, hence has some fixed point that must be a fixed point of f. □

With this powerful tool in hand, Brouwer proved in later articles many other important results: the Invariance of Domain Theorem, the Jordan Separation Theorem in arbitrary dimension, and various dimension properties. Moreover, in the paper [Brouwer 1912b], he defines the *link coefficient* $\ell(K_1, K_2)$ of two disjoint oriented compact manifolds K_1, K_2 in $\mathbb{R}^n$, with complementary dimensions h and $n - h - 1$. This link is the degree of the mapping

$$f : K_1 \times K_2 \to \mathbb{S}^{n-1} : (x, y) \mapsto \frac{y - x}{\|y - x\|},$$

that is,

$$\ell(K_1, K_2) = d(f).$$

We must mention here a contribution made 80 years earlier by Gauss, who in [Gauss 1833] actually computed the link coefficient of two curves.

The Italian mathematician Carlo Miranda proved in [Miranda 1940] that the Brouwer Fixed Point Theorem is equivalent to Poincaré's Intermediate Value Theorem of 1883 (I.1, p. 12). For this reason, some authors call this theorem the *Poincaré-Miranda Theorem*.

3. Accomplishment

This is the place to mention Heinz Hopf (1894–1971). Hopf's interest in degree theory came from the lectures given by Erhard Schmidt (1876–1959) at Breslau in 1917 and at Berlín in 1920 on the invariance of dimension and the proof of the Jordan Theorem in arbitrary dimension using

Brouwer's degree. Hopf himself says in [Hopf 1966] that those results appealed to him deeply:

I was fascinated; this fascination—of the power of the method of the mapping degree—has never left me since, but has influenced major parts of my work.

Hopf's essential contribution to degree theory springs from his Ph.D. thesis, which he defended at Berlin University in 1926. Among other as important geometric questions, he completed Brouwer's results concerning continuous mappings of closed oriented manifolds of the same dimension, with special attention to the case when the target manifold is a sphere. We explain it in the following paragraphs.

Let G and G' be two domains in the Euclidean n-space, and let f, g be two continuous mappings from G into G'. Let $a \in G$ be an *isolated coincidence point of f and g* (that is, $f(a) = g(a)$ and this does not hold true for any other point in a small enough neighborhood $U \subset G$ of a). The *coincidence index of f and g at a* is the degree of the continuous mapping

$$\varphi : S_r(a) \to \mathbb{S}^{n-1} : x \mapsto \frac{f(x) - g(x)}{\|f(x) - g(x)\|},$$

where $S_r(a)$ is the spherical surface of center a and radius $r > 0$ contained in U. By its local nature, this definition extends to mappings of arbitrary manifolds. Modifying Brouwer's proof of the Fixed Point Theorem for spherical surfaces, Hopf proved ([Hopf 1925]):

Theorem. *Let f and g be continuous mappings from a compact oriented manifold M of dimension n into the sphere $\mathbb{S}^n$, and denote by d, d' their respective degrees. Then, if f and g have finitely many concidence points, the sum of the coincidence indices of f and g at those points is $(-1)^n d + d'$.*

Brouwer had proved in [Brouwer 1912a] that continuous homotopic mappings have the same degree and in [Brouwer 1912c] that continuous mappings of $\mathbb{S}^2$ into itself with the same degree are homotopic. Hopf, using the theorem above, proved in [Hopf 1926a] the most general form of this fact:

Hopf Theorem. *Two mappings from a compact oriented manifold M of dimension n into the sphere $\mathbb{S}^n$ that have the same degree are homotopic.*

Later, in [Hopf 1933], Hopf extended this result to the case when the source manifold M is a polyhedron. A further generalization to Banach spaces was to be published by Erich Rothe in [Rothe 1936].

Now, the dual problem that arises in a natural way is the homotopic classification of continuous mappings from a sphere into a topological space. This led to the discovery of *homotopy groups* of higher order by Eduard Čech (1893–1960), who defined them in a very short paper presented at the Zurich ICM, in 1932 [Čech 1932]. However, these groups did not attract much attention, as they are always commutative, in sharp contrast with the behavior of the fundamental group. But in 1935 Witold Hurewicz (1904–1956) rediscovered and studied systematically these homotopy groups ([Hurewicz 1935a], [Hurewicz 1935b], [Hurewicz 1936a], and [Hurewicz 1936b]). The two basic ingredients from which the theory develops are:

(1) the *fundamental group* (now *first homotopy group*), which Poincaré introduced in [Poincaré 1895b], and

(2) the *Hopf invariant,* the means used by Hopf to distinguish infinitely many homotopy classes of mappings $\mathbb{S}^3 \to \mathbb{S}^2$. This he achieves by combinatorial topology techniques in [Hopf 1931]. Some years later, in [Hopf 1935], Hopf extends his result to mappings $\mathbb{S}^{2n-1} \to \mathbb{S}^n$.

On the other hand, in [Poincaré 1885a], Poincaré had proved that if a continuous vector field on a compact orientable surface of genus g has finitely many critical points, the sum of their indices (Poincaré indices) is the topological invariant $2 - 2g$, which is the Euler characteristic of the surface. This was generalized by Brouwer in [Brouwer 1912a] to spherical surfaces of arbitrary dimension. As we have said before, Hadamard announced without proof the same result for compact orientable manifolds of arbitrary dimension [Hadamard 1910]. But it is Hopf who, finally, proved in [Hopf 1926b] the following result:

Poincaré-Hopf Theorem. *Let M be a compact orientable manifold of dimension n, and let X be a continuous vector field with finitely many critical points. Then the sum of the indices of those critical points is the Euler characteristic $\chi(M)$ of M.*

But there is more to tell. In his path to these important results, Hopf gained deeper insight into the topological meaning of the various invariants involved. In fact, he analyzed Kronecker's work on the characteristic of a system of several variables and defined the so-called *integral curvature* of a closed oriented hypersurface M in the Euclidean $(n+1)$-space. That invariant is the degree of the Gauss mapping ν of the hypersurface ($\nu(x)$ is the unitary vector perpendicular to the hypersurface at x). If M is described by an equation $F = 0$, the integral curvature of M is the Kronecker characteristic of the system

$$\left(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_{n+1}}\right).$$

In the articles mentioned, Hopf proved:

Theorem. *The integral curvature of a Jordan hypersurface in $\mathbb{R}^{n+1}$ (one that bounds a domain) coincides with the Euler characteristic of the hypersurface.*

To refine this statement, Hopf introduced the *models* of a closed oriented n-manifold. These are the hypersurfaces in $\mathbb{R}^{n+1}$ that are homeomorphic to M. Then, he proved:

Gauss-Bonnet Theorem in arbitrary dimension. *Let M be a closed oriented manifold of dimension n.*

(1) *If n is odd, the integral curvature of the models of M is not a topological invariant, not even for Jordan models.*

(2) *If n is even, the integral curvature of the models of M is a topological invariant, namely, half the Euler characteristic of M.*

If N is a model of M and $\kappa_1, \ldots, \kappa_n$ are its principal curvatures, then

$$\tfrac{1}{2}\chi(N) = \frac{1}{\mathrm{vol}(\mathbb{S}^n)} \int_N \kappa_1 \cdots \kappa_n \mathrm{d}w.$$

This was obtained by Walter Franz Anton Dyck (1856-1934) in [Dyck 1888] and [Dyck 1890].

