Convection

Material volumes and transport theorem. Let $\rho(\boldsymbol{x},t)$ and $\boldsymbol{u}(\boldsymbol{x},t)$ be the density and velocity fields of a fluid. Suppose $\boldsymbol{u}(\boldsymbol{x},t)$ is given. In principle, trajectories of all fluid particles can be determined, and the density $\rho(\boldsymbol{x},t)$ at any time t can be determined from an initial condition consisting of given values of $\rho(\boldsymbol{x},0)$. To understand this determination, first look at time sequences of regions in space corresponding to the same fluid particles, called *material volumes*. By definition, the total mass of fluid inside a

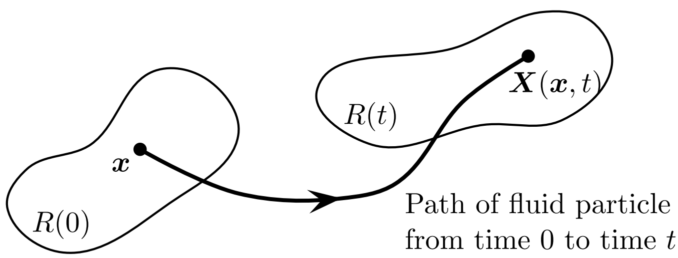

Figure 1.1.

material volume $R(t)$ is independent of t, so

$$\frac{d}{dt}\int_{R(t)} \rho(\boldsymbol{x},t)\, d\boldsymbol{x} = 0. \tag{1.1}$$

In general, given a vector field $\boldsymbol{u}(\boldsymbol{x},t)$ on $\mathbb{R}^n$, one can construct a *flow map* of $\mathbb{R}^n$ into itself. The image of $\boldsymbol{x}$, denoted by $\boldsymbol{X}(\boldsymbol{x},t)$, satisfies the ODE initial value problem

$$\begin{aligned} \dot{\boldsymbol{X}} &= \boldsymbol{u}(\boldsymbol{x},t), \quad \text{all } t, \\ \boldsymbol{X}(\boldsymbol{x},0) &= \boldsymbol{x}, \ \text{given}. \end{aligned} \tag{1.2}$$

The term material volume is henceforth used in a broader sense, referring to the time sequence of images of a fixed region under the flow map.

Let $c(\boldsymbol{x},t)$ be a real-valued function. We say that $c(\boldsymbol{x},t)$ is *convected* by the flow $\boldsymbol{u}(\boldsymbol{x},t)$ if for all material regions $R(t)$ of the flow, we have

$$\frac{d}{dt}\int_{R(t)} c(\boldsymbol{x},t)\,d\boldsymbol{x} = 0. \tag{1.3}$$

The heuristic idea is that $c(\boldsymbol{x},t)$ is the density of some "stuff" which is carried by the flow $\boldsymbol{u}$, such as milk in stirred coffee.

The bulk conservation identity (1.3) implies a PDE for $c(\boldsymbol{x},t)$. The derivation of the PDE asks for a bit of vector calculus, which we review here. Let $f(\boldsymbol{x},t)$ be any real-valued function. Then for any material region $R(t)$ of the flow $\boldsymbol{u}(\boldsymbol{x},t)$,

$$\frac{d}{dt}\int_{R(t)} f(\boldsymbol{x},t)\,d\boldsymbol{x} = \int_{R(t)} \{f_t + \boldsymbol{\nabla}\cdot(f\boldsymbol{u})\}\,d\boldsymbol{x}. \tag{1.4}$$

A standard proof of this *transport theorem* uses the flow map to change the variable of space integration in $\int_{R(t)} f(\boldsymbol{x}',t)\,d\boldsymbol{x}'$ from $\boldsymbol{x}'$ ranging over $R(t)$ to $\boldsymbol{x}$ ranging over $R(0)$, and involves time differentiation of the Jacobian determinant. Here, a heuristic calculation better serves a sense of geometric understanding. We have

$$\frac{d}{dt}\int_{R(t)} f\,d\boldsymbol{x} = \int_{R(t)} f_t\,d\boldsymbol{x} + \int_{\partial R} f\boldsymbol{u}\cdot\boldsymbol{n}\,d\boldsymbol{a} \tag{1.5}$$

Here, $\boldsymbol{n}$ is the outward unit normal on $\partial R(t)$. Figure 1.2 is a cartoon to understand the surface integral. In time increment dt, a "material patch" of area da sweeps out a cylinder of volume $(\boldsymbol{u}\cdot\boldsymbol{n}\,dt)\,da = (\boldsymbol{u}\cdot\boldsymbol{n}\,da)\,dt$. If $\boldsymbol{u}\cdot\boldsymbol{n} > 0$, this cylinder is space "annexed" by the material volume. If

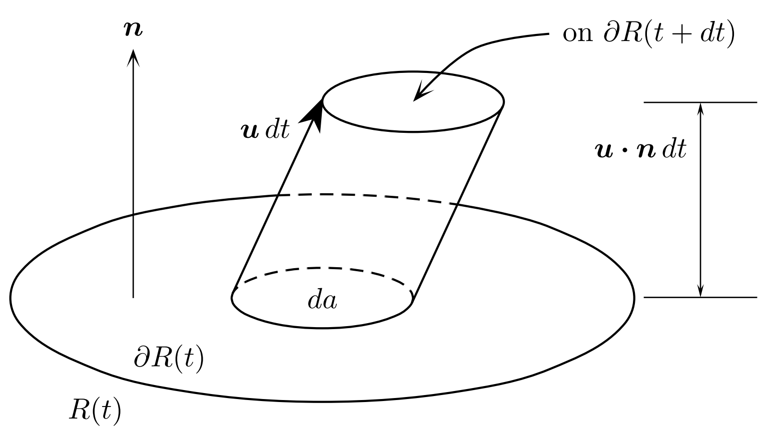

Figure 1.2.

$\boldsymbol{u} \cdot \boldsymbol{n} < 0$, it represents space surrendered. The contribution to $\frac{d}{dt} \int_{R(t)} f\, dx$ due to this cylinder is $f \boldsymbol{u} \cdot \boldsymbol{n}\, da$. Summing over patches da which entirely cover $\partial R(t)$ gives the surface integral in (1.5). The conversion of (1.5) into (1.4) involves an application of the divergence theorem to the surface integral.

A simple corollary of (1.4) identifies the property of the flow $\boldsymbol{u}$ which quantifies local expansion or contraction. In (1.4) take $f \equiv 1$; then we find the rate of change of volume

$$V(t) := \int_{R(t)} 1\, d\boldsymbol{x}$$

with respect to time to be

$$\dot{V}(t) = \int_{R(t)} \boldsymbol{\nabla} \cdot \boldsymbol{u}\, d\boldsymbol{x}.$$

In the limit of a material volume shrinking to a material point $\boldsymbol{x}(t)$,

$$\frac{1}{V} \int_{R(t)} \boldsymbol{\nabla} \cdot \boldsymbol{u}\, d\boldsymbol{x} \to (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{x}(t), t)$$

and hence

$$\frac{\dot{V}}{V} \to (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{x}(t), t). \tag{1.6}$$

Problem 1.1 (Convection along a one-dimensional line). Let x be displacement along the line. The velocity field $u(x, t)$ is a given function of position and time. $(a(t), b(t))$ is a material interval whose endpoints satisfy the ODEs $\dot{a} = u(a, t)$, $\dot{b} = u(b, t)$. Let $f(x, t)$ be any real-valued function of position and time. Formulate the one-dimensional version of the transport theorem (1.4),

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = ?$$

and prove it by use of calculus (the fundamental theorem of calculus and rules of differentiation).

Solution. The one-dimensional version of (1.4) is

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \{f_t + (uf)_x\}dx. \tag{1.1-1}$$

To prove (1.1-1), define

$$F(x,t) := \int_0^x f(x',t)dx'. \tag{1.1-2}$$

This is an antiderivative of f with respect to x, so we have

$$\int_{a(t)}^{b(t)} f(x,t)dx = F(b(t),t) - F(a(t),t).$$

We differentiate this identity with respect to t, using the chain rule for the right-hand side:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = F_t(b,t) - F_t(a,t) + F_x(b,t)\dot{b} - F_x(a,t)\dot{a}. \tag{1.1-3}$$

Time differentiation of (1.1-2) gives

$$F_t(x,t) = \int_0^x f_t(x',t)dx'$$

and hence,

$$F_t(b,t) - F_t(a,t) = \int_0^b f_t(x,t)dx - \int_0^a f_t(x,t)dx = \int_a^b f_t\, dx. \tag{1.1-4}$$

Next,

$$\begin{aligned} F_x(b,t)\dot{b} - F_x(a,t)\dot{a} &= f(b,t)u(b,t) - f(a,t)u(a,t) \\ &= \int_a^b (uf)_x dx. \end{aligned} \tag{1.1-5}$$

The first equality uses $F_x = f$, $\dot{a} = u(a,t)$, and $\dot{b} = u(b,t)$. The second equality uses the fundamental theorem of calculus. From (1.1-3)–(1.1-5), it follows that

$$\frac{d}{dt}\int_a^b f\, dx = \int_a^b \{f_t + (uf)_x\}dx.$$

Problem 1.2 (Jacobian of flow map). Let $\boldsymbol{x} \to \boldsymbol{X}(\boldsymbol{x},t)$ be the flow map of $\mathbb{R}^n$ into itself associated with velocity field $\boldsymbol{u}(\boldsymbol{x},t)$.

a) Show that the flow map's Jacobian determinant $J(\boldsymbol{x},t)$ satisfies

$$J_t(\boldsymbol{x},0) = (\boldsymbol{\nabla}\cdot\boldsymbol{u})(\boldsymbol{x},0). \tag{1.2-1}$$

b) Use (1.2-1) to show that the volume $V(t)$ of the material region $R(t)$ satisfies

$$\dot{V}(0) = \int_{R(0)} (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{x}, 0) d\boldsymbol{x}.$$

This result is invariant under translation of the origin of time:

$$\dot{V}(t) = \int_{R(t)} (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{x}', t) d\boldsymbol{x}'. \tag{1.2-2}$$

c) Use (1.2-2) to show that

$$J_t(\boldsymbol{x}, t) = (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{X}(\boldsymbol{x}, t), t) J(\boldsymbol{x}, t).$$

Solution.

a) We have

$$\boldsymbol{X}(\boldsymbol{x}, h) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}, 0)h + O(h^2)$$

in the limit of $h \to 0$. In component form,

$$X_i(\boldsymbol{x}, h) = x_i + u_i(\boldsymbol{x}, 0)h + O(h^2).$$

Differentiating with respect to x_j gives the ij component of the Jacobian matrix,

$$\partial_j X_i = \delta_{ij} + (\partial_j u_i)(\boldsymbol{x}, 0)h + O(h^2) = \delta_{ij} + A_{ij} h,$$

where A is the matrix with components

$$A_{ij} = (\partial_j u_i)(\boldsymbol{x}, 0) + O(h). \tag{1.2-3}$$

Hence, the Jacobian determinant is

$$J(\boldsymbol{x}, h) = \mathrm{Det}(I + hA) = 1 + h\mathrm{Tr}A + \cdots + h^n \,\mathrm{Det}\, A. \tag{1.2-4}$$

To prove the second equality, recall the characteristic polynomial of A,

$$p(\lambda) := \mathrm{Det}(\lambda I - A) = \lambda^n - \lambda^{n-1}\mathrm{Tr}A + \cdots + (-1)^n \,\mathrm{Det}\, A.$$

We compute

$$\mathrm{Det}(I + hA) = (-h)^n p(-\frac{1}{h}) = 1 + h\mathrm{Tr}A + \cdots + h^n \,\mathrm{Det}\, A.$$

With A as in (1.2-3), (1.2-4) becomes

$$J(\boldsymbol{x}, h) = 1 + h(\partial_{ii} u)(\boldsymbol{x}, 0) + O(h^2),$$

with summation over the repeated index i. In vector notation,

$$J(\boldsymbol{x}, h) = 1 + h(\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{x}, 0) + O(h^2).$$

The coefficient of the 'h' term is $J_t(\boldsymbol{x}, 0)$.

b) We have

$$V(t) = \int_{R(t)} 1\, d\boldsymbol{x} = \int_{R(0)} J(\boldsymbol{x}, t) d\boldsymbol{x}. \tag{1.2-5}$$

Time differentiation and use of (1.2-1) gives

$$\dot{V}(0) = \int_{R(0)} J_t(\boldsymbol{x}, 0) d\boldsymbol{x} = \int_{R(0)} (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{x}, 0) d\boldsymbol{x}.$$

c) In (1.2-2), change the variable of integration $\boldsymbol{x}'$ in $R(t)$ to the pre-image $\boldsymbol{x}$ in $R(0)$. In this way,

$$\dot{V}(t) = \int_{R(0)} (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{X}(\boldsymbol{x}, t), t) J(\boldsymbol{x}, t) d\boldsymbol{x}. \tag{1.2-6}$$

However, time differentiation of (1.2-5) gives

$$\dot{V}(t) = \int_{R(0)} J_t(\boldsymbol{x}, t) d\boldsymbol{x}. \tag{1.2-7}$$

Since (1.2-6) and (1.2-7) are both true for arbitrary $R(0)$, we conclude that

$$J_t(\boldsymbol{x}, t) = (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{X}(\boldsymbol{x}, t), t) J(\boldsymbol{x}, t).$$

Problem 1.3 (Transport theorem by change of variable). Let $R(t)$ be any material region of the flow $\boldsymbol{u}(\boldsymbol{x}, t)$ and let $f(\boldsymbol{x}, t)$ be any real-valued function. Show that

$$\frac{d}{dt} \int_{R(t)} f\, d\boldsymbol{x} = \int_{R(t)} \{f_t + \boldsymbol{\nabla} \cdot (f\boldsymbol{u})\} d\boldsymbol{x}$$

by time differentiation of

$$\int_{R(t)} f(\boldsymbol{x}', t) d\boldsymbol{x}' = \int_{R(0)} f(\boldsymbol{X}(\boldsymbol{x}, t), t) J(\boldsymbol{x}, t) d\boldsymbol{x}.$$

Solution. We have

$$\frac{d}{dt} \int_{R(t)} f(\boldsymbol{x}', t) d\boldsymbol{x}' = \int_{R(0)} \{(f_t + \boldsymbol{X}_t \cdot \boldsymbol{\nabla} f)(\boldsymbol{X}, t) J(\boldsymbol{x}, t) + f(\boldsymbol{X}, t) J_t(\boldsymbol{x}, t)\} d\boldsymbol{x}.$$

In the right-hand side, set $\boldsymbol{X}_t = \boldsymbol{u}(\boldsymbol{X}, t)$ and $J_t(\boldsymbol{x}, t) = (\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{X}, t) J(\boldsymbol{x}, t)$ from Problem 1.2c, to obtain

$$\begin{aligned} \frac{d}{dt} \int_{R(t)} f(\boldsymbol{x}', t) d\boldsymbol{x}' &= \int_{R(0)} \{(f_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} f + (\boldsymbol{\nabla} \cdot \boldsymbol{u}) f)(\boldsymbol{X}, t)\} J(\boldsymbol{x}, t) d\boldsymbol{x} \\ &= \int_{R(0)} \{f_t + \boldsymbol{\nabla} \cdot (f \boldsymbol{u})\}(\boldsymbol{X}, t) J(\boldsymbol{x}, t) d\boldsymbol{x}. \end{aligned}$$

In the right-hand side, change the integration variable from $\boldsymbol{x}$ in $R(0)$ to images $\boldsymbol{x}'$ in $R(t)$, to get the final result,

$$\frac{d}{dt} \int_{R(t)} f(\boldsymbol{x}', t) d\boldsymbol{x}' = \int_{R(t)} \{f_t + \boldsymbol{\nabla} \cdot (f \boldsymbol{u})\} d\boldsymbol{x}'.$$

Dropping the primes gives the transport theorem (1.4).

