Chapter 0

A quick introduction to stratifolds

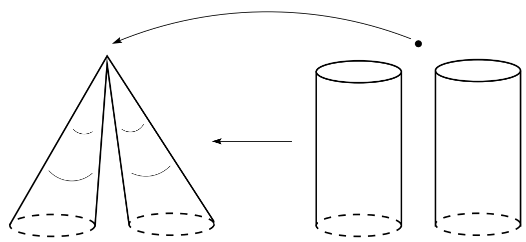
In this chapter we say as much as one needs to say about stratifolds in order to proceed directly to chapter 4 where homology with $\mathbb{Z}/2$-coefficients is constructed. We do it in a completely informal way that does not replace the definition of stratifolds. But some readers might want to see what stratifolds are good for before they study their definition and basic properties.

An n-dimensional **stratifold** $\mathbf{S}$ is a topological space $\mathbf{S}$ together with a class of distinguished continuous functions $f : \mathbf{S} \to \mathbb{R}$ called **smooth functions**. Stratifolds are generalizations of smooth manifolds M where the distinguished class of smooth functions are the C^∞-functions. The distinguished class of smooth functions on a stratifold $\mathbf{S}$ leads to a decomposition of $\mathbf{S}$ into disjoint smooth manifolds $\mathbf{S}^i$ of dimension i where $0 \leq i \leq n$, the dimension of $\mathbf{S}$. We call the $\mathbf{S}^i$ the **strata** of $\mathbf{S}$. An n-dimensional stratifold is a smooth manifold if and only if $\mathbf{S}^i = \varnothing$ for $i < n$.

To obtain a feeling for stratifolds we consider an important example. Let M be a smooth n-dimensional manifold. Then we consider the open cone over M

$$\overset{\circ}{C}M := M \times [0,1)/_{M \times \{0\}},$$

i.e., we consider the half open cylinder over M and collapse $M \times \{0\}$ to a point.



Now, we make $\overset{\circ}{C}M$ an $(n+1)$-dimensional stratifold by describing its distinguished class of smooth functions. These are the continuous functions

$$f : \overset{\circ}{C}M \to \mathbb{R},$$

such that $f|_{M\times(0,1)}$ is a smooth function on the smooth manifold $M \times (0,1)$ and there is an $\epsilon > 0$ such that $f|_{M\times[0,\epsilon)/_{M\times\{0\}}}$ is constant. In other words, the function is locally constant near the cone point $M \times \{0\}/_{M\times\{0\}} \in \overset{\circ}{C}M$. The strata of this $(n+1)$-dimensional stratifold $\mathbf{S}$ turn out to be $\mathbf{S}^0 = M \times \{0\}/_{M\times\{0\}}$, the cone point, which is a 0-dimensional smooth manifold, $\mathbf{S}^i = \varnothing$ for $0 < i < n+1$ and $\mathbf{S}^{n+1} = M \times (0,1)$.

One can generalize this construction and make the open cone over any n-dimensional stratifold $\mathbf{S}$ an $(n+1)$-dimensional stratifold $\overset{\circ}{C}\mathbf{S}$. The strata of $\overset{\circ}{C}\mathbf{S}$ are: $(\overset{\circ}{C}\mathbf{S})^0 = \mathrm{pt}$, the cone point, and for $1 \le i \le n+1$ we have $(\overset{\circ}{C}\mathbf{S})^i = \mathbf{S}^{i-1} \times (0,1)$, the open cylinder over the $(i-1)$-stratum of $\mathbf{S}$.

Stratifolds are defined so that most basic tools from differential topology for manifolds generalize to stratifolds.

- For each covering of a stratifold $\mathbf{S}$ one has a subordinate partition of unity consisting of smooth functions.
- One can define regular values of a smooth function $f : \mathbf{S} \to \mathbb{R}$ and show that if t is a regular value, then $f^{-1}(t)$ is a stratifold of dimension $n-1$ where the smooth functions of $f^{-1}(t)$ are simply the restrictions of the smooth functions of $\mathbf{S}$.
- Sard's theorem can be applied to show that the regular values of a smooth function $\mathbf{S} \to \mathbb{R}$ are a dense subset of $\mathbb{R}$.