In particular, we see that all models of even-dimensional manifolds have even Euler characteristic, and thus we find manifolds which cannot be embedded in codimension 1:

Corollary. *There are closed oriented n-manifolds which have no model in $\mathbb{R}^{n+1}$, even allowing self-intersections.*

An example is the complex projective plane, which has Euler characteristic 3 (as it can be computed with a vector field using the Poincaré-Hopf Theorem).

* * *

Brouwer-Kronecker degree theory and its applications were systematically developed by Hopf in the topology text [Aleksandrov-Hopf 1935] written jointly with Pavel Sergeevich Aleksandrov (1896–1982). The subject is covered in the final three chapters, XII, XIII, and XIV, of that book. The titles of the sections in those chapters give a detailed account of the contents of the theory. We translate them as follows:

CHAPTER XII. The Brouwer degree. The Kronecker characteristic

§1. The order of a point with respect to a cycle. Recall results in link theory. The Poincaré-Bohl theorem. The Rouché theorem. Homology of $\mathbb{R}^n \setminus \{p\}$. The global degree. Relation between order and degree. The winding number in the plane. The Kronecker integral.

§2. The Kronecker characteristic. Local degree of mapping in $\mathbb{R}^n$. The existence theorem for a point. Application: The fundamental theorem of algebra. The characteristic of a function system. The index of a point. The algebraic number at a point. The local degree in $\mathbb{R}^n$. Topological applications. The invariance theorem. The functional determinant.

§3. Special theorems and applications. Relations among vector fields on $\mathbb{R}^n$ and mappings. Vector fields on the sphere. The Brouwer fixed point theorem on an n-dimensional element. Vector fields on a spherical surface of even dimension: A fixed point theorem. Symmetries on the spherical surface. Another fixed point theorem. Borsuk theorem on antipodal mappings. Analytic corollary. Mappings from $\mathbb{S}^n$ to $\mathbb{R}^n$. The covering theorem for the spherical surface $\mathbb{S}^n$.

§4. The degree of mappings between polyhedra. Definition of local degree. Continuous deformations of mappings (homotopies). Cycle mappings. Cycle mappings on acyclic irreducible closed complexes. Coincidence of the local and global degrees. Determination of the global degree by homological invariants. Essential properties of mappings.

Appendix. Comment on the Brouwer link number as a characteristic. The Gauss integral. The number of cuts as a degree. The link number as an order. The Gauss integral.

CHAPTER XIII. Homotopy and mapping extension theorems

§1. More on the Kronecker existence theorem. An extension problem for mappings into $\mathbb{R}^n$. Reduction to mappings into $\mathbb{S}^n$. Elementary lemma on extension and homotopy of mappings. Solution to the extension problem for a simplex. Reduction of Theorem II to a lemma. Algebraic part of the proof of the lemma. Geometric part of the proof of the lemma. Criterion for the local essentiality of mappings on $\mathbb{R}^n$.

§2. Mappings from n-dimensional polyhedra into the spherical n-dimensional surface. Equivalence between homology and homotopy. Listing of the types of mappings in the simplest case. Essential mappings. Resolution theorem.

§3. Mappings from n-dimensional polyhedra into the circle. Mappings from the n-dimensional sphere into the circle. The extension theorem. Types of mappings. Essentiality.

§4. Characterization of the identity and polyhedra borders by deformation properties. A deformation theorem from a polyhedron into itself. The essentiality of a polyhedron into itself. The particular case of polygons. Characterization of the identity. Characterization of the border. Stability of polyhedra. Examples.

Appendix. Mappings that are homologous but not homotopic.

CHAPTER XIV. Fixed points

§1. A fixed point existence theorem. Fixed simplex. A generalization of the Poincaré-Euler formula. The Lefschetz number of a continuous mapping of a polyhedron into itself. Fixed point existence theorem. Examples. Remarks.

§2. The index of fixed points. The index of a zero of a vector field and of a singular point of an oriented field. The index of a fixed point. Index properties. Normal fixed points. Fixed points of affine mappings. Topological invariance of the index of fixed points. The invariance of the index at a singular point of an oriented field.

§3. Algebraic number of fixed points of a continuous mapping from a polyhedron into itself. The algebraic number of regular fixed points. General fixed points theorem. Regular fixed points of simplicial mappings. Reduction to the approximation theorem. Proof of the approximation theorem. Remarks on the concept of number of fixed points.

§4. Oriented fields on closed manifolds. Preliminaries on smooth manifolds. Oriented fields and their singularities in manifolds.

* * *

After this, the next step in degree theory was taken by Jean Leray (1906–1998) and Juliusz Pawel Schauder (1899–1943), who developed degree theory for *completely continuous mappings in Banach spaces.* They published their results in 1934 in the paper [Leray-Schauder 1934]. Previously, a summary of the paper had been presented at the Paris Academy of Sciences [Leray-Schauder 1933] on July 10, 1933. This summary starts as follows:

Let $\mathcal{E}$ be an abstract space, normed, linear, and complete (in the sense of Mr. Banach [Banach 1922]*); let W be an open set of $\mathcal{E}$ (whose boundary we denote by W', and its closure by $\overline{W} = W + W'$); finally, let*

$$y = x - \mathcal{F}(x) = \Phi(x)$$

be a functional transformation defined on $\overline{W}$. We do not suppose $\mathcal{F}(x)$ linear, but only completely continuous *(or* compact, vollstetig*) (that is, $\mathcal{F}(x)$ is continuous and transforms every bounded set into a relatively compact one); furthermore, $\mathcal{F}(x)$ takes values only in $\mathcal{E}$. We have been able to define, after the famous work by Mr. Brouwer* [Brouwer 1912a]*, the* topological degree $d[\Phi, W, b]$ *of the transformation Φ at a point b off $\Phi(W')$, in such a way that the well-known properties of this degree are still valid. We describe briefly our definition.*

The Leray-Schauder construction is this. Let us suppose the point b is the origin $0 \in \mathcal{E}$, and set

$$h = \min\{d(0, \Phi(x)) : x \in W'\}.$$

Consider a new functional transformation $\mathcal{F}_h(x)$ at a distance $< h$ from $\mathcal{F}(x)$ and such that all its values belong to a common linear subset M of $\mathcal{E}$ of finite dimension; denote by W_M the intersection of W with that subset M. Let $\Phi_h(x) = x - \mathcal{F}_h(x)$. We set

$$d(\Phi, W, 0) = d(\Phi_h, W_M, 0),$$

where the right-hand side degree comes from Brouwer's theory.