Problem 1.4 (Time derivative of integral over a changing region). Let $R(t)$ be a time sequence of bounded regions in $\mathbb{R}^n$. The motion of the surface $\partial R(t)$ is quantified by the normal velocity: Think of $\boldsymbol{x}(t)$ as the trajectory of a "bug" trapped on $\partial R(t)$; the projection of its velocity $\dot{\boldsymbol{x}}$ onto the outward unit normal $\boldsymbol{n}$ is the *normal velocity* $v = \dot{\boldsymbol{x}} \cdot \boldsymbol{n}$ of the surface at $\boldsymbol{x}(t)$. For each t, the normal velocity is a function of position on $\partial R(t)$. Show that

$$\frac{d}{dt} \int_{R(t)} f(\boldsymbol{x}, t) d\boldsymbol{x} = \int_{R(t)} f_t \, d\boldsymbol{x} + \int_{\partial R(t)} f v \, da. \tag{1.4-1}$$

Solution. We make the (reasonable) assumption that the given time sequence of regions $R(t)$ is a material region of some velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. Then by Problem 1.3,

$$\begin{aligned} \frac{d}{dt} \int_R f d\boldsymbol{x} &= \int_R \{f_t + \boldsymbol{\nabla} \cdot (f \boldsymbol{u})\} d\boldsymbol{x} \\ &= \int_R f_t \, d\boldsymbol{x} + \int_{\partial R} f \boldsymbol{u} \cdot \boldsymbol{n} \, da. \end{aligned}$$

The last equality uses the divergence theorem. The restriction of $\boldsymbol{u}$ to ∂R gives the velocities of points on ∂R, so $\boldsymbol{u} \cdot \boldsymbol{n} = v$ and we obtain (1.4-1).

The convection PDE. Now let us return to the consequences of the bulk conservation identity (1.3). From (1.3) and (1.4) it follows that

$$\int_{R(t)} \{c_t + \boldsymbol{\nabla} \cdot (c\boldsymbol{u})\} \, d\boldsymbol{x} = 0$$

for all material volumes $R(t)$. Given any fixed region D, there is, at any moment in time, a material volume which coincides with it. Hence, for any fixed region D,

$$\int_{D} \{c_t + \boldsymbol{\nabla} \cdot (c\boldsymbol{u})\} \, d\boldsymbol{x} = 0 \tag{1.7}$$

for all t. This implies the pointwise PDE

$$c_t + \boldsymbol{\nabla} \cdot (c\boldsymbol{u}) = 0. \tag{1.8}$$

Equation (1.8) applies in the fluid case with $c = \rho$, the density field.

Problem 1.5 (Density carried by incompressible flow). A flow $\boldsymbol{u}(\boldsymbol{x}, t)$ is *incompressible* if each of its material regions has volume independent of time. Let $c(\boldsymbol{x}, t)$ be a density convected by such an incompressible flow, and let $S(t)$ be a material surface, that is, a time sequence of images of a given, fixed surface under the flow map. Show that if $S(0)$ is a level surface of $c(x, 0)$, then $S(t)$ is a level surface of $c(\boldsymbol{x}, t)$.

Solution. Let $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ be any two points on $S(t)$. There are $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ on $S(0)$ such that $\boldsymbol{y}_1 = \boldsymbol{X}(\boldsymbol{x}_1, t)$ and $\boldsymbol{y}_2 = \boldsymbol{X}(\boldsymbol{x}_2, t)$. Since $S(0)$ is a level surface of $c(x, 0)$, $c(\boldsymbol{x}_1, 0) = c(\boldsymbol{x}_2, 0)$. Now look at $c_1(t) := c(\boldsymbol{X}(\boldsymbol{x}_1, t), t)$. Compute

$$\begin{aligned} \dot{c}_1(t) &= (c_t + \boldsymbol{X}_t(\boldsymbol{x}_1, t) \cdot \boldsymbol{\nabla} c)(\boldsymbol{X}(\boldsymbol{x}_1, t), t) \\ &= (c_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} c)(\boldsymbol{X}(\boldsymbol{x}_1, t), t). \end{aligned} \tag{1.5-1}$$

Here, we have used $\boldsymbol{X}_t = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{x}_1, t), t)$. For incompressible flow, $\boldsymbol{\nabla} \cdot \boldsymbol{u} \equiv 0$, and the convection PDE reduces to $c_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} c = 0$ for all $\boldsymbol{x}, t$. Hence, the right-hand side of (1.5-1) is zero, i.e., $c_1(t)$ is a constant independent of t, and $c(\boldsymbol{x}_1, 0) = c_1(0) = c_1(t) = c(\boldsymbol{y}_1, t)$. Similarly, $c(\boldsymbol{x}_2, 0) = c(\boldsymbol{y}_2, t)$. Since $c(\boldsymbol{x}_1, 0) = c(\boldsymbol{x}_2, 0)$, it follows that $c(\boldsymbol{y}_1, t) = c(\boldsymbol{y}_2, t)$. Since $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are arbitrary points on $S(t)$, $S(t)$ is a level surface of $c(\boldsymbol{x}, t)$.

Problem 1.6 (Transformation of velocities as a flow map). There is a swarm of stars. Let u denote their x-velocities as seen in a given (inertial) frame of reference. Their velocities v seen in an inertial frame with x-velocity U relative to the first inertial frame is

$$v = \frac{u - U}{1 - \frac{Uu}{c^2}}. \tag{1.6-1}$$

This is a formula from special relativity: c is the speed of light, and we have $|U|$ and all the $|u|$'s less than c. Then it follows from (1.6-1) that the $|v|$'s are all less than c too. Let $\rho(v, U)$ denote the distribution of these v's as seen in the moving frame, i.e., the number of stars with $v_1 < v < v_2$ is $\int_{v_1}^{v_2} \rho(v, U) dv$.

a) Show that $\rho(v, U)$ satisfies a convection PDE, with v as the "space-like" coordinate and U as "time".

b) Determine the solution of the convection PDE which is independent of U. This corresponds to an ensemble of stars whose x-velocity distribution looks the same in all inertial frames moving in the x-direction.

Solution.

a) Think of $u \to v(u, U) := \frac{u-U}{1-\frac{Uu}{c^2}}$ as a "flow map": Think of U as "time" and the star velocity v as "position" so that $v(u, U)$ is position at time U given that position at time 0 was u. The "velocity" at "time" U is

$$v_U(u, U) = -\frac{1 - \frac{u^2}{c^2}}{\left(1 - \frac{Uu}{c^2}\right)^2}. \tag{1.6-2}$$

Expressing this as a function of v gives the "velocity field" of the flow map (i.e., "velocity" as a function of "position" v). Solve $v = \frac{u-U}{1-\frac{Uu}{c^2}}$ for u to get $u = u(v, U) = \frac{v+U}{1+\frac{Uv}{c^2}}$, and substitute this into (1.6-2). The result is

$$V(v, U) := v_U(u(v, U), U) = -\frac{1 - \frac{v^2}{c^2}}{1 - \frac{U^2}{c^2}},$$

so the convection PDE with U, v and $V(v, U)$ in the roles of "time", "position" and "velocity" is

$$\rho_U - \frac{1}{1 - \frac{U^2}{c2}} \partial_v \left\{ \left(1 - \frac{v^2}{c^2}\right) \rho \right\} = 0.$$

b) $\rho_U = 0$ implies that $(1 - \frac{v^2}{c^2})\rho$ is a uniform constant independent of v; so

$$\rho(v) = \frac{\rho(0)}{1 - \frac{v^2}{c^2}}.$$

Notice that this distribution is non-integrable on $-c < v < c$.

Convective flux and boundary conditions. Convective flux is a notion which arises from the conservation identity (1.7) with respect to a *fixed* region D. By using the divergence theorem, convert (1.7) into

$$\frac{d}{dt}\int_D c\, d\boldsymbol{x} = -\int_{\partial D} c\boldsymbol{u}\cdot\boldsymbol{n}\, da. \tag{1.9}$$

If you think of $c(\boldsymbol{x}, t)$ as the density of "stuff", (1.9) says that the rate of stuff entering D is minus the surface integral of the *convective flux* $\boldsymbol{f} = c\boldsymbol{u}$. Figure 1.3 is a cartoon explaining the right-hand side of (1.9). In time increment dt, the stuff inside the cylinder of volume $-(\boldsymbol{u}\cdot\boldsymbol{n}\, da)\, dt$ is carried inside D by the flow $\boldsymbol{u}$, contributing $-(c\boldsymbol{u}\cdot\boldsymbol{n})\, da\, dt$ to $\int_D c\, d\boldsymbol{x}$. The rate of contribution due to the union of patches da entirely covering ∂D is the surface integral on the right-hand side of (1.9).

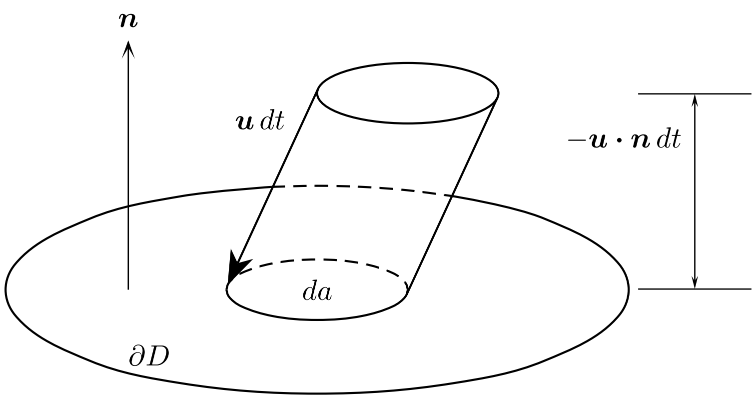

Figure 1.3.

Convective flux often appears in boundary conditions associated with the PDE (1.8). For instance, suppose that the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ has a jump discontinuity on a fixed $(n-1)$-dimensional surface in $\mathbb{R}^n$. Figure 1.4 depicts a two-dimensional surface S in $\mathbb{R}^3$ and particle paths (i.e., solutions of the ODE $\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t)$) which have "kinks" as they cross S. We

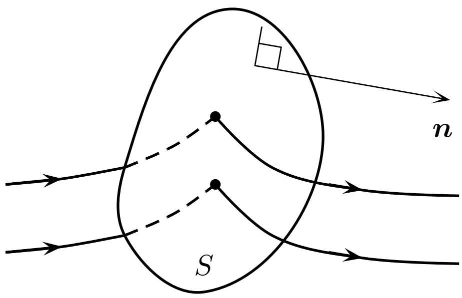

Figure 1.4.

assume that the trajectories really do cross S, instead of converging to S from both sides. Let $(c\boldsymbol{u})^+$ and $(c\boldsymbol{u})^-$ denote the right-hand and left-hand limits of $c\boldsymbol{u}$.[1] Stuff crosses S from the left at rate $(c\boldsymbol{u})^- \cdot \boldsymbol{n}$ per unit area, and emerges on the right at rate $(c\boldsymbol{u})^+ \cdot \boldsymbol{n}$ per unit area. Assuming that the surface S has no capacity to hold a surface density (i.e., nonzero stuff per unit area on S), the left and right rates must be equal, and so

$$[c\boldsymbol{u}] \cdot \boldsymbol{n} = 0, \tag{1.10}$$

where [ ] denotes the right-hand limit minus the left-hand limit of the enclosed quantity, $c\boldsymbol{u}$ in this case. Notice that if the normal component of velocity jumps, i.e., $[\boldsymbol{u} \cdot \boldsymbol{n}] \neq 0$, then c jumps too: If cars slow down at the point on the highway where the highway patrol waits, then clearly the car density downstream from the highway patrol will be greater than the density upstream.

Here is another example of a boundary condition based on convective fluxes.

Elastic rebounds from a moving wall. Consider an ensemble of $N \gg 1$ identical systems: Each consists of a single particle moving along a line. At any time t, the particle is confined to the half line $x \leq \ell(t)$. In $x < \ell(t)$ the particle is free, and its position $x(t)$ and velocity $v(t) := \dot{x}(t)$ satisfy the ODEs

$$\dot{x} = v, \quad \dot{v} = 0. \tag{1.11}$$

$x = \ell(t)$ represents a moving, solid wall. When a particle collides with this wall, it rebounds elastically: One must have $v > \dot{\ell}(t)$ just before collision, so that the incoming relative velocity $v - \dot{\ell}(t)$ is positive. The rebound relative velocity is $\dot{\ell}(t) - v$ and the rebound velocity (in the original frame of reference) is $2\dot{\ell} - v$.

The ensemble is described by an ensemble density function $n(x, v, t)$: The fraction of ensemble members in any region R of the (x, v) *phase plane* is $\int_R n(x, v, t)dx\,dv$. Now, we would really like to consider the density of $N \gg 1$ particles in the half space of $\mathbb{R}^3$ with $x < \ell(t)$. If the particles are

[1] Let $\boldsymbol{p}$ be any point on S. The right-hand (left-hand) limit of a function $f(\boldsymbol{x})$ as $\boldsymbol{x} \to \boldsymbol{p}$ is the limit of $f(\boldsymbol{p} + \boldsymbol{h})$ as $\boldsymbol{h} \to 0$ with $\boldsymbol{h} \cdot \boldsymbol{n} > (<)\, 0$.

so small that particle-particle collisions are rare, then the actual particle density in the space of x-positions and x-velocities evolves in the same way as the ensemble density. The ODEs (1.11) imply that the ensemble density $n(x, v, t)$ is convected by the velocity field

$$\begin{array}{l}(v, 0)\\ \uparrow\ \uparrow\\ x\ v \text{ components.}\end{array}$$

Hence, $n(x, v, t)$ satisfies the convection PDE

$$n_t + (vn)_x + (0n)_v = n_t + vn_x = 0 \tag{1.12}$$

in $x < \ell(t)$.

The kinematics of particles rebounding from the wall is embodied in a boundary condition on n along the line $x = \ell(t)$ in the (x, v) plane. To derive the boundary condition, it is instructive to examine the "absorption" and "emission" of particles from the wall. The constant number of particles is expressed by the integral identity

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\ell} n(x, v, t)dx\,dv = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\ell} n_t\,dx + \dot{\ell}n(\ell, v, t)\right\}dv = 0.$$

Substituting $n_t = -vn_x$ from (1.12), we obtain

$$\int_{-\infty}^{\infty}(\dot{\ell} - v)n(\ell, v, t)dv = 0. \tag{1.13}$$

The integrand of (1.13) has a clear meaning: If $v > \dot{\ell}$, it is negative. Evidently, particles with velocities between $v > \dot{\ell}$ and $v + dv$ are "absorbed" upon collision with the wall at rate

$$(v - \dot{\ell})n(\ell, v, t)dv. \tag{1.14}$$

These particles rebound with velocities between $2\dot{\ell} - v - dv$ and $2\dot{\ell} - v$, and the rate of rebounds is (1.14) with v replaced by $2\dot{\ell} - v$ and dv by $-dv$, that is,

$$(v - \dot{\ell})n(\ell, 2\dot{\ell} - v, t)dv. \tag{1.15}$$

By conservation of particles, the rates (1.14) and (1.15) of collisions and rebounds are equal, so we have the boundary condition

$$n(\ell, v, t) = n(\ell, 2\dot{\ell} - v, t). \tag{1.16}$$

Problems 1.7–1.10 treat the adiabatic expansion of particles in a box. Here, adiabatic means that the sides of the box move much more slowly than the particles inside.

Problem 1.7. We have an ensemble of $N \gg 1$ systems, each consisting of a particle moving along a line segment $-\frac{\ell(t)}{2} < x < \frac{\ell(t)}{2}$ (our box). The particle is free inside the box, and rebounds elastically off the walls $x = \frac{\ell(t)}{2}$ and $x = -\frac{\ell(t)}{2}$. Formulate the boundary value problem for the ensemble density $n(x, v, t)$. Show that this boundary value problem is invariant under simultaneous replacements of x by $-x$ and v by $-v$. (This being the case, there are solutions with $n(x, v, t) = n(-x, -v, t)$, and we will concentrate on these solutions in the remaining problems, 1.8 and 1.9.)

Solution. The ensemble density $n(x, v, t)$ satisfies the PDE

$$n_t + v n_x = 0$$

in $-\frac{\ell(t)}{2} < x < \frac{\ell(t)}{2}$, $-\infty < v < \infty$. The boundary conditions at the walls are

$$\begin{aligned} n(-\tfrac{\ell}{2}, v, t) &= n(-\tfrac{\ell}{2}, -\dot{\ell} - v, t), \\ n(\tfrac{\ell}{2}, v, t) &= n(\tfrac{\ell}{2}, \dot{\ell} - v, t). \end{aligned}$$

The invariance of the PDE under reversal of the signs of x and v is obvious. Let us do the sign reversals in the first boundary condition. We get

$$n(\tfrac{\ell}{2}, -v, t) = n(\tfrac{\ell}{2}, \dot{\ell} + v, t).$$

Since this holds for all v, we can replace v by $-v$, and this gives the second boundary condition. Similarly, the sign reversals applied to the second boundary condition give the first.