As always, when we define mathematical objects like groups, vector spaces, manifolds, etc., we define the "allowed maps" between these objects, like homomorphisms, linear maps, smooth maps. In the case of stratifolds we do the same and call the "allowed maps" morphisms. A **morphism** $f : \mathbf{S} \to \mathbf{S}'$ is a continuous map $f : \mathbf{S} \to \mathbf{S}'$ such that for each smooth function $\rho : \mathbf{S}' \to \mathbb{R}$ the composition $\rho f : \mathbf{S} \to \mathbb{R}$ is a smooth function on $\mathbf{S}$. It is a nice exercise to show that the morphisms between smooth manifolds are precisely the smooth maps. A bijective map $f : \mathbf{S} \to \mathbf{S}'$ is called an **isomorphism** if f and f^{-1} are both morphisms. Thus in the case of smooth manifolds an isomorphism is the same as a diffeomorphism.

Next we consider stratifolds with boundary. For those who know what an n-dimensional manifold W with boundary is, it is clear that W is a topological space together with a distinguished closed subspace $\partial W \subseteq W$ such that $W - \partial W =: \overset{\circ}{W}$ is a n-dimensional smooth manifold and ∂W is a $(n-1)$-dimensional smooth manifold. For our purposes it is enough to imagine the same picture for stratifolds with boundary. An n-dimensional stratifold $\mathbf{T}$ with boundary is a topological space $\mathbf{T}$ together with a closed subspace $\partial \mathbf{T}$, the structure of a n-dimensional stratifold on $\overset{\circ}{\mathbf{T}} = \mathbf{T} - \partial \mathbf{T}$, the structure of an $(n-1)$-dimensional stratifold on $\partial \mathbf{T}$ and an additional structure (a collar) which we will not describe here. We call a stratifold with boundary a **c-stratifold** because of this collar.

The most important example of a smooth n-dimensional manifold with boundary is the half open cylinder $M \times [0, 1)$ over a $(n-1)$-dimensional manifold M, where $\partial M = M \times \{0\}$. Similarly, if $\mathbf{S}$ is a stratifold, then we give $\mathbf{S} \times [0, 1)$ the structure of a stratifold $\mathbf{T}$ with $\partial \mathbf{T} = \mathbf{S} \times \{0\}$. In the world of stratifolds the most important example of a c-stratifold is the closed cone over a smooth $(n-1)$-dimensional manifold M. This is denoted by

$$CM := M \times [0, 1]/_{M \times \{0\}},$$

where $\partial CM := M \times \{1\}$. More generally, for an $(n-1)$-dimensional stratifold $\mathbf{S}$ one can give the closed cone

$$C\mathbf{S} := \mathbf{S} \times [0, 1]/_{\mathbf{S} \times \{0\}}$$

the structure of a c-stratifold with $\partial C\mathbf{S} = \mathbf{S} \times \{1\}$.

If $\mathbf{T}$ and $\mathbf{T}'$ are stratifolds and $f : \partial \mathbf{T} \to \partial \mathbf{T}'$ is an isomorphism one can paste $\mathbf{T}$ and $\mathbf{T}'$ together via f. As a topological space one takes the disjoint union $\mathbf{T} \sqcup \mathbf{T}'$ and introduces the equivalence relation which identifies $x \in \partial \mathbf{T}$ with $f(x) \in \partial \mathbf{T}'$. There is a canonical way to give this space a stratifold structure. We denote the resulting stratifold by

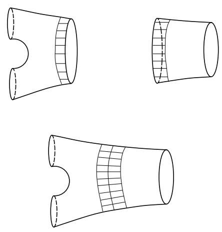
$$\mathbf{T} \cup_f \mathbf{T}'.$$

If $\partial \mathbf{T} = \partial \mathbf{T}'$ and $f = \mathrm{id}$, the identity map, we write

$$\mathbf{T} \cup \mathbf{T}'$$

instead of $\mathbf{T} \cup_f \mathbf{T}'$.

Instead of gluing along the full boundary we can glue along some components of the boundary, as shown below.



If a reader decides to jump from this chapter straight to homology (chapter 4), I recommend that he or she think of stratifolds as mathematical objects very similar to smooth manifolds, keeping in mind that in the world of stratifolds constructions like the cone over a manifold or even a stratifold are possible.

Chapter 12

Cohomology and Poincaré duality

Prerequisites: We assume that the reader knows what a smooth vector bundle is [**B-J**], [**Hi**].

1. Cohomology groups

In this chapter we consider another bordism group of stratifolds which at first glance looks like homology. It is only defined for smooth manifolds (without boundary). Similar groups were first introduced by Quillen [**Q**] and Dold [**D**]. They consider bordism classes of smooth manifolds instead of stratifolds.