Then, the authors prove that the value $d(\Phi, W, 0)$ does not depend on the choice of Φ_h and W_M, after which they develop the main properties of this degree. They introduce the following:

Definition. *The degree at 0 of the transformation*

$$y = x - \mathcal{F}(x) = \Phi(x) \tag{1}$$

is called the total index *of the solutions in W of the equation*

$$x - \mathcal{F}(x) = 0. \tag{2}$$

With this terminology, they prove:

Theorem. *Suppose that equation* (2) *depends continuously on a parameter k varying in a segment K of the real line. Let Ω be an open bounded set of the space $\mathcal{E} \times K$ such that* (i) *$\mathcal{F}$ is defined on $\overline{\Omega}$ and* (ii) *the boundary Ω' of Ω does not contain solutions to* (1)*. Then, the total index of the solutions in the interior of Ω is the same for all values of k.*

Thus, we see that the existence of at least a solution to (2) is guaranteed for every value k as soon as one knows a value k_0 for which the total index is not zero. The authors conclude by stressing that their theorem has a wide range of applications. In particular, it applies to limit (boundary) problems concerning *elliptic second-order partial differential equations,* and they can use it to generalize the well-known existence theorems due to Sergei Natanovich Bernstein (1880–1968).

These important applications to differential equations of the Leray-Schauder methods lead to the consideration of more general operators, in particular linear and non-linear Fredholm or Noether operators. The first investigations in this line are due to Renato Caccioppoli, who in [Caccioppoli 1936] obtained a mod 2 degree theory for continuously differentiable Fredholm mappings of index 0.

4. Renaissance and reformation

After the publication of Brouwer's fundamental papers, much effort was devoted to establishing the main properties of the degree of a continuous mapping, and in particular the Fixed Point Theorem, by analytic methods that did not involve the concepts of combinatorial topology. It is clear that the first satisfactory approach to this question was Hadamard's use of the Kronecker integral. The second attempt is based on the paper [Sard 1942] by ARTHUR SARD (1909–1980). There, we find the famous:

Sard Theorem. *Let G be an open set in $\mathbb{R}^m$, and let $f : \overline{G} \to \mathbb{R}^n$ be a $\mathcal{C}^p$ mapping ($p \geq 1$), with $p \geq m - n + 1$. Let C_f stand for the set of* critical values *of f, that is, $y = f(x) \in C_f$ when the Jacobian matrix of f at x has rank $<$ $\min(m, n)$ (which is the maximum possible rank). Then, C_f has Lebesgue measure zero.*

In particular, C_f has empty interior, hence dense complement, in $\mathbb{R}^n$.

The final conclusion comes from the fact that *a subset of $\mathbb{R}^n$ whose Lebesgue measure is zero does not contain any open set,* a result proved previously by ARTHUR BARTON BROWN in [Brown 1935]. Due to this, the theorem above is often called the *Sard-Brown Theorem.*

Using this theorem and the classical Weierstrass Theorem on polynomial uniform approximation of continuous functions on compact sets in Euclidean spaces, in 1951 MITIO NAGUMO constructed Brouwer's degree theory by elementary analytic means. He published the results in [Nagumo 1951a]; the journal had received the paper on March 6, 1950. In the introduction, the author says:

This paper establishes a theory of degree of mapping for open sets in a Euclidean space of finite dimension, based on the theory of infinitesimal analysis, which is free of the notion of simplicial mapping. Although the results are not new, I hope in this way to make it possible to incorporate the theory of degree of mapping into a course in infinitesimal analysis.

The basic ideas of Nagumo's construction are these:

Step I: Let G be an open bounded set in $\mathbb{R}^n$, $f : \overline{G} \to \mathbb{R}^n$ a $\mathcal{C}^1$ mapping, and $a \in \mathbb{R}^n \setminus \big(f(\overline{G} \setminus G) \cup C_f\big)$. Then, by the Inverse Function Theorem, the derivative $df(x)$ is a linear isomorphism for every $x \in f^{-1}(a)$, and consequently $f^{-1}(a)$ is a finite set, say $f^{-1}(a) = \{x_1, \ldots, x_p\}$. Thus, Nagumo can define

$$d(f,G,a) = \begin{cases} \sum_{i=1}^{p} \operatorname{sign}\det df(x_i) & \text{if } f^{-1}(a) = \{x_1,\dots,x_p\}, \\ 0 & \text{if } f^{-1}(a) = \emptyset. \end{cases}$$

Step II: Let G be an open bounded set in $\mathbb{R}^n$, $f : \overline{G} \to \mathbb{R}^n$ a continuous mapping, and $a \in \mathbb{R}^n \setminus f(\overline{G} \setminus G)$. Then $d(g,G,a)$ has the same value for all $\mathcal{C}^2$ mappings $g : \overline{G} \to \mathbb{R}^n$ such that

(1) $|g(x) - f(x)| < \operatorname{dist}\big(a, f(\overline{G} \setminus G)\big)$ for all $x \in \overline{G}$, and

(2) $a \in \mathbb{R}^n \setminus C_g$.

Notice here that condition (1) implies $a \notin g(\overline{G} \setminus G)$; hence Step I applies to g, and Nagumo can also define the *degree of f at a* by setting

$$d(f,G,a) = d(g,G,a)$$

for any g as above.

We remark that Step II is where Nagumo used the Sard-Brown and Weierstrass Theorems. On the other hand, it is worth stressing that the degree $d(f,G,a)$ is the Hadamard index of $f|_{\partial G}$, in case $\partial G = \overline{G} \setminus G$ is a hypersurface.

This same method leads to the construction of degree for continuous mappings of boundaryless compact oriented manifolds, once one extends the Sard and Weierstrass Theorems to those manifolds. It is easy to see that the degree obtained this way coincides with Brouwer's.