Problem 1.8. We want to determine approximate solutions of the boundary value problem in Problem 1.7 in the *adiabatic limit* of slowly moving walls. A preliminary integral identity is crucial. Show that in the limit $\dot{\ell} \to 0$,

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} n_t \, dx - \dot{\ell} v n_v(\tfrac{\ell}{2}, v, t) = O(\dot{\ell}^2).$$

Remember that we are working in the class of solutions with $n(x, v, t) = n(-x, -v, t)$.

Solution. The x-integral of the PDE over $-\frac{\ell}{2} < x < \frac{\ell}{2}$ gives

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} n_t\, dx + v\{n(\tfrac{\ell}{2}, v, t) - n(-\tfrac{\ell}{2}, v, t)\} = 0. \tag{1.8-1}$$

By the boundary condition at $x = \frac{\ell}{2}$ and the $n(x, v, t) = n(-x, -v, t)$ symmetry, we have

$$n(-\tfrac{\ell}{2}, v, t) = n(-\tfrac{\ell}{2}, -\dot{\ell} - v, t) = n(\tfrac{\ell}{2}, \dot{\ell} + v, t),$$

and (1.8-1) becomes

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} n_t\, dx + v\{n(\tfrac{\ell}{2}, v, t) - n(\tfrac{\ell}{2}, v + \dot{\ell}, t)\} = 0.$$

The Taylor expansion of $n(\frac{\ell}{2}, v + \dot{\ell}, t)$ in $\dot{\ell}$ gives

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} n_t\, dx - \dot{\ell} v n_v(\tfrac{\ell}{2}, v, t) = O(\dot{\ell}^2).$$

Problem 1.9. In the adiabatic limit, we expect that there are solutions for $n(x, v, t)$ with weak dependence on x. We make the heuristic approximation $n \approx n(v, t)$ independent of x. The *velocity distribution*, defined by

$$\rho(v, t) := \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} n\, dx,$$

is approximated by $\rho(v, t) \approx \ell(t) n(v, t)$. Derive an approximate convection PDE in v-space for $\rho(v, t)$. Determine the flow map of v-space into itself.

Solution. Assuming n to be independent of x, the integral identity of Problem 1.8 reduces to

$$n_t(v, t) - \frac{\dot{\ell}(t)}{\ell(t)} v n_v(v, t) = 0,$$

where we have dropped the $O(\dot{\ell}^2)$ from the right-hand side. We convert this into a PDE for the velocity distribution by substituting $n = \frac{\rho}{\ell}$. We get

$$\rho_t + \left(-\frac{\dot{\ell}}{\ell} v \rho\right)_v = 0.$$

The "velocity field" in v-space is

$$u = -\frac{\dot{\ell}}{\ell} v.$$

Hence, the flow map $v \to V(v,t)$ is the solution of the initial value problem

$$\begin{aligned} V_t &= -\frac{\dot{\ell}}{\ell}V \quad \text{in } t > 0, \\ V(v,0) &= v. \end{aligned}$$

The solution is $V(v,t) = \frac{\ell(0)v}{\ell(t)}$, which implies that

$$\ell(t)V(v,t) = \ell(0)v = \text{constant}.$$

Problem 1.10. The flow map in v-space has a nice interpretation within classical mechanics. Assume that the speed of the particle is large compared to the speed of the walls. The "orbit" of the particle in the (x,v) phase plane is approximated by an oriented rectangle, depicted in Figure 1.5. The horizontal segments represent the particle moving back and forth with speed V between the walls, and the vertical segments, the (instantaneous) collisions which reverse the direction of motion. As the wall moves slowly, the collisions do work on the particle, and the speed V changes in response to the change in the width ℓ of the box. Although the actual changes in speed are discrete, we will approximate the time dependence of V due to the cumulative effect of many collisions by a smooth function $V(t)$. Derive an approximate ODE for $V(t)$, and show that it predicts that the area of the rectangular orbit in the (x,v) phase plane is a constant independent of time.

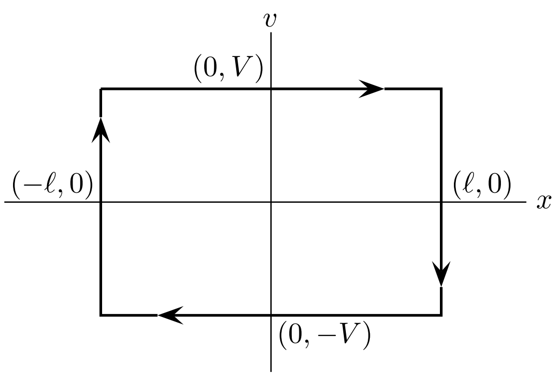

Figure 1.5.

Solution. One "orbit" around the (x,v) rectangle takes time $\Delta t = \frac{2\ell}{V}$, and in that time, there is a collision with the right wall $x = \frac{\ell}{2}$, which changes the velocity from V to $\dot{\ell} - V$, and a second collision with the left wall $x = \frac{-\ell}{2}$, which changes the velocity $\dot{\ell} - V$ to $-\dot{\ell} - (\dot{\ell} - V) = -2\dot{\ell} + V$. So the change

ΔV of speed in time Δt is $-2\dot{\ell}$. Hence, we propose the approximate ODE

$$\dot{V} = \frac{\Delta V}{\Delta t} = \frac{-2\dot{\ell}}{\left(\frac{2\ell}{V}\right)} = -\frac{\dot{\ell}}{\ell}V.$$

This is the ODE of the flow map in Problem 1.8, and it implies that $V(t)\ell(t)$ is a constant independent of t.

Convective derivative. Let $\boldsymbol{x}(t)$ be any material point in a flow $\boldsymbol{u}(\boldsymbol{x}, t)$, so

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t). \tag{1.17}$$

The time derivative of a function $f(\boldsymbol{x}, t)$ seen at the material point $\boldsymbol{x}(t)$ is, by the chain rule,

$$\frac{d}{dt} f(\boldsymbol{x}(t), t) = (f_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} f)(\boldsymbol{x}(t), t). \tag{1.18}$$

The directional derivative of $f(\boldsymbol{x}, t)$ in $(\boldsymbol{x}, t)$ spacetime, which appears on the right-hand side, is called the *convective derivative* of f,

$$\frac{Df}{Dt} := f_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} f. \tag{1.19}$$

The convective derivative informs an alternative derivation of the PDE (1.8) for the density c carried in a flow $\boldsymbol{u}$. Let us start with the conservation identity (1.3) which says that the total amount of stuff in material region $R(t)$ is a constant independent of t. In the limit of $R(t)$ shrinking to the single material point $\boldsymbol{x}(t)$, (1.3) reduces to

$$\frac{1}{V(t)} \frac{d}{dt}(c(\boldsymbol{x}(t), t)V(t)) \to 0, \tag{1.20}$$

where $V(t)$ is the volume of $R(t)$. Carry out time differentiation of the product cV and use the chain rule and definition (1.19) of the convective derivative to rewrite (1.20) as

$$\frac{Dc}{Dt}(\boldsymbol{x}(t), t) + \frac{\dot{V}(t)}{V(t)}\, c(\boldsymbol{x}(t), t) \to 0. \tag{1.21}$$

Now recall (1.6), which says that $\frac{\dot{V}}{V}$ converges to $(\boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{x}(t), t)$ as $R(t)$ shrinks to a point $\boldsymbol{x}(t)$. Hence, (1.6) and (1.21) lead to

$$\left\{ \frac{Dc}{Dt} + (\boldsymbol{\nabla} \cdot \boldsymbol{u})c \right\} (\boldsymbol{x}(t), t) = 0. \tag{1.22}$$

For any fixed point $(\boldsymbol{x}, t)$ in spacetime, there is a material point which matches the given $\boldsymbol{x}$ at the given t, so (1.22) holds for all fixed $(\boldsymbol{x}, t)$, i.e.,

$$\frac{Dc}{Dt} + (\boldsymbol{\nabla} \cdot \boldsymbol{u})c = 0. \tag{1.23}$$

Rewrite (1.23) as

$$c_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} c + (\boldsymbol{\nabla} \cdot \boldsymbol{u})c = 0$$

and use the product rule identity

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} c + (\boldsymbol{\nabla} \cdot \boldsymbol{u})c = \boldsymbol{\nabla} \cdot (c\boldsymbol{u})$$

to see the equivalence between (1.8) and (1.23).

Convected scalars. The real-valued function $f(\boldsymbol{x}, t)$ is called a *convected scalar* in the flow $\boldsymbol{u}$ if

$$\frac{Df}{Dt} = f_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})f = 0. \tag{1.24}$$

Unlike a convected density, there is no adjustment in its value at a material point due to local expansion or concentration of material regions. Convected scalars *do* arise in the real, common-sense world. For instance, suppose two kinds of particles A and B are carried in the flow $\boldsymbol{u}$, so that we have two convected densities c_A and c_B. The fraction of A particles is given by

$$f_A := \frac{c_A}{c_A + c_B} \tag{1.25}$$

and similarly for the fraction of B particles. These fractions are convected scalars, as we will show in Problem 1.11. In ideal gas dynamics with no dissipative process, the *entropy per unit mass* is a convected scalar. In the special case of *incompressible* flow (in which all material regions have volumes independent of t), it follows that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. Hence, (1.23) reduces to $\frac{Dc}{Dt} = 0$, and we see that a convected density in incompressible flow is a convected scalar as well.

Problem 1.11 (A convected scalar). Let c_A and c_B be convected densities of "A" and "B" particles in a flow $\boldsymbol{u}$. Show that the fraction of A particles as defined in (1.25) is a convected scalar.

Solution. Logarithmic time differentiation of (1.25) at a material point gives

$$\frac{1}{f_A}\frac{Df_A}{Dt} = \frac{1}{c_A}\frac{Dc_A}{Dt} - \frac{1}{c_A + c_B}\frac{D}{Dt}(c_A + c_B). \tag{1.11-1}$$

Now substitute

$$\frac{1}{c_A}\frac{Dc_A}{Dt} = \frac{1}{c_B}\frac{Dc_B}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{u},$$

and the right-hand side of (1.11-1) becomes

$$-\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0.$$

Gradient of a convected scalar. The notion of convective derivative leads to a clear analysis of how the *gradient* of a convected scalar or density evolves. For simplicity, consider a convected scalar which satisfies (1.24), i.e.,

$$\frac{Df}{Dt} = \partial_t f + u_j \partial_j f = 0. \tag{1.26}$$

Here, u_j are the components of the velocity field $\boldsymbol{u}$ and $\partial_j c$ is the partial derivative of c with respect to x_j. The twice repeated index j in (1.26) means summation over $j = 1, \ldots, n$. Compute the ith partial derivative of (1.26),

$$\partial_t(\partial_i f) + u_j \partial_j(\partial_i f) = -(\partial_i u_j)\partial_j f,$$

or

$$\frac{D(\partial_i f)}{Dt} = -(\partial_i u_j)\partial_j f, \quad i = 1, \ldots, n. \tag{1.27}$$

Equation (1.27) holds for all events in $(\boldsymbol{x}, t)$ spacetime, and in particular along the world line of a material point $\boldsymbol{x}(t)$. Along this world line, $\partial_i f$ are values at events $(\boldsymbol{x}(t), t)$ and, as such, functions of time. Likewise for the velocity field derivatives $\partial_i u_j$. The left-hand side is the time derivative of $(\partial_i f)(\boldsymbol{x}(t), t)$. Hence, (1.27) is a system of ODEs for the values of $\partial_1 f, \ldots, \partial_n f$ at $(\boldsymbol{x}(t), t)$.

Example. In polar coordinates (r, ϑ) of the plane, the flow $\boldsymbol{u}$ due to a point vortex in incompressible two-dimensional fluid is

$$\boldsymbol{u} = \frac{\Gamma}{2\pi r}\,\boldsymbol{e}_\vartheta.$$

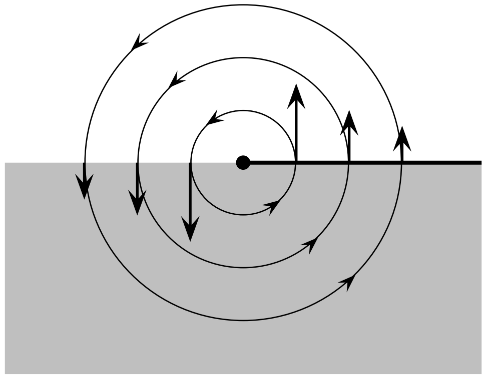

Figure 1.6.

The constant Γ measures the strength of the vortex. Figure 1.6 depicts the circular orbits of material points in this vortex flow. The vertical arrows represent the restriction of $\boldsymbol{u}$ to the x-axis. Let $c(r, \vartheta, t)$ be a convected density in this vortex flow. Since the vortex flow is incompressible, c is a convected scalar with $\frac{Dc}{Dt} = c_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} c = 0$. The polar coordinate form of this PDE is

$$c_t + \left(0\boldsymbol{e}_r + \frac{\Gamma}{2\pi r}\,\boldsymbol{e}_\vartheta\right) \cdot \left(c_r \boldsymbol{e}_r + \frac{1}{r}\,c_\vartheta \boldsymbol{e}_\vartheta\right) = 0,$$

or

$$\frac{Dc}{Dt} = c_t + \frac{\Gamma}{2\pi r^2}\,c_\vartheta = 0. \tag{1.28}$$

We want to track the gradient $\boldsymbol{\nabla} c = c_r \boldsymbol{e}_r + \frac{1}{r} c_\vartheta \boldsymbol{e}_\vartheta$ along the path of a material point. Hence, we need evaluations of c_r and c_ϑ. Differentiation of (1.28) with respect to r and ϑ gives

$$\frac{D(c_r)}{Dt} = (c_r)_t + \frac{\Gamma}{2\pi r^2}(c_r)_\vartheta = \frac{\Gamma}{\pi r^3} c_\vartheta, \tag{1.29}$$

$$\frac{D(c_\vartheta)}{Dt} = (c_\vartheta)_t + \frac{\Gamma}{2\pi r^2}(c_\vartheta)_\vartheta = 0. \tag{1.30}$$

Now (1.29) and (1.30) are special cases of the ODE (1.27) in polar coordinates. They give the time evolutions of c_r and c_ϑ at a material point whose polar coordinates $r(t), \vartheta(t)$ satisfy

$$\dot{r} = 0, \quad \dot{\vartheta} = \frac{\Gamma}{2\pi r^2}. \tag{1.31}$$

From (1.31) we see that material points move in circles of various radii r, with uniform angular velocities $\frac{\Gamma}{2\pi r^2}$. From the ODEs (1.29) and (1.30) it follows that at any such material point, $c_\vartheta = \gamma$, a constant independent of

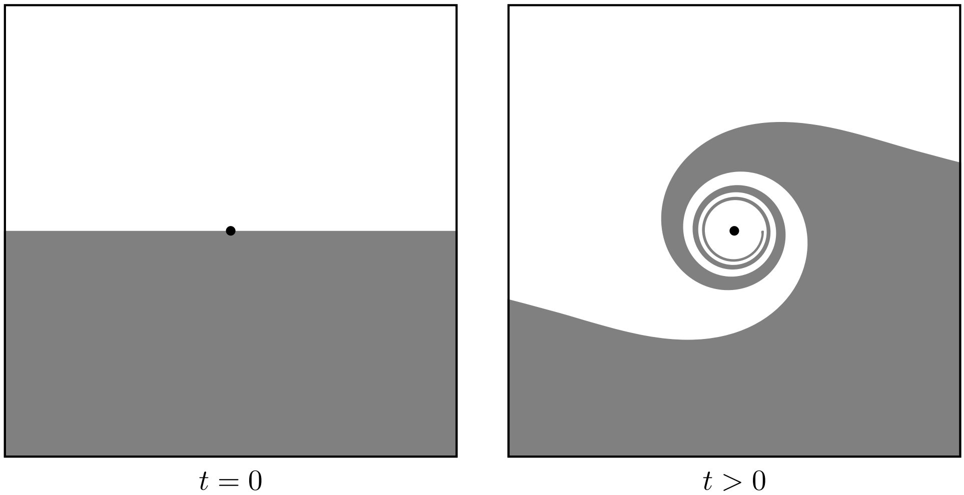

Figure 1.7.

time, and then

$$\frac{D(c_r)}{Dt} = \frac{\Gamma\gamma}{\pi r^3}.$$

A material point has $r(t) = r =$ constant, so the right-hand side is independent of time. Thus, the value of c_r seen at a fluid particle circulating around the origin grows indefinitely at a uniform rate. Figure 1.7 gives a visual sense of this growth in c_r. In the first panel, the shading represents stuff in the lower half plane at time $t = 0$. The second panel shows the stuff after it has been stirred by the vortex flow for time $t > 0$. There are spiral bands of stuff which become very fine and nearly circular as $t \to \infty$. Points of high and low concentration become very close to each other, and this indicates the growth of c_r.

