INTRODUCTION

In this book we present some basic concepts and results from algebraic and differential topology. We do this in the framework of differential topology. Homology groups of spaces are one of the central tools of algebraic topology. These are abelian groups associated to topological spaces which measure certain aspects of the complexity of a space.

The idea of homology was originally introduced by Poincaré in 1895 [**Po**] where homology classes were represented by certain global geometric objects like closed submanifolds. The way Poincaré introduced homology in this paper is the model for our approach. Since some basics of differential topology were not yet far enough developed, certain difficulties occurred with Poincaré's original approach. Three years later he overcame these difficulties by representing homology classes using sums of locally defined objects from combinatorics, in particular singular simplices, instead of global differential objects. The singular and simplicial approaches to homology have been very successful and up until now most books on algebraic topology follow them and related elaborations or variations.

Poincaré's original idea for homology came up again many years later, when in the 1950's Thom [**Th 1**] invented and computed the bordism groups of smooth manifolds. Following on from Thom, Conner and Floyd [**C-F**] introduced singular bordism as a generalized homology theory of spaces in the 1960's. This homology theory is much more complicated than ordinary homology, since the bordism groups associated to a point are complicated abelian groups, whereas for ordinary homology they are trivial except in degree 0. The easiest way to simplify the bordism groups of a point is to generalize manifolds in an appropriate way, such that in particular the cone over a closed manifold of dimension > 0 is such a generalized manifold. There are several approaches in the literature in this direction but they are at a more advanced level. We hope it is useful to present an approach to ordinary homology which reflects the spirit of Poincaré's original idea and is written as an introductory text. For another geometric approach to (co)homology see [**B-R-S**].

As indicated above, the key for passing from singular bordism to ordinary homology is to introduce generalized manifolds that are a certain kind of stratified space. These are topological spaces **S** together with a decomposition of **S** into manifolds of increasing dimension called the strata of **S**. There are many concepts of stratified spaces (for an important paper see [**Th 2**]), the most important examples being Whitney stratified spaces. (For a nice tour through the history of stratification theory and an alternative concept of smooth stratified spaces see [**Pf**].) We will introduce a new class of stratified spaces, which we call **stratifolds**. Here the decomposition of **S** into strata will be derived from another structure. We distinguish a certain algebra **C** of continuous functions which plays the role of smooth functions in the case of a smooth manifold. (For those familiar with the language of sheaves, **C** is the algebra of global sections of a subsheaf of the sheaf of continuous functions on **S**.) Others have considered such algebras before (see for example [**S-L**]), but we impose stronger conditions. More precisely, we use the language of differential spaces [**Si**] and impose on this additional conditions. The conditions we impose on the algebra **C** provide the decomposition of **S** into its strata, which are smooth manifolds.

It turns out that basic concepts from differential topology like Sard's theorem, partitions of unity and transversality generalize to stratifolds and this allows for a definition of homology groups based on stratifolds which we call "stratifold homology". For many spaces this agrees with the most common and most important homology groups: singular homology groups (see below). It is rather easy and intuitive to derive the basic properties of homology groups in the world of stratifolds. These properties allow computation of homology groups and straightforward constructions of important homology classes like the fundamental class of a closed smooth oriented manifold or, more generally, of a compact stratifold. We also define stratifold cohomology groups (but only for smooth manifolds) by following an idea of Quillen [**Q**], who gave a geometric construction of cobordism groups, the cohomology theory associated to singular bordism. Again, certain important cohomology classes occur very naturally in this description, in particular the characteristic classes of smooth vector bundles over smooth oriented manifolds. Another useful aspect of this approach is that one of the most fundamental results, namely Poincaré duality, is almost a triviality. On the other hand, we do not develop much homological algebra and so related features of homology are not covered: for example the general Künneth theorem and the universal coefficient theorem.

From (co)homology groups one can derive important invariants like the Euler characteristic and the signature. These invariants play a significant role in some of the most spectacular results in differential topology. As a highlight we present Milnor's exotic 7-spheres (using a result of Thom which we do not prove in this book).

We mentioned above that Poincaré left his original approach and defined homology in a combinatorial way. It is natural to ask whether the definition of stratifold homology in this book is equivalent to the usual definition of singular homology. Both constructions satisfy the Eilenberg-Steenrod axioms for a homology theory and so, for a large class of spaces including all spaces which are homotopy equivalent to CW-complexes, the theories are equivalent. There is also an axiomatic characterization of cohomology for smooth manifolds which implies that the stratifold cohomology groups of smooth manifolds are equivalent to their singular cohomology groups. We consider these questions in chapter 20. It was a surprise to the author to find out that for more general spaces than those which are homotopy equivalent to CW-complexes, our homology theory is different from ordinary singular homology. This difference occurs already for rather simple spaces like the one-point compactifications of smooth manifolds!

The previous paragraphs indicate what the main themes of this book will be. Readers should be familiar with the basic notions of point set topology and of differential topology. We would like to stress that one can start reading the book if one only knows the definition of a topological space and some basic examples and methods for creating topological spaces and concepts like Hausdorff spaces and compact spaces. From differential topology one only needs to know the definition of smooth manifolds and some basic examples and concepts like regular values and Sard's theorem. The author has given introductory courses on algebraic topology which start with the presentation of these prerequisites from point set and differential topology and then continue with chapter 1 of this book. Additional information like orientation of manifolds and vector bundles, and later on transversality, was explained was explained when it was needed. Thus the book can serve as a basis for a combined introduction to differential and algebraic topology.

It also allows for a quick presentation of (co)homology in a course about differential geometry.

As with most mathematical concepts, the concept of stratifolds needs some time to get used to. Some readers might want to see first what stratifolds are good for before they learn the details. For those readers I have collected a few basics about stratifolds in chapter 0. One can jump from there directly to chapter 4, where stratifold homology groups are constructed.

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