Nagumo also wrote a second important article, [Nagumo 1951b], where we find the first axiomatic approach to degree theory. This second article was also received by the journal on March 6, 1950, and published in the same volume right after the first one. In the introduction, the author explains that the goal of the paper is to complete the details of Leray-Schauder's construction of degree and that following their ideas, he will describe a degree theory for locally convex topological spaces and prove the Invariance of Domain Theorem for locally convex complete linear spaces. From this paper, we extract paragraph 1.1, relevant for the axiomatic developement of degree theory. Nagumo starts:

First we shall explain the notion of degree of mapping in a finite dimensional Euclidean space.

And he presents the following existence statement:

Nagumo axiomatization. *Let G be an open set in $\mathbb{R}^m$ and let $f : \overline{G} \to \mathbb{R}^m$ be such that $f(x) - x$ is bounded for $x \in \overline{G}$. Let $a \notin f(\overline{G} \setminus G)$. Then there will be determined an integer $d(f, G, a)$, called the* degree of mapping *of G at a by f, with the following properties:*

(i) $d(\mathrm{Id}_{\overline{G}}, G, a) = 1$ *for* $a \in G$; $d(\mathrm{Id}_{\overline{G}}, G, a) = 0$ *for* $a \notin G$.

(ii) *If* $d(f, G, a) \neq 0$, *then there is an* $x \in G$ *such that* $f(x) = a$.

(iii) *If* $G \supset \bigcup_i G_i$, $\overline{G} \supset \bigcup_i \overline{G_i}$, *for finitely many disjoint open sets* G_i, *and* $a \notin f(\overline{G_i} \setminus G_i)$ *for any* i, *then*

$$d(f, G, a) = \sum_i d(f_i, G, a).$$

(iv) *If* $f_t(x) - x$ *is a bounded continuous function of* (t, x) *for* $0 \leq t \leq 1$, $x \in \overline{G}$, *if* $a(t) \in \mathbb{R}^n$ *is continuous, and if* $a(t) \notin f_t(\overline{G} \setminus G)$ *for* $0 \leq t \leq 1$, *then* $d(f_t, G, a(t))$ *is constant for* $0 \leq t \leq 1$.

(v) *Let* X *be the set of all roots of the equation* $f(x) = a$ *in* G, *and let* G_0 *be any open set such that* $X \subset G_0 \subset G$. *Then*

$$d(f, G, a) = d(f, G_0, a).$$

Then, Nagumo remarks that (v) follows from (ii) and (iii) and concludes:

The existence and uniqueness of $d(f, G, a)$ satysfying the above conditions can be verified, if we use simplicial mappings for approximations of f (at first for bounded G and then for general G). But I may refer to [Nagumo 1951a] *in which the existence of $d(f, G, a)$ is given, based on infinitesimal analysis but free from the notion of simplicial mapping.*

Only one objection can be made here: Nagumo does not prove uniqueness!

* * *

We close this section by recalling two additional ways to define the Brouwer-Kronecker degree.

De Rham cohomology. On one hand, Georges de Rham (1903–1990) published the book [de Rham 1955], where he developed homology and degree theory from a completely different viewpoint. He used flows in connection with the theory of distributions due to LAURENT SCHWARTZ (1915–2002). We can describe this method very roughly as follows.

Every differentiable mapping $f : M \to N$ of boundaryless compact connected oriented n-manifolds induces in (de Rham) cohomology a linear mapping $f^* : H^n(N, \mathbb{R}) \to H^n(M, \mathbb{R})$. Since $H^n(\cdot, \mathbb{R}) \equiv \mathbb{R}$, that linear mapping is just multiplication times some number, *but this number is an integer* d*, which we call the degree of* f*.* Among other things, two important results are behind this approach: Stokes' Theorem and the Change of Variables Formula for Integrals.

Heinz Integral Formula. On the other hand, and related to the above method, in 1959 ERHARD HEINZ presented a more elementary construction, which appears in [Heinz 1959] and runs as follows.

Let $y = y(x)$ be a $\mathcal{C}^1$ mapping defined on a bounded open set $\Omega \subset \mathbb{R}^m$, which is continuous on $\overline{\Omega}$. Furthermore, assume we are given a point $z \in \mathbb{R}^m$ such that $y(x) \neq z$ for all $x \in \overline{\Omega} \setminus \Omega$, and let $\Phi(r)$ be a real-valued function such that the following hold:

(1) $\Phi(r)$ is continuous on the interval $0 \leq r < +\infty$ and vanishes on a neighborhood of $r = 0$ and on $\varepsilon \leq r < +\infty$, where

$$0 < \varepsilon < \min\{y(x) - z : x \in \overline{\Omega} \setminus \Omega\}.$$

(2) $\int_{\mathbb{R}^m} \Phi(|x|)dx = 1$.

Then the Brouwer degree $d(y(x), \Omega, z)$ is uniquely defined by

$$d(y(x), \Omega, z) = \int_{\Omega} \Phi(|y(x) - z|)J(y(x))dx.$$

5. Axiomatization

The development of degree theory was fully completed in 1971, with the axiomatic characterization of the Brouwer-Kronecker topological degree. Such a characterization was obtained by LUTZ FÜHRER in his Ph.D. thesis [Führer 1971], presented at the Freie Universität Berlin. These results were published in the paper [Führer 1972], received by the journal on October 28, 1971. Let us also mention that Führer quoted Nagumo.

However, the axiomatic characterization of degree is often attributed to Herbert Amann and Stanley A. Weiss. In fact, in the paper [Amann-Weiss 1973] (received by the journal on August 21, 1972) they characterize axiomatically not only the Brouwer-Kronecker degree, but also the Leray-Schauder degree. The authors conclude their introduction with the following acknowledgement:

> *After having finished the first draft of this paper the authors learned of the thesis of Führer* [Führer 1971] *in which the uniqueness of the Brouwer degree has been proved also.*

Thus, priorities are clear: the axiomatization of the Brouwer-Kronecker degree is due to Führer, that of the Leray-Schauder degree to Amann-Weiss.