Problem 1.12 (Flows that preserve gradients).

a) Let $c(\boldsymbol{x}, t)$ be a density convected in an incompressible flow $\boldsymbol{u}(\boldsymbol{x}, t)$ in $\mathbb{R}^n$. Show that $\frac{D}{Dt}|\boldsymbol{\nabla} c|^2 = -2\boldsymbol{\nabla} c \cdot (S\boldsymbol{\nabla} c)$, where S is the *strain matrix* of the flow $\boldsymbol{u}$, with cartesian components $S_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.

b) What is the most general incompressible flow for which $|\boldsymbol{\nabla} c|^2$ is always time-independent at material points?

Solution.

a) We calculate, with the help of (1.27),

$$\begin{aligned}\frac{D}{Dt}|\boldsymbol{\nabla} c|^2 &= \frac{D}{Dt}(\partial_i c\, \partial_i c)\\ &= 2\partial_i c \frac{D}{Dt}\partial_i c = -2\partial_i c(\partial_i u_j)\partial_j c.\end{aligned}$$

The double summation (over i, j) is invariant under interchange of i and j. Hence,

$$\begin{aligned}\frac{D}{Dt}|\boldsymbol{\nabla} c|^2 &= -2\partial_i c\Big\{\frac{1}{2}(\partial_i u_j + \partial_j u_i)\Big\}\partial_j c\\ &= -2\partial_i c S_{ij}\partial_j c\\ &= -2\boldsymbol{\nabla} c \cdot (S\boldsymbol{\nabla} c).\end{aligned}$$

b) For arbitrary $\boldsymbol{v} := \boldsymbol{\nabla} c$ we require $\boldsymbol{v} \cdot (S\boldsymbol{v}) = 0$. First, take $\boldsymbol{v} = \boldsymbol{e}_i$, the ith unit vector with components

$$(\boldsymbol{e}_i)_j = \delta_{ij} := \left[\begin{array}{ll} 1, & i = j, \\ 0, & i \neq j. \end{array}\right.$$

We find that $0 = \boldsymbol{e}_i \cdot (S\boldsymbol{e}_i) = S_{ii} = \partial_i u_i$ (with no summation over i); so for each i, u_i *is independent of* x_i. Next, take $\boldsymbol{v} = \boldsymbol{e}_i + \boldsymbol{e}_j$, where $\boldsymbol{e}_i \neq \boldsymbol{e}_j$. We have $0 = (\boldsymbol{e}_i + \boldsymbol{e}_j) \cdot S(\boldsymbol{e}_i + \boldsymbol{e}_j) = S_{ii} + S_{ij} + S_{ji} + S_{jj} = 2S_{ij} = \partial_i u_j + \partial_j u_i$, so

$$\partial_j u_i = -\partial_i u_j. \tag{1.12-1}$$

In the right-hand side, $\partial_i u_j$ is independent of x_j, and it follows that u_i is a linear combination of the x_j with $j \neq i$, plus an additive constant. In mathematical notation,

$$u_i = \sum_{j \neq i} w_{ij} x_j + v_i, \tag{1.12-2}$$

where the w_{ij} and v_i are constants. Substituting (1.12-2) into (1.12-1), we find that $w_{ij} = -w_{ji}$. The summary of equation (1.12-2) in vector notation is

$$\boldsymbol{u} = W\boldsymbol{x} + \boldsymbol{v},$$

where W is the antisymmetric matrix with components w_{ij} and $\boldsymbol{v}$ is the constant vector with components v_i. Hence, the flow $\boldsymbol{u}$ is a sum of solid body rotation ($W\boldsymbol{x}$) and translation at uniform velocity $\boldsymbol{v}$.

Problem 1.13 (Vector analog of a convected scalar). First, the analog is *not* $\frac{D}{Dt}\boldsymbol{v} = 0$. Instead, paint little oriented line segments in cookie dough to represent the initial values of a vector field; then deform the dough, and the induced displacement, rotation and stretching of the line segments will represent "convection" of the original vector field. Here is the mathematical formulation (Figure 1.8). Let $\boldsymbol{u}(\boldsymbol{x}, t)$ be a given velocity field and $\boldsymbol{v}(\boldsymbol{x}, t)$ another time-dependent vector field with a specific connection to $\boldsymbol{u}$. Let $C : \boldsymbol{x}(s, t)$ be a material curve of the flow $\boldsymbol{u}$, with s labeling material points. If at any one time t the restriction of $\boldsymbol{v}$ to C is the tangent vector $\boldsymbol{\tau}(s, t) := \boldsymbol{x}_s(s, t)$, then $\boldsymbol{v}$ restricted to C is the tangent vector $\boldsymbol{\tau}(s, t)$ for all time. In this case, we say that the *vector field* $\boldsymbol{v}(\boldsymbol{x}, t)$ is *convected by the flow* $\boldsymbol{u}$.

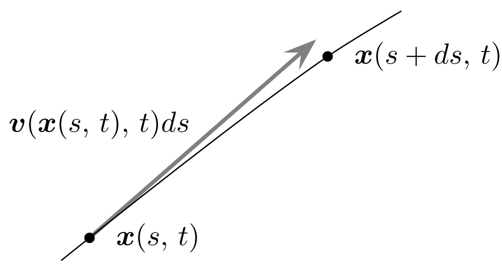

Figure 1.8.

We want to derive an evolution PDE for $\boldsymbol{v}(\boldsymbol{x}, t)$.

a) First, a little geometric preamble: Let $\boldsymbol{x}(s, t)$ be a parametric representation of a material curve in a flow $\boldsymbol{u}(\boldsymbol{x}, t)$, such that s labels material points. Show that the tangent vector $\boldsymbol{\tau}(s, t) := \boldsymbol{x}_s(s, t)$ satisfies

(1.13-1) $$\boldsymbol{\tau}_t = (\boldsymbol{\tau} \cdot \boldsymbol{\nabla})\boldsymbol{u}.$$

b) Show that

(1.13-2) $$\boldsymbol{v}_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u}.$$

Solution.

a) Since fixed values of s label material points, $\boldsymbol{x}(s, t)$ satisfies

$$\boldsymbol{x}_t(s, t) = \boldsymbol{u}(\boldsymbol{x}(s, t), t),$$

or, in component notation,

$$\partial_t x_i = u_i(\boldsymbol{x}(s, t), t).$$

Differentiation with respect to s gives

$$\partial_t(\partial_s x_i)(s, t) = (\partial_s x_j(s, t))\partial_j u_i(\boldsymbol{x}(s, t), t),$$

or

$$\partial_t \tau_i(s, t) = ((\tau_j(s, t)\partial_j)u_i)(\boldsymbol{x}(s, t), t)$$

with summation over the repeated index j. In vector notation, $\boldsymbol{\tau}_t = (\boldsymbol{\tau} \cdot \boldsymbol{\nabla})\boldsymbol{u}$.

b) Substitute $\boldsymbol{\tau}(s,t) = \boldsymbol{v}(\boldsymbol{x}(s,t),t)$ into (1.13-1) to get

$$\frac{D\boldsymbol{v}}{dt} = \boldsymbol{v}_t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u}.$$

Problem 1.14 (Vector analogs of a convected density). Recall that the volume integral of a *convected density* over any material region is conserved in time. There are two vector analogs in spatial $\mathbb{R}^3$. First, in electrically conducting fluid, the magnetic flux, or surface integral of the magnetic field over any material surface, is conserved in time. So, in this analog, the surface integral of a vector field takes on the role of mass, and the vector field, whose integrals over material surfaces are conserved, takes on the role of density. Second, we can consider vector fields whose *line* integral over any material curve is conserved. We begin with this second analog and an example of it.

a) The vector field $\boldsymbol{g}(\boldsymbol{x},t)$ in $\mathbb{R}^3$ has

$$\frac{d}{dt}\int_{C(t)} \boldsymbol{g}(\boldsymbol{x},t)\cdot d\boldsymbol{x} = 0 \tag{1.14-1}$$

for any material curve $C(t)$. Give a class of vector fields which automatically satisfy equation (1.14-1). Derive an evolution PDE for $\boldsymbol{g}(\boldsymbol{x},t)$, first in *coordinate* notation. Next, convert your equations to coordinate-free vector notation. This makes it easier for you to see that the equation of $\boldsymbol{g}$ is *not* the same as that of a convected vector, as in Problem 1.13.

b) Now for the first analog: Let $\boldsymbol{\omega}(\boldsymbol{x},t)$ be a vector field on $\mathbb{R}^3$ whose surface integral

$$\int_{S(t)} \boldsymbol{\omega}\cdot\boldsymbol{n}\, da$$

over any material surface $S(t)$ is conserved in time. Derive the evolution PDE for $\boldsymbol{\omega}(\boldsymbol{x},t)$. Here is an outline of an argument that cuts to the essence and avoids vector identities that everybody looks up but nobody remembers. First, we consider regions which are "bundles" of integral curves of $\boldsymbol{\omega}$, as depicted in Figure 1.9. Argue that if a material region is such a bundle at

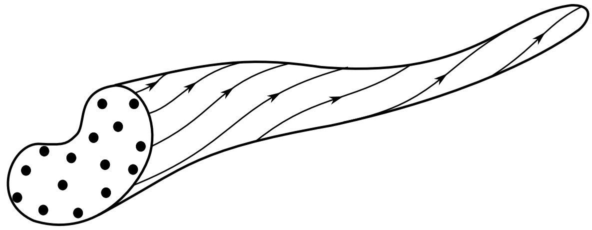

Figure 1.9.

one time, it is so for all times. Argue that integral curves of $\boldsymbol{\omega}$ are material curves. In this case, $\boldsymbol{\omega}$ is everywhere parallel to a convected vector $\boldsymbol{v}$ in the sense of Problem 1.13. You will have to come up with some idea for the local proportionality coefficient between $\boldsymbol{\omega}$ and $\boldsymbol{v}$.

Solution.

a) For any smooth function $f(\boldsymbol{x}, t)$,

$$\int_{C(t)} \boldsymbol{\nabla} f \cdot d\boldsymbol{x} = f(\boldsymbol{x}(1,t),t) - f(\boldsymbol{x}(0,t),t).$$

The right-hand side represents the difference of f values at the endpoints of $C(t)$. If $f(\boldsymbol{x}, t)$ is a convected scalar, the endpoint values are time-independent, so

$$\frac{d}{dt}\int_{C(t)} \boldsymbol{\nabla} f \cdot d\boldsymbol{x} = 0.$$

In summary, (1.14-1) is satisfied if $\boldsymbol{g}$ is a gradient of a convected scalar.

We derive the PDE for $\boldsymbol{g}(\boldsymbol{x}, t)$: Let $\boldsymbol{x}(s, t)$ be a parametric representation of $C(t)$ with s in $[0, 1]$ labeling material points. We rewrite (1.14-1) as

$$\begin{aligned} 0 &= \frac{d}{dt}\int_0^1 \boldsymbol{g}(\boldsymbol{x}(s,t),t)\cdot\boldsymbol{\tau}(s,t)\,ds \\ &= \int_0^1 \left\{\frac{D\boldsymbol{g}}{Dt}\cdot\boldsymbol{\tau} + \boldsymbol{g}\cdot((\boldsymbol{\tau}\cdot\boldsymbol{\nabla})\boldsymbol{u})\right\} ds. \end{aligned} \tag{1.14-2}$$

Here, $\boldsymbol{\tau} := \boldsymbol{x}_s(s, t)$ is the tangent vector of $C(t)$ and the second equality uses $\boldsymbol{\tau}_t = (\boldsymbol{\tau} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ as in (1.13-1). In component notation, (1.14-2) reads

$$\int_0^1 \left\{\frac{Dg_i}{Dt} + g_j \partial_i u_j\right\} \tau_i\, ds = 0,$$

and our equations for the g_i are

$$\frac{Dg_i}{Dt} = -(\partial_i u_j) g_j, \tag{1.14-3}$$

where there is summation over the repeated index j.

We write these equations in vector notation. First, rewrite (1.14-3) as

$$\frac{D}{Dt} g_i + g_j \partial_j u_i = -g_j(\partial_i u_j - \partial_j u_i). \tag{1.14-4}$$

The left-hand side of (1.14-4) is the ith component of the vector

$$\frac{D\boldsymbol{g}}{Dt} + (\boldsymbol{g}\cdot\boldsymbol{\nabla})\boldsymbol{u}.$$

It remains to determine what vector has components as in the right-hand side. We simplify by reorienting our cartesian coordinates so that $\boldsymbol{e}_1$ is parallel to $\boldsymbol{g}$, and so $\boldsymbol{g} = g_1 \boldsymbol{e}_1$. Then the right-hand side of (1.14-4) reduces to

$$-g_1(\partial_i u_1 - \partial_1 u_i),$$

and the vector with these components is

$$-g_1 \left\{ (\partial_2 u_1 - \partial_1 u_2)\boldsymbol{e}_2 + (\partial_3 u_1 - \partial_1 u_3)\boldsymbol{e}_3 \right\}. \tag{1.14-5}$$

We recognize components of the *curl* of $\boldsymbol{u}$,

$$\begin{aligned} \boldsymbol{\omega} := \boldsymbol{\nabla} \times \boldsymbol{u} &= w_1 \boldsymbol{e}_1 + w_2 \boldsymbol{e}_2 + w_3 \boldsymbol{e}_3 \\ &= (\partial_1 u_2 - \partial_2 u_1)\boldsymbol{e}_3 + (\partial_2 u_3 - \partial_3 u_2)\boldsymbol{e}_1 + (\partial_3 u_1 - \partial_1 u_3)\boldsymbol{e}_2. \end{aligned}$$

(Memory aid: Write out the term $(\partial_1 u_2 - \partial_2 u_1)\boldsymbol{e}_3$ and then the other two are obtained by cyclic permutation of integers $1, 2, 3$.) Hence, (1.14-5) becomes

$$\begin{aligned} -g_1 \left\{ -w_3 \boldsymbol{e}_2 + w_2 \boldsymbol{e}_3 \right\} &= -g_1 \left\{ -w_3 \boldsymbol{e}_3 \times \boldsymbol{e}_1 + w_2 \boldsymbol{e}_1 \times \boldsymbol{e}_2 \right\} \\ &= -g_1 \boldsymbol{e}_1 \times \left\{ w_2 \boldsymbol{e}_2 + w_3 \boldsymbol{e}_3 \right\} \\ &= -g_1 \boldsymbol{e}_1 \times \left\{ w_1 \boldsymbol{e}_1 + w_2 \boldsymbol{e}_2 + w_3 \boldsymbol{e}_3 \right\} \\ &= -\boldsymbol{g} \times \boldsymbol{\omega}. \end{aligned}$$

(Another memory aid: $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$ and, by cyclic permutation, $\boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1$ and $\boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2$. Also, there is antisymmetry, $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$, and $\boldsymbol{e}_1 \times \boldsymbol{e}_1 = 0$ as a special case.) In summary, the vector form of (1.14-4) is

$$\frac{D\boldsymbol{g}}{Dt} = -(\boldsymbol{g} \cdot \boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{g} \times \boldsymbol{\omega}. \tag{1.14-6}$$

This is *not* the convected vector equation

$$\frac{D\boldsymbol{v}}{Dt} = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u}. \tag{1.14-7}$$

Why is (1.14-6) different from (1.14-7)? In elementary vector analysis, we are used to calling the gradient of a function "a vector field". But geometers (rightly) beg to differ: In its essence, the gradient is really a *linear function*, telling you the *change* in a scalar field when you make a small displacement. For geometers, the prototypical vector is *spatial displacement*, a different object, really. Take our streaks in cookie dough: When we stretch the dough in the direction parallel to the streaks, they *elongate*. That is why $+(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ appears in the right-hand side of (1.14-7). On the other hand, suppose the sugar concentration is increasing in the direction of our streaks, so that the gradient "vector" is parallel to the streaks. When we stretch the dough, the gradient *decreases* and hence $-(\boldsymbol{g} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ appears in (1.14-6).

b) Consider a material region $R(t)$ which at $t = 0$ is a bundle of integral curves of $\boldsymbol{\omega}(\boldsymbol{x}, 0)$, as depicted in Figure 1.9. The surface integral $\int_{S(t)} \boldsymbol{\omega} \cdot \boldsymbol{n}\, da$ through any material patch $S(t)$ on the walls is constant in time. At time zero, we have $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$ on the walls, so $\int_{S(0)} \boldsymbol{\omega} \cdot \boldsymbol{n}\, da = 0$. Hence, $\int_{S(t)} \boldsymbol{\omega} \cdot \boldsymbol{n}\, da = 0$ for all material patches on the walls, and $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$ on the walls for all time. The surface of $R(t)$ is made up of integral curves of $\boldsymbol{\omega}(\boldsymbol{x},t)$, and the interior of $R(t)$ is filled with integral curves which don't break the surface. In summary, $R(t)$ is a bundle of integral curves for all time. We also note that the surface integral through material cross-sections, generally nonzero, is constant in time (but not necessarily the same for different cross-sections, unless we further assume $\boldsymbol{\nabla}\cdot\boldsymbol{\omega}=0$). By considering the limit of such a bundle shrinking down to a material curve, we see that integral curves of $\boldsymbol{\omega}$ are also material curves.