* * *

Let us start with the following:

Führer Characterization. *There exists a unique mapping*

$$d : \left\{ (f, D, y) : \begin{array}{l} D \subset \mathbb{R}^n \quad \textit{bounded open}, \\ f : \overline{D} \to \mathbb{R}^n \quad \textit{continuous}, \\ y \in \mathbb{R}^n \setminus f(\partial D) \end{array} \right\} \longrightarrow \mathbb{Z}$$

such that we have the following:

(1) Homotopy invariance: *For every bounded open set $D \subset \mathbb{R}^n$ and all continuous mappings $F : [0,1] \times \overline{D} \to \mathbb{R}^n$ and $\gamma : [0,1] \to \mathbb{R}^n$ such that*

$$\gamma(t) \in \mathbb{R}^n \setminus F(\{t\} \times \partial D) \quad \textit{for} \quad 0 \leq t \leq 1,$$

the following formula holds:

$$d(F(t, \cdot), D, \gamma(t)) = d(F(0, \cdot), D, \gamma(0)) \quad \textit{for} \quad 0 \leq t \leq 1.$$

(2) Normality: *For every bounded open set $D \subset \mathbb{R}^n$ and every point $p \in D$,*

$$d(\mathrm{Id}_{\overline{D}}, D, p) = 1.$$

(3) Additivity: *For every bounded open set $D \subset \mathbb{R}^n$, every disjoint union $D = D_1 \cup D_2$ of two open sets, every continuous mapping $f : \overline{D} \to \mathbb{R}^n$, and every point $p \in \mathbb{R}^n \setminus f(\partial D \cup \partial D_1 \cup \partial D_2)$,*

$$d(f, D, p) = d(f, D_1, p) + d(f, D_2, p).$$

(4) Existence of solutions: *For every bounded open set* $D \subset \mathbb{R}^n$*, every continuous mapping* $f : \overline{D} \to \mathbb{R}^n$*, and every point* $p \in \mathbb{R}^n \setminus f(\partial D)$*, such that* $d(f, D, p) \neq 0$*, the equation* $f(x) = p$ *has some solution in* D*.*

This is the full description of the Brouwer-Kronecker degree. This on one hand has the virtue that the theory becomes clearer and systematic, but on the other hand, and this is very important, this also shows that all different definitions of degree invented by the 1960s coincide, no matter which topological or analytical means were involved.

As often happens, once a collection of complete axioms is found, there is a search for simplicity that aims to distill the collection as much as possible. In this respect, it is worth quoting here the following result obtained by Klaus Deimling in [Deimling 1985]:

Deimling Characterization. *There exists a unique mapping*

$$d : \left\{ (f, D, y) : \begin{array}{l} D \subset \mathbb{R}^n \quad \textit{bounded open,} \\ f : \overline{D} \to \mathbb{R}^n \quad \textit{continuous,} \\ y \in \mathbb{R}^n \setminus f(\partial D) \end{array} \right\} \longrightarrow \mathbb{Z}$$

such that we have the following:

(1) Normality: *For every bounded open set* $D \subset \mathbb{R}^n$ *and every point* $p \in D$*,*

$$d(\mathrm{Id}_{\overline{D}}, D, p) = 1.$$

(2) Additivity: *For every bounded open set* $D \subset \mathbb{R}^n$*, every pair of two disjoint open sets* $D_1, D_2 \subset D$*, every continuous mapping* $f : \overline{D} \to \mathbb{R}^n$*, and every point* $p \notin f(\overline{D} \setminus D_1 \cup D_2)$*,*

$$d(f, D, p) = d(f, D_1, p) + d(f, D_2, p).$$

(3) Homotopy invariance: *For every bounded open set* $D \subset \mathbb{R}^n$ *and all continuous mappings* $H : [0, 1] \times \overline{D} \to \mathbb{R}^n$ *and* $y : [0, 1] \to \mathbb{R}^n$ *such that*

$$y(t) \notin H(\{t\} \times \partial D) \quad \textit{for} \quad 0 \leq t \leq 1,$$

the following formula holds:

$$d(H(t, \cdot), D, y(t)) = d(H(0, \cdot), D, y(0)) \quad \textit{for} \quad 0 \leq t \leq 1.$$

* * *

Next we describe the axiomatization of the Leray-Schauder degree according to Amann-Weiss.

Let $\mathcal{E}$ be a separated topological linear space, and consider a family $\mathcal{W}$ of open subsets including $\emptyset$ (but not only $\emptyset$). To each $\Omega \in \mathcal{W}$, attach a set $\mathcal{M}(\Omega)$ of continuous mappings $\overline{\Omega} \to \mathcal{E}$. We equip this set $\mathcal{M}(\Omega)$ with the uniform convergency topology.

Definition 1. *The family*

$$\mathbb{M}(\mathcal{W}) = \{\mathcal{M}(\Omega) : \ \Omega \in \mathcal{W}\}$$

is called an admisible class *of mappings on $\mathcal{E}$ when the following properties hold true:*

(1) $\mathrm{Id}_{\overline{\Omega}} \in \mathcal{M}(\Omega)$ *for every non-empty open set $\Omega \in \mathcal{W}$.*

(2) *For every pair of two open sets $\Omega_1 \subset \Omega$ in $\mathcal{W}$ and every f in $\mathcal{M}(\Omega)$, the restriction $f|_{\overline{\Omega_1}}$ belongs to $\mathcal{M}(\Omega_1)$.*

Now, fix such an admissible class $\mathbb{M}(\mathcal{W})$ for the given family $\mathcal{W}$ of open sets of $\mathcal{E}$. For every $\Omega \in \mathcal{W}$, consider the following subspace of $\mathcal{M}(\Omega)$:

$$\mathcal{M}_0(\Omega) = \{f \in \mathcal{M}(\Omega) : \ 0 \notin \overline{f(\partial\Omega)}\},$$

where as usual $\partial\Omega = \overline{\Omega} \setminus \Omega$ stands for the boundary of Ω. Then:

Definition 2. *A* topological degree d *for* $\mathbb{M}(\mathcal{W})$ *is a family of mappings*

$$d = \{d(\cdot, \Omega) : \mathcal{M}_0(\Omega) \to \mathbb{Z} : \ \Omega \in \mathcal{W}\}$$

for which the following conditions hold true:

(1) Normality: $d(\mathrm{Id}_{\overline{\Omega}}, \Omega) = 1$ *for every $\Omega \in \mathcal{W}$ with $0 \in \Omega$.*

(2) Additivity: *For every non-empty open set $\Omega \in \mathcal{W}$, every pair of two disjoint open subsets of Ω, $\Omega_1, \Omega_2 \in \mathcal{W}$, and every $f \in \mathcal{M}(\Omega)$ such that $0 \notin f(\overline{\Omega} \setminus \Omega_1 \cup \Omega_2)$,*

$$d(f, \Omega) = d(f|_{\overline{\Omega_1}}, \Omega_1) + d(f|_{\overline{\Omega_2}}, \Omega_2).$$

(3) Homotopy invariance: *For every non-empty open set $\Omega \in \mathcal{W}$ and every continuous mapping $h : [0, 1] \to \mathcal{M}_0(\Omega)$,*

$$d(h(t), \Omega) = d(h(0), \Omega) \quad \textit{for} \quad 0 \leq t \leq 1.$$

Once the setting is fixed in this way, the authors prove their main result:

Theorem. *Let $\mathcal{E}$ be a locally convex linear space, and let $\mathcal{W}$ be either* (i) *the family of all open sets of $\mathcal{E}$ or* (ii) *the family of all bounded open sets of $\mathcal{E}$. Then, there exists a* unique *topological degree for the admissible class*

$$\mathbb{M}(\mathcal{W}) = \{K(\Omega) : \Omega \in \mathcal{W}\},$$

where $K(\Omega)$ consists of all continuous mappings $f : \overline{\Omega} \to \mathcal{E}$ such that the image $(\mathrm{Id}_{\overline{\Omega}} - f)(\overline{\Omega})$ is relatively compact (f is a compact vector field*).*

Of course, this implies the uniqueness of both the Brouwer-Kronecker and the Leray-Schauder degrees.

6. Further developments

There are several lines along which generalizations have been developed, with various aims and scope. We discuss here: (i) the case of spaces of *infinite dimension,* (ii) the case when *source and targent have different dimensions,* and (iii) the *equivariant* case.

* * *

Degree theory in infinite dimension. This line of research comes from Leray-Schauder theory. Its main purpose is to construct degrees for vector fields other than the compact ones, i.e., Fredholm, monotonous, contractive,..., and to find the corresponding axiomatizations. The resulting theories apply to problems on partial differential equations and bifurcation in functional equations. As we mentioned before (I.3, p. 34), this was started by Caccioppoli in 1936. Years later, the idea was rediscovered and presented in a more rigorous general way by Stephen Smale in [Smale 1965], making use of the non-oriented cobordism rings invented by René Thom in his outstanding foundational paper [Thom 1954]. Smale's definition can be summarized as follows.

Let $f : M \to V$ be a proper Fredholm mapping of index $p \geq 0$ and class $q > p+1$. Then the non-oriented cobordism class of the inverse image $f^{-1}(a)$ of a regular value a is a well-defined invariant $\gamma(f)$, which vanishes if f is not surjective. In case the index p is zero, then $f^{-1}(a)$ is a finite set and $\gamma(f)$ coincides with Caccioppoli's mod 2 degree.

When orientations are taken into account for index 0 Fredholm operators, the result is a degree theory with integral values, a construction made by K. David Elworthy and Anthony J. Tromba in [Elworthy-Tromba 1970a] and [Elworthy-Tromba 1970b]. In the first of these two references, the authors present an oriented degree theory for proper Fredholm mappings of index n and class $r = n+2$ using framed cobordism as introduced by Lev Semenovich Pontryagin in [Pontryagin 1955] (a gem of topology).

Then, in [Nirenberg 1971], Louis Nirenberg produced a generalized topological degree theory for compact perturbations of Fredholm operators, using stable homotopy groups and their analogous version in infinite dimension. These results are extended by E. Norman Dancer in [Dancer 1983]. Another interesting contribution to this topic is due to Jorge Ize, who used cohomotopy groups in [Ize 1981].

* * *

Mapping degree for source and target of distinct dimensions. From the preceding discussion, we see how bifurcation theory motivates the introduction of oriented degree theories for Fredholm mappings of positive index and therefore brings in the notion of topological degree for *mappings between spaces of different dimensions*. In such theories the so-called degree is not an integer anymore, but some homotopy class in a suitable homotopy group (of a sphere, because spheres are compactifications of Euclidean spaces). An important development of this theory is presented by Kazimierz Geba, Ivar Massabò, and Alfonso Vignoli in [Geba *et al.* 1986].

The goal is to construct a generalized degree (the *Geba-Massabò-Vignoli degree*) for continuous mappings $f : \overline{U} \to \mathbb{R}^n$, where U is a bounded open subset of $\mathbb{R}^m$, with $m \geq n$, under the assumption that f does not vanish on $\partial U = \overline{U} \setminus U$. The authors support their theory by exploring the $m = n$ case, that is, by reformulating the Brouwer-Kronecker degree in a way suitable for generalization. Indeed, let f be given as above, with $m = n$, and denote by $f_0 : \partial U \to \mathbb{R}^n \setminus \{0\}$ the restriction of f to ∂U. We look at $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{0\}$ inside the *Alexandroff compactification* $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$. Since $(\mathbb{R}^n)^* \setminus \{0\}$ is homeomorphic to $\mathbb{R}^n$, by the Tietze Extension Theorem, f_0 extends to a continuous mapping $f_\infty : (\mathbb{R}^n)^* \setminus U \to (\mathbb{R}^n)^* \setminus \{0\}$. Thus one gets a continuous mapping

$$f^* : (\mathbb{R}^n)^* \to (\mathbb{R}^n)^* : x \mapsto \begin{cases} f(x) & \text{if } x \in \overline{U}, \\ f_\infty(x) & \text{if } x \in (\mathbb{R}^n)^* \setminus U. \end{cases}$$

Such an extension is called *admissible,* but note that it is not unique. Next, we consider a homeomorphism $\sigma_n : (\mathbb{R}^n)^* \to \mathbb{S}^n$ such that

$$\begin{aligned}
&\sigma_n(\infty) = (1, 0, \dots, 0),\\
&\sigma_n((\mathbb{R}^n)^*_+) = \{x \in \mathbb{S}^n : x_{n+1} \geq 0\},\\
&\sigma_n((\mathbb{R}^n)^*_-) = \{x \in \mathbb{S}^n : x_{n+1} \leq 0\},\\
&\sigma_{n+1}(x) = \sigma_n(x) \quad \text{for all } x \in \mathbb{R}^n.
\end{aligned}$$

It can then be shown that the homotopy class of $\sigma_n \circ f^* \circ \sigma_n^{-1}$ does not depend on the admissible extension f^*, and thus we have a well-defined element $[\sigma_n \circ f^* \circ \sigma_n^{-1}]$ in the n-th homotopy group $\pi_n(\mathbb{S}^n)$. Next, via an algebraic isomorphism $\varphi_n : \pi_n(\mathbb{S}^n) \to \mathbb{Z}$, set

$$d^*(f, U, 0) = \varphi_n([\sigma_n \circ f^* \circ \sigma_n^{-1}]).$$

Once d^* is thus constructed, it is checked that the axioms of degree theory hold true, and by the axiomatic characterization (I.5, p. 38), d^* is indeed the Brouwer-Kronecker degree.