Let $\boldsymbol{v}(\boldsymbol{x},t)$ be the convected vector field, satisfying (1.14-7), whose initial values at $t=0$ are $\boldsymbol{\omega}(\boldsymbol{x},0)$. Integral curves of $\boldsymbol{v}$ are material curves, and since they coincide with the integral curves of $\boldsymbol{\omega}$ at $t=0$, they are the same as the integral curves of $\boldsymbol{\omega}$. Let $\boldsymbol{x}(s,t)$ be a parametric representation of one of these integral curves $C(t)$, with s labeling material points. We can choose the labeling s so that $\boldsymbol{v}(\boldsymbol{x}(s,0),0)=\boldsymbol{\tau}(s,0):=\boldsymbol{x}_s(s,0)$. By the definition of a convected vector as in Problem 1.13,

$$\boldsymbol{v}(\boldsymbol{x}(s,t),t)=\boldsymbol{\tau}(s,t) \tag{1.14-8}$$

for all t. We also have

$$\boldsymbol{\omega}(\boldsymbol{x}(s,t),t)=\lambda(s,t)\boldsymbol{v}(\boldsymbol{x}(s,t),t). \tag{1.14-9}$$

The local coefficient $\lambda(s,t)$ remains to be determined. Assuming for now that it is given, time differentiation of (1.14-9) and use of (1.13-2) gives

$$\begin{aligned}\frac{D\boldsymbol{\omega}}{Dt}&=\lambda_t\boldsymbol{v}+\lambda\frac{D\boldsymbol{v}}{Dt}\\&=\lambda_t\boldsymbol{v}+\lambda(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u}\\&=\frac{\lambda_t}{\lambda}\boldsymbol{\omega}+(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}.\end{aligned} \tag{1.14-10}$$

Now we deal with λ: Consider a "thin" bundle of integral curves of $\boldsymbol{\omega}$ about $C(t)$ and, in particular, a short material section of it, depicted in Figure 1.10. The face of this oblique cylinder through $\boldsymbol{x}(s,t)$ has area da and unit normal $\boldsymbol{n}$. The geometric parameters da, $\boldsymbol{n}$, and $\boldsymbol{\tau}(s,t)$ (the displacement from $\boldsymbol{x}(x,t)$ to $\boldsymbol{x}(s+ds,t)$ is $\boldsymbol{\tau}(s,t)\,ds$) can vary in time, subject to a constraint: The volume of the material cylinder is $(\boldsymbol{\tau}\,ds)\cdot(\boldsymbol{n}\,da)=(\boldsymbol{\tau}\cdot\boldsymbol{n}\,da)\,ds$, and its time rate of change satisfies (1.6), that is,

$$(\boldsymbol{\tau}\cdot\boldsymbol{n}\,da)_t=(\boldsymbol{\nabla}\cdot\boldsymbol{u})(\boldsymbol{\tau}\cdot\boldsymbol{n}\,da), \tag{1.14-11}$$

where $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ is evaluated at $\boldsymbol{x}=\boldsymbol{x}(s,t)$ and we have dropped the ds, which is constant in time.

Next, the surface integral of $\boldsymbol{\omega}$ over the cross-section of the bundle through $\boldsymbol{x}(s,t)$, namely

$$\boldsymbol{\omega}(\boldsymbol{x}(s,t),t)\cdot\boldsymbol{n}\,da,$$

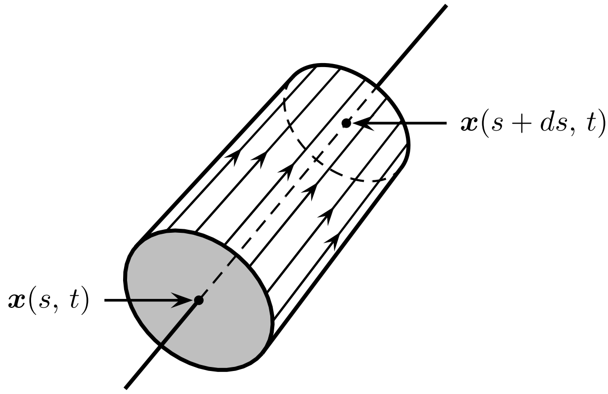

Figure 1.10.

is a constant independent of time. By (1.14-8) and (1.14-9),

$$\boldsymbol{\omega}(\boldsymbol{x}(s,t),t) = \lambda(s,t)\boldsymbol{\tau}(s,t),$$

so $\lambda\boldsymbol{\tau}\cdot\boldsymbol{n}\,da$ is a constant independent of time. Hence,

$$\lambda_t\boldsymbol{\tau}\cdot\boldsymbol{n}\,da + \lambda(\boldsymbol{\tau}\cdot\boldsymbol{n}\,da)_t = 0, \tag{1.14-12}$$

and it follows from (1.14-11) and (1.14-12) that

$$\frac{\lambda_t}{\lambda} = -\boldsymbol{\nabla}\cdot\boldsymbol{u}.$$

Substituting this result for $\frac{\lambda_t}{\lambda}$ into (1.14-10), we obtain the evolution PDE for $\boldsymbol{\omega}$,

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{\omega}. \tag{1.14-13}$$

Geometric postscript. In the geometry world, densities are called *forms*. In $\mathbb{R}^3$, an ordinary density as the amount of "stuff" per unit volume is called a *three-form*. A "vector field" which determines values of surface integrals over two-dimensional surfaces is a *two-form*, and a "vector field" which determines line integrals over one-dimensional curves is a *one-form*. As such, their physical units contain $1 \div (\text{length})^3$, $1 \div (\text{length})^2$ and $1 \div$ length, respectively. In particular, the $1 \div$ length unit of a one-form is a dead giveaway that it is not the same thing as a displacement with the unit of length. It is a coincidence that one- and two-forms in $\mathbb{R}^3$ are specified by three real numbers (their "components") so that in many "everyday" uses, one- and two-forms are "mapped" into vectors.

Guide to bibliography. Recommended references for this chapter are Batchelor [**2**], Chorin & Mardsen [**4**], Landau & Lifshitz [**17**], and Ockendon & Ockendon [**20**].

In all these references, analysis of the convection PDE is embedded in the bigger picture of fluid mechanics. Ockendon & Ockendon [**20**] base their (clear, elegant) derivations on moving material regions and the transport theorem. Chorin & Mardsen [**4**] also present the transport theorem. Batchelor's [**2**] discussion of transport into or from a material volume is based on heuristic vector analysis, but the reader needs to be self-sufficient in supplying the visual insight (such as Figure 1.2). Batchelor [**2**] emphasizes the local kinematics of a fluid about a material point, and he likes convective derivatives. Landau & Lifshitz [**17**] introduce and use the term "flux" as we do. This makes sense, because many of their derivations are based on fixed control volumes rather than moving material volumes.

Chapter 7

Free surface waves

The nominal subject of this chapter is the surface waves on oceans and rivers. Their phenomena are close to "everyday experience" and very rich. Their intrinsic interest granted, we come to the real purpose: of "descending" from a fully inclusive model down through a sequence of scaling-based reductions. The reduced models are fundamental sources of intuition in continuum mechanics.

Basic equations. Incompressible fluid of uniform density ρ fills an ocean. In cartesian coordinates (x_1, x_2, x_3), $x_3 = 0$ represents a solid bottom. The upper free surface is represented by

$$x_3 = h(x_1, x_2, t). \tag{7.1}$$

The velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and pressure field $p(\boldsymbol{x}, t)$ in $0 < x_3 < h(x_1, x_2, t)$ satisfy incompressible flow equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{7.2}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\nabla p - g\boldsymbol{e}_3. \tag{7.3}$$

In (7.3), $-g\boldsymbol{e}_3$ is the force per unit mass due to gravity. There are boundary conditions. The flow is tangential on the bottom, so

$$u_3 = 0 \quad \text{on } x_3 = 0. \tag{7.4}$$

On the free surface there is a so-called *kinematic* boundary condition: Let $\boldsymbol{x}(t)$ be a material point on the surface. The continuity of the flow map guarantees that material points initially on the surface remain on the surface

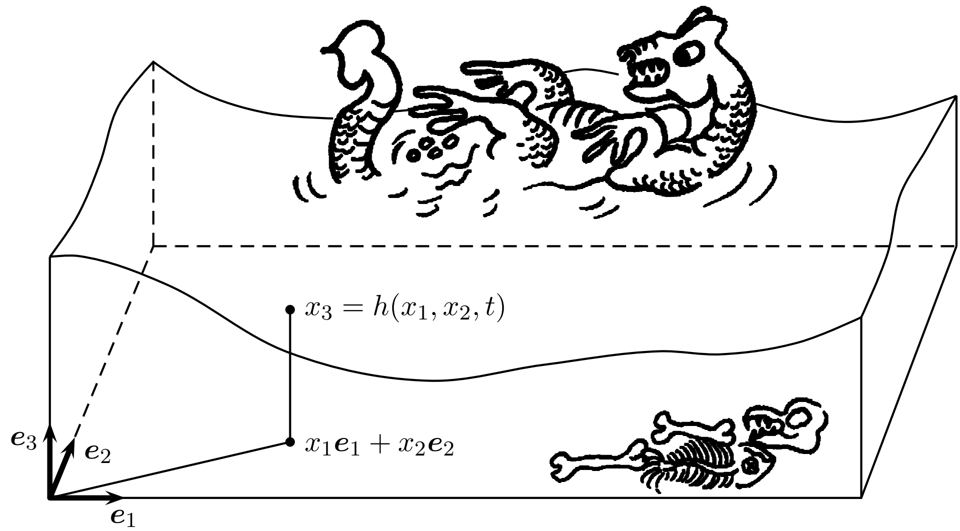

Figure 7.1.

for all time. Hence,

$$x_3(t) = h(x_1(t), x_2(t), t) \tag{7.5}$$

for all t. Differentiating (7.5) with respect to t and setting $\dot{x}_i(t) = u_i(\boldsymbol{x}(t), t)$ gives a relation

$$u_3 = h_t + u_1\partial_1 h + u_2\partial_2 h, \tag{7.6}$$

which holds at each $\boldsymbol{x}(t)$ on the surface. Hence, (7.6) holds on $x_3 = h(x_1, x_2, t)$ for all t, and this is the kinematic boundary condition. There is also a *dynamical* boundary condition. Assume that the medium above the ocean has uniform pressure, which without loss of generality we can take to be zero. If the pressure just below the surface is nonzero, an infinitesimal layer of fluid experiences a finite force per unit area and we get an infinite acceleration of the surface. This doesn't happen. Hence, the pressure field satisfies the dynamical boundary condition

$$p(x_1, x_2, h(x_1, x_2, t), t) = 0 \tag{7.7}$$

for all x_1, x_2 and t. The PDEs (7.2) and (7.3) together with the boundary conditions (7.4), (7.6) and (7.7) govern the evolution of the surface elevation h, velocity $\boldsymbol{u}$, and pressure p in $0 < x_3 < h(x_1, x_2, t)$.

It is convenient to distinguish the horizontal and vertical dimensions of space in these equations. Write the position vector in $\mathbb{R}^3$ as $\boldsymbol{x} + x_3\boldsymbol{e}_3$, where $\boldsymbol{x}$ now denotes horizontal displacement, $\boldsymbol{x} := x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$. Similarly, write the velocity field as $\boldsymbol{u} + u_3\boldsymbol{e}_3$, where $\boldsymbol{u} := u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2$. The gradient operator is represented as $\boldsymbol{\nabla} + \partial_3 \boldsymbol{e}_3$, where $\boldsymbol{\nabla} := \partial_1 \boldsymbol{e}_1 + \partial_2 \boldsymbol{e}_2$. The PDEs (7.2) and (7.3) are reformulated as

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} + \partial_3 u_3 = 0, \tag{7.8}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla} + u_3 \partial_3)\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p, \tag{7.9}$$

$$\partial_t u_3 + (\boldsymbol{u} \cdot \boldsymbol{\nabla} + u_3 \partial_3) u_3 = -\frac{1}{\rho}\partial_3 p - g \tag{7.10}$$

in $0 < x_3 < h(\boldsymbol{x}, t)$, and the boundary conditions (7.4), (7.6) and (7.7) become

$$u_3 = 0 \tag{7.11}$$

on $x_3 = 0$ and

$$h_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} h = u_3, \tag{7.12}$$

$$p = 0 \tag{7.13}$$

on $x_3 = h(\boldsymbol{x}, t)$.

Linearized waves. The simplest solutions of (7.8)–(7.13) are rest states with $h \equiv H$, a uniform constant, $\boldsymbol{u} \equiv 0$, and $p = \rho g(H - x_3)$. The physics of the pressure field is simple. The pressure on the surface $x_3 = H$ is zero, and the pressure at x_3, $0 < x_3 < H$, is what is required to support the weight of the water above the given x_3. *Perturbations* from the rest state are represented by

$$\begin{aligned} \varepsilon h' &:= h - H, \\ \varepsilon \boldsymbol{u}' &:= \boldsymbol{u}, \quad \varepsilon u_3' := u_3, \\ \varepsilon p' &:= p - \rho g(H - x_3). \end{aligned} \tag{7.14}$$

Here $\varepsilon > 0$ is a gauge parameter, and we consider the limit $\varepsilon \to 0$ with the primed variables fixed (so that perturbations of the dependent variables from their rest values are $O(\varepsilon)$). Writing the PDEs (7.8)–(7.10) in terms of primed variables and taking the limit $\varepsilon \to 0$ gives

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{u}' + \partial_3 u_3' &= 0, \\ \boldsymbol{u}_t' &= -\frac{1}{\rho}\boldsymbol{\nabla} p', \\ \partial_t u_3 &= -\frac{1}{\rho}\partial_3 p' \end{aligned} \tag{7.15}$$

in the $\varepsilon \to 0$ *limit region*, $0 < x_3 < H$. The boundary condition on the bottom is $u_3 = 0$ on $x_3 = 0$ as in (7.11). The $\varepsilon \to 0$ limits of the surface boundary conditions (7.12) and (7.13) require some care. The kinematic boundary condition (7.12) in terms of primed variables is

$$(\varepsilon h'_t + \varepsilon^2 \boldsymbol{u}' \cdot \boldsymbol{\nabla} h')(\boldsymbol{x}, H + \varepsilon h', t) = \varepsilon u_3(\boldsymbol{x}, H + \varepsilon h', t).$$

Taking the limit $\varepsilon \to 0$ gives

$$h'_t = u'_3 \quad \text{on } x_3 = H. \tag{7.16}$$

The dynamical boundary condition (7.13) translates into

$$\rho g(H - (H + \varepsilon h'(\boldsymbol{x}, H + \varepsilon h', t))) + \varepsilon p'(\boldsymbol{x}, H + \varepsilon h', t) = 0,$$

and the $\varepsilon \to 0$ limit is

$$p' = \rho g h' \quad \text{on } x_3 = H. \tag{7.17}$$

Notice that the linearized free surface boundary conditions apply on the *unperturbed* free surface $x_3 = H$. In particular, notice that p' is *not* zero on $x_3 = H$. If $h' > 0$, then $\varepsilon p' = \varepsilon \rho g h'$ represents the pressure field on $x_3 = H$ due to the weight of the water in $0 < x_3 < \varepsilon h'$. If $h' < 0$, then $x_3 = H$ is *above* the free surface, and $\varepsilon p' = \varepsilon \rho g h'$ represents a linear extrapolation of the pressure field to $x_3 = H$.

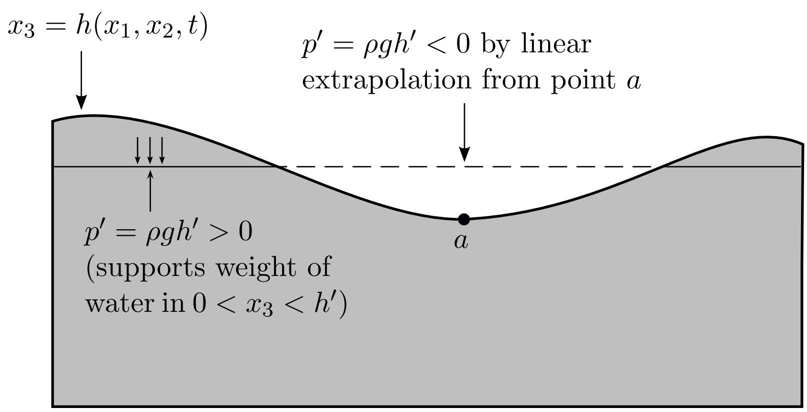

Figure 7.2.

Plane waves and the dispersion relation. The linearized equations (7.15)–(7.17) have plane wave solutions with

$$\begin{aligned} h' &= h^0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \\ \boldsymbol{u}' &= \boldsymbol{u}^0(x_3) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \\ u'_3 &= u_3^0(x_3) e^{\boldsymbol{k}\cdot\boldsymbol{x} - \omega t}, \\ p' &= p^0(x_3) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}. \end{aligned} \tag{7.18}$$

Since $\partial_t e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = -i\omega e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ and $\boldsymbol{\nabla} e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = i\boldsymbol{k}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, we obtain reduced equations for h^0, $\boldsymbol{u}^0(x_3)$, $u_3^0(x_3)$ and $p^0(x_3)$ via the replacements $\boldsymbol{\nabla} \to i\boldsymbol{k}$ and $\partial_t \to -i\omega$ in the linearized equations (7.15)–(7.17). The reduced equations are

$$i\boldsymbol{k}\cdot\boldsymbol{u}^0 + \partial_3 u_3^0 = 0,$$

$$-i\omega\boldsymbol{u}^0 = -\frac{1}{\rho} i\boldsymbol{k}p^0, \tag{7.19}$$

$$-i\omega u_3^0 = -\frac{1}{\rho}\partial_3 p^0 \tag{7.20}$$

in $0 < x_3 < H$, with boundary conditions

$$u_3^0 = 0$$

on $x_3 = 0$ and

$$\begin{aligned} u_3^0 &= -i\omega h^0, \\ p^0 &= \rho g h^0 \end{aligned} \tag{7.21}$$

on $x_3 = H$. Assuming $\omega \neq 0$, we can eliminate $\boldsymbol{u}^0$ and u_3^0 to extract a boundary value problem for $p^0(x_3)$ which contains h^0 as a parameter:

$$\begin{aligned} \partial_{33}p^0 - (\boldsymbol{k}\cdot\boldsymbol{k})p^0 = 0 \quad &\text{in } 0 < x_3 < H, \\ \partial_3 p^0 = 0 \quad &\text{on } x_3 = 0, \\ p^0 = \rho g h^0 \quad &\text{on } x_3 = H. \end{aligned}$$

The solution for $p^0(x_3)$ is

$$p^0(x_3) = \rho g h_0 \frac{\cosh kx_3}{\cosh kH}. \tag{7.22}$$

Here, $k := \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$. It follows from (7.20) and (7.21) that

$$\frac{1}{\rho}\partial_3 p^0 = \omega^2 h^0$$

on $x_3 = H$, and upon substituting (7.22) for $p^0(x_3)$, we get the *dispersion relation*

$$\omega^2 = gk \tanh kH, \tag{7.23}$$

which gives two "branches" of ω as functions of k. For the branch with $\omega > 0$, the phase speed $v := \frac{\omega}{k}$ as a function of k is

$$v = \sqrt{gH\frac{\tanh kH}{kH}}. \tag{7.24}$$

Figure 7.3 is a graph of the phase speed as a function of k. In the *long wave limit* $kH \to 0$, v asymptotes to the uniform constant $\sqrt{gH}$. In the deep water limit $kH \to \infty$, v asymptotes to $\sqrt{g/k}$, represented by the dashed line in Figure 7.3.

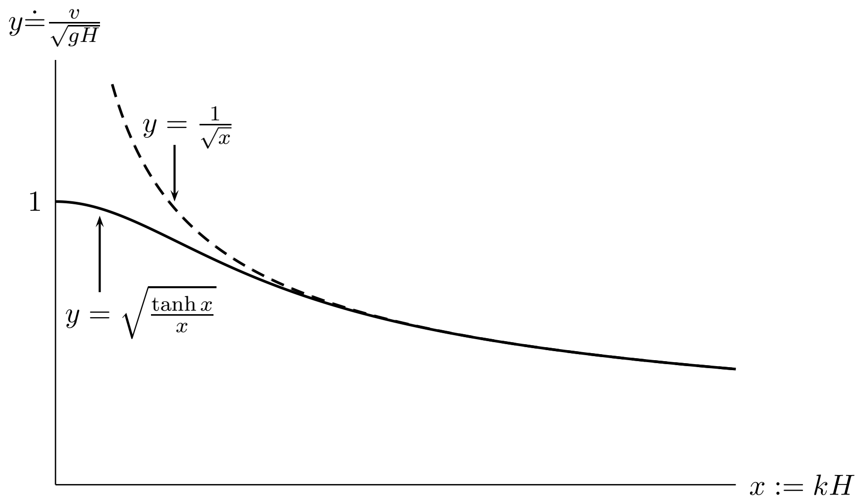

Figure 7.3.

We look at the particle kinematics of linearized plane waves. Assume that the wave propagates in the $\boldsymbol{e}_1$ direction, with $\boldsymbol{k} = k\boldsymbol{e}_1$. From (7.19), (7.20) and (7.22) we find that the $\mathbb{R}^3$ velocity field is

$$\begin{aligned} \boldsymbol{u}' &= \mathrm{Re}\{(\boldsymbol{u}^0 + u_3^0\boldsymbol{e}_3)e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\} \\ &= \frac{1}{\omega\rho}\mathrm{Re}\{(kp^0\boldsymbol{e}_1 - i\partial_3 p^0\boldsymbol{e}_3)e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\} \\ &= \frac{g}{\omega}\frac{h^0}{H}\frac{kH}{\cosh kH}\{\cosh kx_3\cos(kx_1-\omega t)\boldsymbol{e}_1 \\ &\qquad + \sinh kx_3 \sin(kx_1-\omega t)\boldsymbol{e}_3\}. \end{aligned} \tag{7.25}$$

This velocity field translates with the wave. Figure 7.4a is a visualization of $\boldsymbol{u}$ at a fixed moment in time.

Particle trajectories $\boldsymbol{x}(t)$ are solutions of the ODE $\dot{\boldsymbol{x}}(t) = \varepsilon\boldsymbol{u}'(\boldsymbol{x}(t), t)$. In the limit $\varepsilon \to 0$, with t fixed, the variation $\delta\boldsymbol{x}(t)$ of $\boldsymbol{x}(t)$ is $O(\varepsilon)$. Hence, in the limit $\varepsilon \to 0$, an approximation to the variation is determined by integrating $\varepsilon\boldsymbol{u}'(\boldsymbol{x}, t)$ in t with $\boldsymbol{x}$ fixed. Thus,

$$\delta\boldsymbol{x}(t) = \frac{g}{\omega^2}\frac{\varepsilon h^0}{H}\frac{kH}{\cosh kH}\{\cosh kx_3\sin(\omega t - kx_1)\boldsymbol{e}_1 + \sinh kx_3\cos(\omega t - kx_1)\boldsymbol{e}_3\}.$$

The prefactor can be simplified by substituting for ω^2 from the dispersion relation (7.23). Upon doing this we obtain

$$\delta\boldsymbol{x}(t) = \varepsilon h^0\left\{\frac{\cosh kx_3}{\sinh kH}\sin(\omega t - kx_1)\boldsymbol{e}_1 + \frac{\sinh kx_3}{\sinh kH}\cos(\omega t - kx_1)\boldsymbol{e}_3\right\}. \tag{7.26}$$

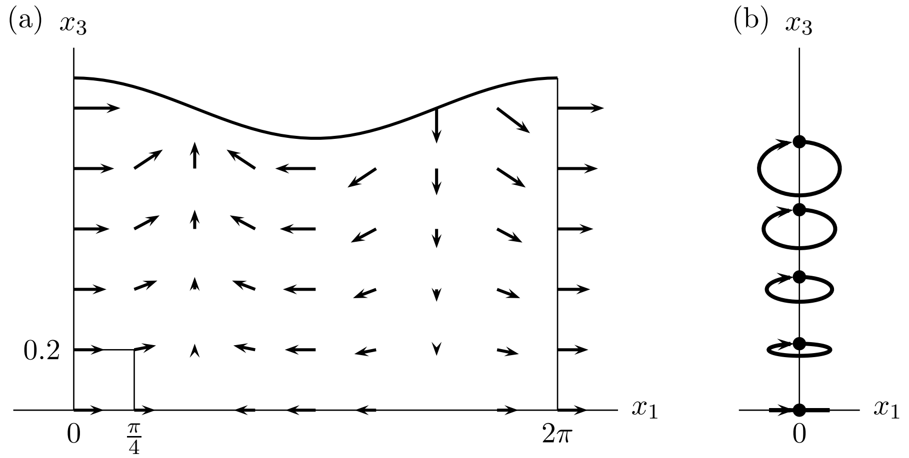

Figure 7.4.

Note that at the surface ($x_3 = H$), the vertical component of displacement reduces to $\varepsilon h^0 \cos(\omega t - kx_1)$, as it should. The trajectories described by (7.26) are clockwise ellipses. Figure 7.4b depicts the ellipses corresponding to particles which started from the x_3-axis at $t = 0$. With a little imagination, you can see that these clockwise ellipses are consistent with the velocity field in Figure 7.4a translating to the right.

Problem 7.1 (Internal waves, again). In Problem 6.11 we analyzed stationary waves in stably stratified fluid. We eventually specialized to waves on the interface between dense fluid below and light fluid above. The analysis was based on the asymptotic limit of small density difference. Here, we re-examine these stationary interfacial waves as a free boundary problem, but this time with *no* assumption on the smallness of the density difference.

a) Formulate the free boundary problem for two-dimensional, time-independent, incompressible ideal fluid flow, with density $\rho = \rho_-$, a positive constant, in $0 < x_2 < h(x_1)$, and density $\rho = \rho_+$ in $x_2 > h(x_1)$, where ρ_+ is another positive constant with $\rho_+ < \rho_-$. There is a gravity force of $-g\boldsymbol{e}_2$ per unit mass. The plane $x_2 = 0$ is a solid bottom.

Linearize about the uniform flow solution $\boldsymbol{u} \equiv U\boldsymbol{e}_1$ in $x_2 > 0$ and $h(x_1) \equiv H$, a positive constant. Derive a "reduced" solution of the linearized free boundary problem containing only the interface and pressure perturbations.

b) Analyze the reduced free boundary problem in Part a for stationary wave solutions whose x_1-dependence is the exponential e^{ikx_1}. Determine the relation between the unperturbed flow velocity U and the wavenumber k. Compare this to the result in Problem 6.11b. Assuming we have U such that k is real, construct approximate streamlines of the flow and graph them in the (x_1, x_2) plane.

Solution.

a) The PDEs in $0 < x_2 < h(x_1)$ and in $x_2 > h(x_1)$ are

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -g\boldsymbol{e}_2 - \frac{1}{\rho}\boldsymbol{\nabla} p,$$

where $\rho \equiv \rho_-$ in $0 < x_2 < h(x_1)$ and $\rho \equiv \rho_+$ in $x_2 > h(x_1)$. The "bottom" boundary condition is

$$u_2 = 0$$

on $x_2 = 0$. The kinematic boundary conditions on the interface are

$$u_2(x_1, h(x_1)^+) = h'(x_1)u_1(x_1, h(x_1)^+),$$
$$u_2(x_1, h(x_1)^-) = h'(x_1)u_1(x_1, h(x_1)^-).$$

In addition, the pressure is continuous across the interface, so

$$p(x_1, h(x_1)^+) = p(x_1, h(x_1)^-).$$

The pressure field associated with the uniform flow solution is

$$p = p(x_2) := \left[\begin{array}{ll} -g\rho_-(x_2 - H), & 0 < x_2 < H, \\ -g\rho_+(x_2 - H), & \quad x_2 > H. \end{array} \right.$$

Next, define perturbation quantities

$$\varepsilon h' := h - H,$$
$$\varepsilon \boldsymbol{u}' := \boldsymbol{u} - U\boldsymbol{e}_1,$$
$$\varepsilon p' := p - p(x_2).$$

Here, ε is a positive gauge parameter, and we will consider the limit $\varepsilon \to 0$. Writing the PDE in terms of perturbation variables and taking $\varepsilon \to 0$ gives

$$\partial_1 u_1' + \partial_2 u_2' = 0',$$
$$U\partial_1 u_1' = -\frac{1}{\rho}\partial_1 p', \qquad (7.1\text{-}1)$$
$$U\partial_1 u_2' = -\frac{1}{\rho}\partial_2 p'.$$

Since the interface perturbation from $h = H$ vanishes as $\varepsilon \to 0$, the density ρ in (7.1-1) is ρ_- in $0 < x_2 < H$ and ρ_+ in $x_2 > H$. The bottom boundary condition translates into

$$u_2'(x_1, 0) = 0, \tag{7.1-2}$$

and the kinematic boundary conditions in terms of perturbation variables reduce to

$$u_2(x_1, H^+) = u_2(x_1, H^-) = U\partial_1 h'(x) \tag{7.1-3}$$

in the limit $\varepsilon \to 0$. The continuity of pressure across the interface is expressed exactly by

$$-\varepsilon g\rho_- h'(x_1) + \varepsilon p'(x_1, (H + \varepsilon h'(x_1))^-) = -\varepsilon g\rho_+ h'(x_1) + \varepsilon p'(x_1, (H + \varepsilon h'(x_1))^+).$$

In the limit $\varepsilon \to 0$ this reduces to

$$p'(x_1, H^+) - p'(x_1, H^-) = g(\rho_+ - \rho_-)h'(x_1). \tag{7.1-4}$$

The full set of linearized equations consists of the PDEs (7.1-1) and the boundary conditions (7.1-2)–(7.1-4).