Now, this construction can be mimicked for arbitrary $m \geq n$ to obtain a generalized degree:

$$d^*(f, U) = d^*(f, U, 0) = [\sigma_n \circ f^* \circ \sigma_m^{-1}] \in \pi_m(\mathbb{S}^n),$$

which, for $m > n$, is not an integer any more. Moreover, we see why the case $m < n$ was neglected: in that case $\pi_m(\mathbb{S}^n) = 0$.

In the article mentioned, the authors prove for this degree the basic properties, namely:

(1) Homotopy invariance. *If $h : [0, 1] \times (U, \partial U) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is a continuous mapping, then $d^*(h_t, U)$ is well defined for all t and does not depend on t.*

(2) Excision. *Let $f : (U, \partial U) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ be a continuous mapping. Then, for every open set $V \subset U$ such that f has no zeros in $U \setminus V$, we have $d^*(f, V) = d^*(f, U)$.*

(3) Existence of solutions. *Let $f : (U, \partial U) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ be a continuous mapping with $d^*(f, U) \neq 0 \in \pi_m(\mathbb{S}^n)$. Then, there is an $x \in U$ such that $f(x) = 0$.*

(4) Suspension. *Let U be an open subset of $\mathbb{R}^{m+1}$ and let*

$$f : (U, \partial U) \to (\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\})$$

be a continuous mapping such that

$$f(\overline{U} \cap \mathbb{R}^{m+1}_{+}) \subset \mathbb{R}^{n+1}_{+}, \quad f(\overline{U} \cap \mathbb{R}^{m+1}_{-}) \subset \mathbb{R}^{n+1}_{-}.$$

Then, setting $U_0 = U \cap \mathbb{R}^m \equiv U \cap (\mathbb{R}^m \times \{0\})$ *and* $f_0 = f|_{\overline{U}_0}$*, we have* $f_0 : (U_0, \partial U_0) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ *and*

$$d^*(f, U) = \Sigma(d^*(f_0, U_0)),$$

where $\Sigma : \pi_m(\mathbb{S}^n) \to \pi_{m+1}(\mathbb{S}^{n+1})$ *is the suspension homomorphism (an isomorphism for* $m < 2n-1$ *and an epimorphism for* $m = 2n-1$*).*

(5) Additivity. *Let* $f : (U, \partial U) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ *be a continuous mapping, and let* $U_1, U_2 \subset U$ *be two open disjoint sets such that* f *has no zero in* $U \setminus U_1 \cup U_2$*. Then* $d^*(f, U) = d^*(f, U_1) + d^*(f, U_2)$*, whenever* $m - n \leq n - 4$.

Later Francisco Romero Ruiz del Portal showed in his Ph.D. thesis [Ruiz del Portal 1991] that additivity also holds for $m - n \leq n - 2$, and this is definitive: there is a counterexample for $m = 2n + 1$. This appeared in [Ruiz del Portal 1992]. Another counterexample was published afterwards by Ize, Massabò, and Vignoli in [Ize *et al.* 1992], a paper that deals with equivariant degree as explained below.

To conclude, it must be noted that one major problem of this theory is the computation of the homotopy groups of the spheres, a question which is wide open today.

* * *

Equivariant degree theory. The purpose here is to define a suitable topological degree for mappings that are *invariant under the action of a Lie group* G on the given spaces. The case most studied is that of $G = \mathbb{S}^1$, that is, the so-called $\mathbb{S}^1$-*equivariant topological degree.*

Let us recall that Poincaré used what later would be called the Brouwer-Kronecker degree to study the critical points of a differential equation. But, as is well known, other very important elements for the understanding of differential equations are periodic orbits. It was to count their number that in 1965 F. Brock Fuller introduced an invariant of flows that today we call the *Fuller index*. The definition and properties of this index are given in detail in [Fuller 1967]. A careful analysis of the constructions behind the Fuller index and the generalized Geba-Massabò-Vignoli degree for $m = n+1$ led the above-mentioned Geba and Grzegorz Dylawerski, Jerzy Jodel, and Wacław Marzantowicz, from Gdansk University, to define in a preprint in 1987 (later published as [Dylawerski *et al.* 1991]) a new homotopy invariant for $\mathbb{S}^1$-equivariant continuous mappings, which they called $\mathbb{S}^1$-*degree*. Let us describe this briefly.

Let ρ be a *finite representation of* $\mathbb{S}^1$, that is, a continuous homomorphism $\rho : \mathbb{S}^1 \to GL(V)$ into the linear group $GL(V)$ of a real linear space V of finite dimension; ρ determines an $\mathbb{S}^1$-*action on* V by $(g, v) \mapsto \rho(g)(v)$. For such a pair (V, ρ), we say the following:

(1) A set $X \subset V \times \mathbb{R}$ is *invariant* if

$$(\rho(g)(v), \lambda) \in X \quad \text{for all } g \in \mathbb{S}^1,\ (v, \lambda) \in X.$$

(2) A continuous mapping $f : X \to V$ with invariant domain X is an $\mathbb{S}^1$-*mapping* if

$$f(\rho(g)(v), \lambda) = \rho(g)(f(v, \lambda)) \quad \text{for all } g \in \mathbb{S}^1,\ (v, \lambda) \in X.$$

We will denote by $\mathfrak{A}$ the abelian group of all finite sequences $\alpha = (\alpha_r)_{r \geq 0}$, with $\alpha_0 \in \mathbb{Z}_2$ and $\alpha_r \in \mathbb{Z}$ for $r \geq 1$ (sum defined componentwise).