We derive reduced equations for the pressure and interface perturbations p' and h': First, from (7.1-1) it follows that p' is harmonic, i.e.,

$$\Delta p' = 0 \tag{7.1-5}$$

in $0 < x_2 < H$ and in $x_2 > H$. From the bottom boundary condition (7.1-2) and the $\boldsymbol{e}_2$-component of the linearized Euler equation (the last equation of (7.1-1)), we find that

$$\partial_2 p'(x, 0) = 0. \tag{7.1-6}$$

From the kinematic boundary conditions (7.1-3) and the $\boldsymbol{e}_2$-component of the linearized Euler equation, we further have

$$U^2 \partial_{11} h(x_1) = -\frac{1}{\rho_-}\partial_2 p'(x_1, H^-) = -\frac{1}{\rho_+}\partial_2 p'(x_1, H^+). \tag{7.1-7}$$

In summary, the harmonic pressure perturbation is subject to the "bottom" boundary condition (7.1-6) and *three* interface conditions: the two in (7.1-7), and (7.1-4). We will soon see that we have exactly the right number of boundary conditions.

b) We seek linearized stationary waves with

$$h'(x_1) = h_0 e^{ikx_1}, \tag{7.1-8}$$

$$p'(x_1, x_2) = p_0(x_2) e^{ikx_1}. \tag{7.1-9}$$

Substituting (7.1-9) into Laplace's equation gives an ODE for $p_0(x_2)$,

$$p_0''(x_2) - k^2 p_0(x_2) = 0$$

in $0 < x_2 < H$ and in $x_2 > H$. Solutions which satisfy the bottom boundary condition (7.1-6) and don't blow up as $x_2 \to +\infty$ are

$$p_0(x_2) = \begin{cases} p_- \dfrac{\cosh kx_2}{\cosh kH}, & 0 < x_2 < H, \\ p_+ e^{-k(x_2-H)}, & x_2 > H, \end{cases} \tag{7.1-10}$$

where p_-, p_+ are constants. Notice "details of craftsmanship" such as dividing by $\cosh kH$ in $0 < x_2 < H$ and use of the argument $x_2 - H$ in $x_2 > H$. Substituting (7.1-10) into the three interface conditions in (7.1-4) and (7.1-7) gives three homogeneous linear equations for the constants h_0, p_+, p_-:

$$\begin{aligned} p_1 - p_+ &= g(\rho_- - \rho_+)h_0, \\ kU^2 h_0 &= \frac{p_-}{\rho_-} \tanh kH, \\ kU h_0 &= -\frac{p_+}{\rho_+}. \end{aligned} \tag{7.1-11}$$

The consistency condition is

$$kH \frac{\rho_- \cosh kH + \rho_+ \sinh kH}{\bar{\rho} \sinh kH} = \frac{\rho_- - \rho_+}{\bar{\rho}} \frac{gH}{U^2}, \tag{7.1-12}$$

where $\bar{\rho} := \frac{\rho_+ + \rho_-}{2}$. This is the relation between the unperturbed flow speed U and the wavenumber k.

In the limit $\rho_+, \rho_- \to \rho_0$, it is easy to see that (7.1-12) reduces to its counterpart (6.11-10) from our first crack at internal waves. The qualitative features of the k versus U relation are very little changed. In particular, kH decreases to zero ("long waves") as U approaches the speed limit

$$U_{\max} := \sqrt{\frac{\rho_- - \rho_+}{\rho_-}} \sqrt{gH}$$

from below.

Streamlines $x_2 = x_2(x_1)$ are integral curves of the velocity field, and as such, they satisfy the ODE

$$\frac{dx_2}{dx_1} = \frac{\varepsilon u_2'}{U + \varepsilon u_1'} \sim \frac{\varepsilon}{U} u_2' \tag{7.1-13}$$

as $\varepsilon \to 0$. The u_2' in the right-hand side can be related to p', which is now known through (7.1-9) and (7.1-10). In the third equation of (7.1-1) ($\boldsymbol{e}_2$-component of the linearized Euler equation), set $p' = p_0(x_2)e^{ikx_1}$ and integrate with respect to x_1 to get

$$u_2' = -\frac{p_0'(x_2)}{\rho U} \frac{e^{ikx_1}}{ki}.$$

Substituting the *real* part of u_2' into (7.1-13) gives

(7.1-14) $$\frac{dx_2}{dx_1} \sim -\varepsilon \frac{p_0'(x_2)}{\rho U^2} \frac{\sin kx_1}{k},$$

with $p_0(x_2)$ as in (7.1-10). The asymptotic solutions of this ODE as $\varepsilon \to 0$ are

(7.1-15) $$x_2(x_1) \sim \bar{x}_2 + \varepsilon \frac{p_0'(\bar{x}_2)}{\rho U^2} \frac{\cos kx_1}{k^2},$$

where $\bar{x}_2$ is a constant. In $0 < \bar{x}_2 < H$, (7.1-15) reduces to

$$x_2(x_1) \sim \bar{x}_2 + \varepsilon \frac{p_-}{\rho_- U^2} \frac{\sinh k\bar{x}_2}{k \cosh kH} \cos kx_1.$$

By the second equation of (7.1-11) we can simplify the prefactor of the second term on the right-hand side, so $x_2(x_1) \sim \bar{x}_2 + \varepsilon h_0 \cos kx_1$. Similarly,

$$x_2(x_1) \sim \bar{x}_2 + \varepsilon h_0 e^{-k(x_2 - H)} \cos kx_1$$

in $\bar{x}_2 > H$. Figure 7.5 is the graph of the streamlines.

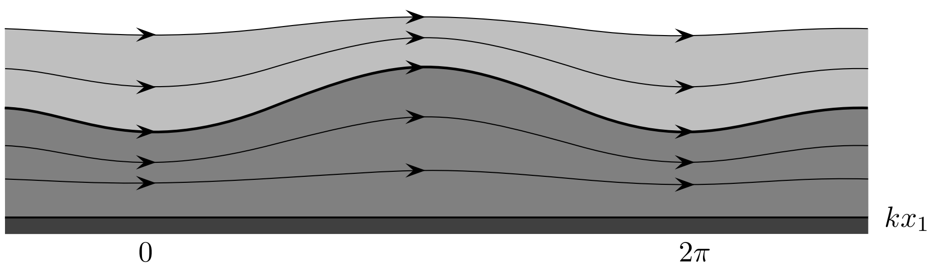

Figure 7.5.

Shallow water equations. In the long wave limit $kH \to 0$, the phase speed of the plane wave reduces to $\sqrt{gH}$, as noted before, and the velocity field (7.25) reduces to

$$\boldsymbol{u}' = \frac{g}{\omega} kh^0 \cos(kx_1 - \omega t)\boldsymbol{e}_1.$$

This velocity field is horizontal and independent of x_3. This feature of linearized long waves suggests a limit process that reduces the full free boundary problem to *shallow water* PDEs describing long but fully nonlinear waves.

Let H denote the depth of the ocean at rest, as before, and let L be the characteristic horizontal scale of the surface waves. The gauge parameter is $\varepsilon := \frac{H}{L}$ and the long wave limit is $\varepsilon \to 0$. In shallow water theory the characteristic variation of depth is not small compared to H as in the linearized theory, but has the same magnitude as H itself. Hence, the unit of horizontal displacement is L, and the unit of vertical displacement h is $H = \varepsilon L$, as in the scaling table below:

(7.27)

Variable	$\boldsymbol{x}$	x_3, h	t	p	$\boldsymbol{u}$	u_3
Unit	L	εL	$\sqrt{\frac{L}{\varepsilon g}}$	$\varepsilon \rho g L$	$\sqrt{\varepsilon g L}$	$\varepsilon\sqrt{\varepsilon g L}$

The unit of time is $[t] := \frac{L}{\sqrt{gH}} = \sqrt{\frac{L}{\varepsilon g}}$, the time it takes a linearized wave to travel distance L. Of course, nonlinear waves don't necessarily travel at the linearized speed, but the linearized wave speed does give the correct order of magnitude.

It remains to determine the units of the intensive variables p, $\boldsymbol{u}$ and u_3. The hydrostatic pressure at rest, $p = \rho g(H - x_3)$, indicates the correct order of magnitude for pressure, so the unit of pressure is $\rho g H = \varepsilon \rho g L$. The unit $[\boldsymbol{u}]$ of horizontal velocity follows by an order-of-magnitude balance between the horizontal acceleration of fluid particles and the horizontal pressure force per unit mass. That is,

$$\frac{[\boldsymbol{u}]}{[t]} = \frac{[\boldsymbol{u}]}{\sqrt{L/\varepsilon g}} = \frac{1}{\rho L}(\varepsilon \rho g L),$$

so $[\boldsymbol{u}] = \sqrt{\varepsilon g L}$, the same as the long wave speed $\sqrt{gH}$. Finally, $[u_3]$ should be the characteristic variation of depth divided by the unit of time, so

$$[u_3] = \frac{H}{[t]} = \frac{\varepsilon L}{\sqrt{L/\varepsilon g}} = \varepsilon\sqrt{\varepsilon g L}.$$

The above units of p, $\boldsymbol{u}$ and u_3 are entered in the scaling table (7.27).

The dimensionless versions of the basic equations (7.8)–(7.13) are

$$\nabla \cdot \boldsymbol{u} + \partial_3 u_3 = 0, \tag{7.28}$$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla + u_3 \partial_3)\boldsymbol{u} = -\nabla p, \tag{7.29}$$

$$\varepsilon^2\{\partial_t u_3 + (\boldsymbol{u} \cdot \nabla + u_3 \partial_3) u_3\} = -\partial_3 p - 1 \tag{7.30}$$

in $0 < x_3 < h(\boldsymbol{x}, t)$,

$$u_3 = 0 \tag{7.31}$$

on $x_3 = 0$, and

$$p = 0, \tag{7.32}$$

$$u_3 = \partial_t h + (\boldsymbol{u} \cdot \nabla) h \tag{7.33}$$

on $x_3 = h$. In the limit $\varepsilon \to 0$, only the vertical momentum equation (7.30) has an obvious reduction to

$$\partial_3 p = -1. \tag{7.34}$$

The remaining equations formally retain all their terms. But the stage is set for massive reduction. It follows from (7.34) and $p = 0$ on $x_3 = h$ that

$$p = h(\boldsymbol{x}, t) - x_3. \tag{7.35}$$

Hence, the hydrostatic approximation to pressure applies asymptotically in the shallow water dynamics. Substituting (7.35) into the horizontal momentum PDE (7.29) gives

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla} + u_3 \partial_3)\boldsymbol{u} = -\boldsymbol{\nabla} h. \tag{7.36}$$

The right-hand side $\boldsymbol{\nabla} h$ is independent of x_3. Hence, the horizontal acceleration of fluid particles is independent of x_3. Suppose a region of ocean is at rest at time $t = 0$, before waves arrive. Then fluid particles in any vertical line remain in a vertical line for all $t > 0$. This happens only if the horizontal velocity $\boldsymbol{u}$ is independent of x_3, i.e., $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$. More generally, if $\boldsymbol{u}$ is initially independent of x_3, it remains independent of x_3 for all time. Under these conditions, (7.36) reduces to

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} h. \tag{7.37}$$

Given $\boldsymbol{u}$ independent of x_3, it now follows from the incompressibility equation (7.28) and the boundary condition $u_3 = 0$ on $x_3 = 0$ that

$$u_3(\boldsymbol{x}, h, t) = -x_3 \boldsymbol{\nabla} \cdot \boldsymbol{u},$$

and substituting this result into the kinematic boundary condition (7.33) gives

$$-h\boldsymbol{\nabla} \cdot \boldsymbol{u} = \partial_t h + (\boldsymbol{u} \cdot \boldsymbol{\nabla})h,$$

or

$$h_t + \boldsymbol{\nabla} \cdot (h\boldsymbol{u}) = 0. \tag{7.38}$$

We recognize (7.37) and (7.38) as a pair of PDEs for the ocean's elevation $h(\boldsymbol{x}, t)$ and horizontal velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. These are the famous *shallow water equations*. Notice that they are formally equivalent to the two-dimensional ideal fluid PDEs (6.18) and (6.19) with h in the role of density and $\frac{1}{2}h^2$ in the role of pressure.

Problem 7.2 (Variable-depth ocean and "slow currents"). We consider an ocean whose depth at rest is a function of horizontal position $\boldsymbol{x}$. The characteristic length of $\boldsymbol{x}$-variations in the bottom topography is L, and the characteristic depth is $H \ll L$, so we have the small gauge parameter $\varepsilon := \frac{H}{L}$ (the same as before). The bottom is given by

$$\frac{x_3}{H} = -\ell\left(\frac{\boldsymbol{x}}{L}\right). \tag{7.2-1}$$

a) Derive modified shallow water equations which include the effect of the variable bottom topography.

b) We propose a simplified model of "slow currents" in our variable-depth ocean. The characteristic horizontal scale of the currents is L, the same as that of the bottom topography. The scaling unit $[u]$ of horizontal velocity is much smaller than the characteristic speed $\sqrt{gH}$ of surface waves. The characteristic time is $L/[u]$.

We could perform a "slow current" scaling reduction, starting from the full dimensional equations. But it is simpler to do a "secondary scaling" of the variable-depth shallow water equations from Part a. The formal limit process begins by assuming that the dimensionless horizontal velocity $\boldsymbol{u}$ has order of magnitude μ, where μ is a small gauge parameter (independent of $\varepsilon := \frac{H}{L}$). The brief physical description of the "slow current" limit in the preceding paragraph indicates what the collateral scaling of $\boldsymbol{x}$, t and h with μ are. Derive asymptotic $\mu \to 0$ equations of "slow current flow".

Solution.

a) The dimensionless PDEs for $\boldsymbol{u}$, u_3 and p are (7.28)–(7.30), the same as before, but now they apply in $-\ell(\boldsymbol{x}) < x_3 < h(\boldsymbol{x}, t)$. The free surface conditions (7.32) and (7.33) also apply with no change. The only change is the replacement of the bottom boundary condition (7.31) by

$$u_3 = -(\boldsymbol{u} \cdot \boldsymbol{\nabla})\ell \tag{7.2-2}$$

on $x_3 = -\ell(\boldsymbol{x})$. The argument leading to the "shallow water Euler equation" (7.37) is completely unchanged, so

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} h \tag{7.2-3}$$

as before. Where, then, does $\ell(\boldsymbol{x})$ appear? Given the new bottom boundary condition (7.2-2), x_3-integration of $\partial_3 u_3 = -\boldsymbol{\nabla} \cdot \boldsymbol{u}$ (from (7.28)) gives

$$u_3 = -(\boldsymbol{u} \cdot \boldsymbol{\nabla})\ell - (\boldsymbol{\nabla} \cdot \boldsymbol{u})(x_3 + \ell),$$

and substituting this result for u_3 into the kinematic boundary condition (7.33) on the free surface, we get

$$h_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})h = -(\boldsymbol{u} \cdot \boldsymbol{\nabla})\ell - (\boldsymbol{\nabla} \cdot \boldsymbol{u})(h + \ell),$$

which consolidates into

$$h_t + \boldsymbol{\nabla} \cdot \{(\ell + h)\boldsymbol{u}\} = 0. \tag{7.2-4}$$

The modified shallow water equations are (7.2-3) and (7.2-4). The only change is in replacing the volume flux $h\boldsymbol{u}$ in the volume conservation PDE (7.38) by $(\ell + h)\boldsymbol{u}$.

b) We are given μ as the scaling unit of $\boldsymbol{u}$, i.e., $[u] = \mu$. Since the characteristic horizontal scale of the currents is the same as that of the bottom topography, there is *no* scaling of $\boldsymbol{x}$ with μ. The characteristic time to make a unit horizontal displacement at speed $[u] = \mu$ is $[t] = \frac{1}{\mu}$. The scaling unit $[h]$ of h is determined by dominant balance in the rescaled version of the "shallow water Euler equation" (7.2-3). Following the nondimensionalization procedure, we have

$$\frac{[u]}{[t]}\boldsymbol{u}_t + [u]^2(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -[h]\boldsymbol{\nabla} h$$

or, using $[u] = \mu$ and $[t] = \frac{1}{\mu}$,

$$\mu^2\{\boldsymbol{u}_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\} = -[h]\boldsymbol{\nabla} h;$$

so the scaling unit of h is $[h] = \mu^2$, and thus (7.2-3) is *invariant* under our slow current rescaling. It is the volume conservation PDE (7.2-4) which simplifies. Applying nondimensionalization to (7.2-4) gives

$$\frac{[h]}{[t]}h_t + [u]\boldsymbol{\nabla} \cdot \{(\ell + [h]h)\boldsymbol{u}\} = 0,$$

or

$$\mu^2 h_t + \boldsymbol{\nabla} \cdot \{(\ell + \mu^2 h)\boldsymbol{u}\} = 0,$$

and in the limit $\mu \to 0$,

$$\boldsymbol{\nabla} \cdot (\ell\boldsymbol{u}) = 0. \tag{7.2-5}$$

In summary, the slow current equations are (7.2-3) and (7.2-5). The reduced volume conservation PDE (7.2-5) is what we should get when the departure of the free surface from "rest" ($x_3 \equiv 0$) can be neglected.

Problem 7.3 (What bottom topography does to vortices). We present a "vorticity formulation" of the slow current equations (7.2-3) and (7.2-5).

a) Let $\omega := \partial_1 u_2 - \partial_2 u_1$ be the vorticity associated with the two-dimensional flow $\boldsymbol{u}$. Show that $\frac{\omega}{\ell}$ is a convected scalar.

b) To determine a "vorticity evolution equation", which specifies ω_t as a functional of ω, it is sufficient to determine the velocity field $\boldsymbol{u}$ as a functional of ω. The key idea is a generalized stream function: We assume that our "ocean" spans a bounded, simply connected region R of $\mathbb{R}^2$. Let $\boldsymbol{x}_0$ be a fixed point on ∂R. Let $\boldsymbol{x}$ be any point in the interior of R, and let C be any non-self-intersecting curve in R connecting $\boldsymbol{x}_0$ to $\boldsymbol{x}$. Explain why the volume of fluid crossing C per unit time depends only on the endpoint $\boldsymbol{x}$, so that it is a well-defined function $\psi = \psi(\boldsymbol{x})$ of $\boldsymbol{x}$. This is our generalized stream function. Explain why ψ vanishes on ∂R. How do you recover $\boldsymbol{u}$ from ψ? Express the vorticity ω in terms of ψ. Identify the boundary value problem whose solution determines ψ as a functional of ω.

Solution.

a) The "slow current" Euler equation (7.2-3) expresses the acceleration of fluid particles as the gradient of a scalar, $\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} h$. This is the essential ingredient in the proof of the circulation theorem (Problem 6.4), so the circulation theorem applies here: The circulation

$$P := \int_{C(t)} \boldsymbol{u} \cdot d\boldsymbol{x} \tag{7.3-1}$$

about a closed material curve $C(t)$ is time-independent. By Green's theorem, we convert the line integral (7.3-1) into an area integral of vorticity ω over the interior $S(t)$ of $C(t)$, so we have

$$\frac{d}{dt}\int_{S(t)} \omega \, d\boldsymbol{x} = 0,$$

and in the limit of $S(t)$ shrinking to a point, we deduce that

$$\frac{D\omega}{Dt} = -(\boldsymbol{\nabla} \cdot \boldsymbol{u})\omega. \tag{7.3-2}$$

Next, rewrite (7.2-5) as

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = -\frac{\boldsymbol{u} \cdot \boldsymbol{\nabla}\ell}{\ell}. \tag{7.3-3}$$

Since $\ell = \ell(\boldsymbol{x})$ is independent of t,

$$\frac{D\ell}{Dt} = \boldsymbol{u} \cdot \boldsymbol{\nabla}\ell.$$

Hence, (7.3-3) becomes

$$\nabla \cdot \boldsymbol{u} = -\frac{1}{\ell}\frac{D\ell}{Dt}, \tag{7.3-4}$$

and substituting (7.3-4) for $\nabla \cdot \boldsymbol{u}$ into (7.3-2), we obtain

$$\frac{1}{\ell}\frac{D\ell}{Dt} = \frac{1}{\omega}\frac{D\omega}{Dt}.$$

It follows that

$$\frac{D}{Dt}\left(\frac{\omega}{\ell}\right) = 0,$$

so $\frac{\omega}{\ell}$ is constant along particle paths.

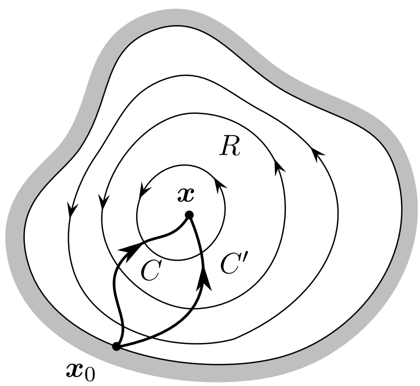

Figure 7.6.

b) Figure 7.6 depicts a portion of R, covered by integral curves of the velocity field $\boldsymbol{u}$, and two curves C, C' from $\boldsymbol{x}_0$ to $\boldsymbol{x}$.

If the rates of fluid volume crossing C and C' were different, volume would accumulate in the interior S of $C - C'$; but within the slow current approximation, the free surface is flat, and the volume of fluid in S is $\int_S \ell(\boldsymbol{x})\, d\boldsymbol{x}$, a fixed constant. If C and C' intersect at other points besides $\boldsymbol{x}_0$ and $\boldsymbol{x}$, we could repeat the argument for each closed "loop" of $C - C'$. So we have our generalized stream function $\psi(\boldsymbol{x})$. Suppose $\boldsymbol{x}$ is on ∂R. Take C to be a curve in R connecting $\boldsymbol{x}_0$ to $\boldsymbol{x}$ and dividing R into two pieces. The volume of fluid in each piece is constant, so the rate of fluid volume crossing C is zero, and hence $\psi(\boldsymbol{x}) = 0$.

The line integral representation of $\psi(\boldsymbol{x})$ is

$$\psi(\boldsymbol{x}) = \int_C \ell \boldsymbol{u} \cdot \boldsymbol{n}\, ds = \int_C -\ell u_2\, dx_1 + \ell u_1\, dx_2, \tag{7.3-5}$$

where C is any curve in R going from $\boldsymbol{x}_0$ to $\boldsymbol{x}$. In the right-hand side, we must have $-\ell u_2 = \partial_1 \psi$ and $\ell u_1 = \partial_2 \psi$, so

$$u_1 = \frac{1}{\ell}\partial_2 \psi, \quad u_2 = -\frac{1}{\ell}\partial_1 \psi. \tag{7.3-6}$$

From (7.3-6), we further deduce that

$$\omega := \partial_1 u_2 - \partial_2 u_1 = -\partial_1\left(\frac{1}{\ell}\partial_1 \psi\right) - \partial_2\left(\frac{1}{\ell}\partial \psi\right),$$

or finally,

$$\nabla \cdot \left(\frac{\nabla \psi}{\ell} \right) = -\omega \tag{7.3-7}$$

in R. The boundary value problem for ψ consists of the PDE (7.3-7) and the zero boundary condition on ψ along ∂R.

Problem 7.4 (A cautionary tale about point vortices). This problem deals with a specific example of the vortex dynamics in Problem 7.3.

a) The "ocean" space is $\mathbb{R}^2$, and the depth function is $\ell(\boldsymbol{x}) = e^{-x_1}$. Compute the generalized stream function ψ due to a point vortex of unit circulation at $\boldsymbol{x} = 0$, assuming $\boldsymbol{u} \to 0$ at ∞. (Hint: The analysis of the "smoke plume" in Problem 4.3 should help you get started; but this problem is happening in $\mathbb{R}^2$, not $\mathbb{R}^3$. You might want to recall Problem 4.8, in which we did a "descent" from $\mathbb{R}^3$ to $\mathbb{R}^2$. The result of your analysis is going to be an integral representation of ψ.)

b) Compute the average of the velocity field over a circle of radius ε about $\boldsymbol{x} = 0$, in the limit $\varepsilon \to 0$. (Hint: The analysis in Problem 4.9 suggests the first steps.)

c) What we really want is nearly uniform vorticity inside the disk $r < \varepsilon$ about $\boldsymbol{x} = 0$. We make the assumption that the flow due to this vortex patch in $r < \varepsilon$ is well approximated by the point vortex flow analyzed in Part a. Give a reasonable conjecture for the motion of this vortex patch as induced by the nonuniform bottom. What happens as $\varepsilon \to 0$?

Solution.

a) For $\ell = e^{-x_1}$ and $\omega = \delta(\boldsymbol{x})$, the PDE (7.3-7) takes the specific form

$$\Delta\psi + \partial_1\psi = -\delta(\boldsymbol{x}). \tag{7.4-1}$$

The "smoke plume" in Problem 4.3 suggests a new dependent variable $g(\boldsymbol{x})$, related to $\psi(\boldsymbol{x})$ by

$$\psi = e^{-\frac{x_1}{2}}\, g, \tag{7.4-2}$$

where $g(\boldsymbol{x})$ satisfies the PDE

$$\Delta g - \frac{g}{4} = -\delta(\boldsymbol{x}).$$

The relevant solution is circularly symmetric and vanishes at ∞. If we were in $\mathbb{R}^3$, the solution would be

$$g = \frac{e^{-\frac{r}{2}}}{4\pi r}.$$

In $\mathbb{R}^2$, we form a superposition of such three-dimensional solutions along a line, like in Problem 4.8. This gives the integral representation of the two-dimensional solution,

$$g = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}\,dz = \frac{1}{2\pi}\int_0^{\infty} e^{-\frac{r}{2}\cosh\zeta}\,d\zeta. \tag{7.4-3}$$

b) The velocity field components are given by (7.3-6) with $\ell = e^{-x_1}$, so

$$\boldsymbol{u} = e^{x_1}(\partial_2\psi\boldsymbol{e}_1 - \partial_1\psi\boldsymbol{e}_2) = -e^{x_1}J\boldsymbol{\nabla}\psi, \tag{7.4-4}$$

where $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents rotation by $\frac{\pi}{2}$ radians. For $g = g(r)$, taking the gradient of (7.4-2) gives

$$\boldsymbol{\nabla}\psi = \left(g_r\boldsymbol{e}_r - \frac{g}{2}\boldsymbol{e}_1\right)e^{-\frac{x_1}{2}},$$

and substituting this into (7.4-4) gives

$$\boldsymbol{u} = e^{\frac{x_1}{2}}\left(-g_r\boldsymbol{e}_\vartheta + \frac{g}{2}\boldsymbol{e}_2\right). \tag{7.4-5}$$

The average of u_1 over $r = \varepsilon$ vanishes:

$$\langle u_1\rangle = g_r(\varepsilon)\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\varepsilon}{2}\cos\vartheta}\sin\vartheta\,d\vartheta = 0.$$

The average of u_2 requires more work:

$$\langle u_2\rangle = -g_r(\varepsilon)\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\varepsilon}{2}\cos\vartheta}\cos\vartheta\,d\vartheta + \frac{1}{2}g(\varepsilon)\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\varepsilon}{2}\cos\vartheta}\,d\vartheta,$$

or

$$\langle u_2\rangle = -g_r(\varepsilon)\left\{\frac{\varepsilon}{4} + O(\varepsilon^3)\right\} + \frac{1}{2}g(\varepsilon)\{1 + O(\varepsilon^2)\}. \tag{7.4-6}$$

Finally, we must extract $r \to 0$ asymptotic formulas for $g(r)$ and $g_r(r)$ from the integral representation (7.4-3). We start with the change of variable $x = re^{\zeta}$, like in Problem 4.9b, and (7.4-3) becomes

$$g = \frac{1}{2\pi}\int_r^{\infty} e^{-\frac{1}{2}(x+\frac{r^2}{x})}\,\frac{dx}{x}. \tag{7.4-7}$$

An integration by parts advances us to

$$g = -\frac{1}{2\pi}e^{-r}\log r + \frac{1}{4\pi}\int_0^{\infty} e^{-\frac{1}{2}(x+\frac{r^2}{x})}\left(1 - \frac{r^2}{x^2}\right)\log x\,dx. \tag{7.4-8}$$

How do we know to do *this* integration by parts? We note that the $r \to 0$ behavior of g should be like that of the point source solution $-\frac{1}{2\pi}\log r$ of

Laplace's equation in $\mathbb{R}^2$. So doing the integration by parts of (7.4-7) with $dv = \frac{dx}{x}$ gives us our logarithm. In the limit $r \to 0$ (7.4-8) reduces to

$$g = -\frac{1}{2\pi} \log r + \frac{1}{4\pi} \int_0^\infty e^{-\frac{x}{2}} \log x \, dx + o(r). \tag{7.4-9}$$

The notation $o(r)$ denotes a truncation error that vanishes as $r \to 0$. In (7.4-9), $e^{-\frac{x}{2}} \log x$ is the formal limit of the integrand in (7.4-8) as $r \to 0$ with x fixed. It is nonuniformly valid for x comparable to r, but its contribution to the integral vanishes as $r \to 0$. In summary,

$$g(r) = -\frac{1}{2\pi} \log r + \frac{\gamma}{4\pi} + o(r), \tag{7.4-10}$$

where

$$\gamma := \int_0^\infty e^{-\frac{x}{2}} \log x \, dx \tag{7.4-11}$$

is the Euler constant.

We also need an appropriate approximation to $g_r(r)$ as $r \to 0$. Look back at the formula (7.4-6) for $\langle u_2 \rangle$: By (7.4-10), its second term is

$$\frac{1}{2} g(\varepsilon)(1 + O(\varepsilon^2)) = -\frac{1}{4\pi} \log \varepsilon + \frac{\gamma}{8\pi} + o(\varepsilon). \tag{7.4-12}$$

The smallest explicit term on the right-hand side is the constant $\frac{\gamma}{8\pi}$. So we need to resolve the first term of (7.4-6), $-g_r(\varepsilon)\{\frac{\varepsilon}{4} + O(\varepsilon^3)\}$, up to a constant term, and hence $g_r(\varepsilon)$ to a $\frac{1}{\varepsilon}$ term. Differentiating the $\log r$ term in (7.4-10) with respect to r and setting $r = \varepsilon$ gives $-\frac{1}{2\pi\varepsilon}$. But could there be some strong singularity in the r-derivative of the unresolved $o(r)$ term in (7.4-10)? The simplest check is to differentiate (7.4-7) with respect to r, obtaining

$$g_r = -\frac{e^{-r}}{2\pi r} - \frac{r}{2\pi} \int_r^\infty e^{-\frac{1}{2}(x + \frac{r^2}{x})} \frac{dx}{x^2}.$$

It is easy to show that the integral is bounded in absolute value by a constant independent of r. So we have

$$g_r(\varepsilon) = -\frac{1}{2\pi\varepsilon} + O(1),$$

and now (7.4-6) reduces to

$$\langle u_2 \rangle = \frac{1}{4\pi} \log \frac{1}{\varepsilon} + \frac{1}{8\pi}(\gamma + 1) + O(\varepsilon). \tag{7.4-13}$$

c) It seems that the vortex patch translates with the uniform velocity

$$\left\{ \frac{1}{4\pi} \log \frac{1}{\varepsilon} + \frac{1}{8\pi}(\gamma + 1) + O(\varepsilon) \right\} \boldsymbol{e}_2.$$

Our ocean gets shallower as x_1 increases, so a *rough* physical analogy would be an impenetrable boundary wall at some $x_1 > 0$. According to the analysis of the "ground effect" in Problem 6.8, the wall induces a vortex of positive circulation to travel in the $+\boldsymbol{e}_2$ direction. But notice the $\log \frac{1}{\varepsilon}$ divergence of velocity as the vortex is concentrated to a vanishingly small disk. We expect that the shallow water approximation will break down if the vortex is too concentrated. More generally, the approximation of "point" vortices in two-dimensional flow, while appealing, usually cannot be retained whenever we perturb the physics away from purely two-dimensional, incompressible ideal fluid. Even more broadly, Coulomb interactions between localized objects (such as electric charges or vortices) become infinitely strong when the objects shrink down to points or curves.

Guide to bibliography. Recommended references for this chapter are Courant & Friedrichs [**7**], Lighthill [**18**], Ockendon & Ockendon [**20**], Saffman [**21**], Stoker [**22**], and Whitham [**24**].

Lighthill [**18**], Stoker [**22**], and Whitham [**24**] carry out linearized analysis of free surface waves, using the velocity potential for irrotational flow and Bernoulli's equation (a first integral of Euler's equation for irrotational flow). Our choice to retain velocity components as state variables is motivated by this chapter's emphasis on shallow water theory, in which horizontal velocity components are most natural state variables. The cost is a (superficially) longer discussion of linearized theory.

Stoker [**22**] presents a scaling-based reduction of the full water wave equations to shallow water theory, similar to ours. There is also a tradition of informal derivations based on the hydrostatic approximation of pressure, the independence of horizontal velocity upon depth, and retention of depth dependence in the (small) vertical component. This kind of derivation is presented in Courant & Friedrichs [**7**] and Whitham [**24**].

The logarithmic divergence of vortex velocity induced by a sloping bottom as the vortex radius goes to zero is reminiscent of similar divergences in the long-standing theory of thin vortex rings, reviewed in Saffman [**21**].