With this terminology fixed, the authors prove:

Theorem. *Let* (V, ρ) *run through finite representations of* $\mathbb{S}^1$*,* Ω *through the family of all bounded, invariant open subsets of* $V \times \mathbb{R}$*, and* $f : X \to V$ *through* $\mathbb{S}^1$*-mappings such that* X *is invariant,* $\overline{\Omega} \subset X$*, and* $f(\partial\Omega) \subset V \setminus \{0\}$*. Then there exists an* $\mathfrak{A}$*-valued function* $\mathrm{Deg}(f, \Omega)$*, called the* $\mathbb{S}^1$-degree, *satisfying the following conditions:*

(a) *If* $\mathrm{Deg}(f, \Omega) \neq 0$*, then* $f^{-1}(0) \cap \Omega \neq \emptyset$*.*

(b) *If* $\Omega_0 \subset \Omega$ *is open and invariant and* $f^{-1}(0) \cap \Omega \subset \Omega_0$*, then* $\mathrm{Deg}(f, \Omega) = \mathrm{Deg}(f, \Omega_0)$*.*

(c) *If* Ω_1, Ω_2 *are two open invariant subsets of* Ω *such that* $\Omega_1 \cap \Omega_2 = \emptyset$ *and* $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$*, then* $\mathrm{Deg}(f, \Omega) = \mathrm{Deg}(f, \Omega_1) + \mathrm{Deg}(f, \Omega_2)$*.*

(d) *If* $h : (\overline{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V \setminus \{0\})$ *is an* $\mathbb{S}^1$*-homotopy, then* $\mathrm{Deg}(h_0, \Omega) = \mathrm{Deg}(h_1, \Omega)$*.*

(e) *Suppose* (W, η) *is another representation of* $\mathbb{S}^1$ *and let* U *be an open bounded, invariant subset of* W *such that* $0 \in U$*. Define* $F : \overline{U} \times \overline{\Omega} \to W \times V$ *by* $F(x, y) = (x, f(y))$*. Then* $\mathrm{Deg}(F, U \times \Omega) = \mathrm{Deg}(f, \Omega)$*.*

The construction of $\mathrm{Deg}(f, \Omega)$ begins with two particular cases. The first case is when we have the trivial representation $V = \mathbb{R}^n$ of $\mathbb{S}^1$. Then the $\mathbb{S}^1$-mappings in the theorem are the continuous mappings $f : (\Omega, \partial\Omega) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, where Ω is a bounded open subset of $\mathbb{R}^{n+1}$. In this case we have a homomorphism $\Sigma : \pi_{n+1}(\mathbb{S}^n) \to \mathbb{Z}_2$ (*suspension*) that is an isomorphism for $n \geq 3$, and the authors define

$$\mathrm{Deg}(f, \Omega) = \Sigma(d^*(f, \Omega)) \in \mathbb{Z}_2,$$

where d^* is the Geba-Massabò-Vignoli generalized degree. In this case, the proof of the theorem above follows from the results in [Geba *et al.* 1986].

Secondly, suppose there exists a point $a \in \Omega$ such that

$$f^{-1}(0) = \{\rho(g)(a) :\, g \in \mathbb{S}^1\};$$

that is, $f^{-1}(0)$ is an *orbit* of the action of $\mathbb{S}^1$ over V. Then the set

$$\mathbb{S}^1 * a = \{g \in \mathbb{S}^1 :\, \rho(g)(a) = a\}$$

is finite, hence a subgroup of $\mathbb{S}^1$ consisting of k-th roots of unit. We can furthermore assume without loss of generality that (V, ρ) is orthogonal (with respect to some inner product on V) and decompose

$$V = W \oplus W^\perp, \quad W = \{x \in V :\, \rho(g)(x) = x\}.$$

Then if in that decomposition f takes the form $f(x, y, \lambda) = (f_1(x, y, \lambda), y)$, the authors define

$$\mathrm{Deg}(f, \Omega) = (\alpha_r), \quad \alpha_r = \begin{cases} \deg(f, D) & \text{for } r = k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg(f, D)$ is the Brouwer-Kronecker degree and D is a closed disc in the linear space W, contained in $\Omega \cap (W \times \mathbb{R})$, transversal to $\mathbb{S}^1 * a = f^{-1}(0)$, and oriented so that it can be identified with the unit disc in W.

In the general case, the construction of Deg requires (i) the classical theorem that gives the full classification of all finite-dimensional representations of $\mathbb{S}^1$ ([Adams 1969]) and (ii) a quite non-trivial homotopy argument.

Another version of this degree for mappings defined on spheres has been studied using cohomological obstruction theory by Ize, Massabò, and Vignoli, in [Ize *et al.* 1986]. Afterwards, the same authors presented in the two papers [Ize *et al.* 1989] and [Ize *et al.* 1992] an equivariant degree theory for mappings defined on the closure of an arbitrary open set of the ambient space and replaced the circle group $\mathbb{S}^1$ by an arbitrary compact Lie group. In addition, they avoided the use of obstruction theory. This is different, in the non-equivariant setting, from the Geba-Massabò-Vignoli degree described earlier.

Namely, let U be a bounded open subset of $\mathbb{R}^m$ and let $f : \overline{U} \to \mathbb{R}^n$ $(m \geq n)$ be a continuous mapping such that $f(x) \neq 0$ for $x \in \partial U$. Let $\widehat{f} : B \to \mathbb{R}^n$ be a continuous extension of f to a closed ball $\overline{D}$ containing U. Let V be a bounded open neighborhood of ∂U with $f(x) \neq 0$ for $x \in \overline{V}$. Consider a Uryshon function $\varphi : \overline{D} \to [0, 1]$ which is $\equiv 1$ off $U \cup V$ and $\equiv 0$ on $\overline{U}$. Define a mapping

$$F : [0, 1] \times \overline{D} \to \mathbb{R}^{n+1} : (t, x) \mapsto F(t, x) = (2t + 2\varphi(x) - 1, \widehat{f}(x)).$$

It is easy to see that $F(t, x) = 0$ only if $x \in \overline{U}, f(x) = 0$, and $t = \frac{1}{2}$. Thus F maps $\partial([0, 1] \times \overline{D})$ into $\mathbb{R}^{n+1} \setminus \{0\}$, which defines an element of the homotopy group $\pi_m(\mathbb{S}^n)$: this is the generalized degree of f with respect to U. When the action of a compact Lie group G is present, this construction extends without difficulty, and the equivariant degree is an element of the equivariant homotopy group $\pi_m^G(\mathbb{S}^n)$. Then, in [Ize *et al.* 1992] the authors proved that this formulation for $G = \mathbb{S}^1$ is the $\mathbb{S}^1$-degree of [Dylawerski *et al.* 1991].

All of this is revised in systematic form in the book [Ize-Vignoli 2003].

Also, we mention that Dancer defined an $\mathbb{S}^1$-degree for gradient mappings in [Dancer 1985]. In this respect, the paper [Geba *et al.* 1990], by Geba, Massabò, and Vignoli, is remarkable. Later, Sławomir Rybicki discussed an $\mathbb{S}^1$-degree for orthogonal mappings (which include gradient mappings, [Rybicki 1994]), drawing upon the earlier work in [Dylawerski *et al.* 1991]. To end these comments, let us add that Dancer, Geba, and Rybicki obtained in [Dancer *et al.* 2005] a complete classification of equivariant gradient mappings up to homotopy, and the corresponding equivariant homotopy classes can be seen as equivariant degrees.