The main difference between the new groups and homology is that we consider bordism classes of non-compact stratifolds. To obtain something non-trivial we require that the map $g : \mathbf{T} \to M$ is a proper map. We recall that a map between paracompact spaces is **proper** if the preimage of each compact space is compact. A second difference is that we only consider smooth maps. For simplicity we only define these bordism groups for oriented manifolds. (Each manifold is canonically homotopy equivalent to an oriented manifold, namely the total space of the tangent bundle, so that one can extend the definition to non-oriented manifolds using this trick, see the exercises in chapter 13.)

Definition: *Let M be an oriented smooth m-dimensional manifold without boundary. Then we define the* **integral cohomology group** $SH^k(M)$ *as* *the group of bordism classes of proper smooth maps $g : \mathbf{S} \to M$, where $\mathbf{S}$ is an oriented regular stratifold of dimension $m-k$, addition is by disjoint union of maps and the inverse of $[\mathbf{S}, g]$ is $[-\mathbf{S}, g]$ (of course we also require that the maps for bordisms are proper and smooth and that the stratifolds are oriented and regular).*

The reader might wonder why we required that M be oriented. The definition seems to work without this condition. This will become clear when we define induced maps. Then we will understand the relationship between $SH^k(M)$ and $SH^k(-M)$ better.

The relation between the grade, k, of $SH^k(M)$ and the dimension $m-k$ of representatives of the bordism classes looks strange but we will see that it is natural for various reasons.

If M is a point then $g : \mathbf{S} \to \mathrm{pt}$ is proper if and only if $\mathbf{S}$ is compact. Thus

$$SH^k(\mathrm{pt}) = SH_{-k}(\mathrm{pt}) \cong \mathbb{Z}, \text{ if } k = 0, \text{ and } 0 \text{ if } k \neq 0.$$

In order to develop an initial feeling for cohomology classes, we consider the following situation. Let $p : E \to N$ be a k-dimensional, smooth, oriented vector bundle over an n-dimensional oriented smooth manifold. Then the total space E is a smooth $(k+n)$-dimensional manifold. The orientations of M and E induce an orientation on this manifold. The 0-section $s : N \to E$ is a proper map since $s(N)$ is a closed subspace. Thus

$$[N, s] \in SH^k(E)$$

is a cohomology class. This is the most important example we have in mind and will play an essential role when we define characteristic classes. A special case is given by a 0-dimensional vector bundle where $E = N$ and $p = \mathrm{id}$. Thus we have for each smooth oriented manifold N the class $[N, \mathrm{id}] \in SH^0(N)$, which we call $1 \in SH^0(N)$. Later we will define a multiplication on the cohomology groups and it will turn out that multiplication with $[N, \mathrm{id}]$ is the identity, justifying the notation.

Is the class $[N, s]$ non-trivial? We will see that it is often non-trivial but it is zero if E admits a nowhere vanishing section $v : N \to E$. Namely then we obtain a zero bordism by taking the smooth manifold $N \times [0, \infty)$ and the map $G : N \times [0, \infty) \to E$ mapping $(x, t) \mapsto tv(x)$. The fact that v is nowhere vanishing implies that G is a proper map. Thus we have shown:

Proposition 12.1. *Let $p : E \to N$ be a smooth, oriented k-dimensional vector bundle over a smooth oriented manifold N. If E has a nowhere vanishing section v then $[N, s] \in SH^k(E)$ vanishes.*

In particular, if $[N, s]$ is non-trivial, then E does not admit a nowhere vanishing section.

In the following considerations and constructions it will be helpful for the reader to look at the cohomology class $[N, s] \in SH^k(E)$ and test the situation with this class.

2. Poincaré duality

Cohomology groups are, as indicated for example in Proposition 12.1, a useful tool. To apply this tool one has to find methods for their computation. We will do this in two completely different ways. The fact that they are so different is very useful since one can combine the information to obtain very surprising results like the vanishing of the Euler characteristic of odd-dimensional, compact, smooth manifolds.

The first tool, the famous Poincaré duality isomorphism, only works for compact, oriented manifolds and relates their cohomology groups to the homology groups. Whereas in the classical approach to (co)homology the duality theorem is difficult to prove, it is almost trivial in our context. The second tool is the Kronecker pairing which relates the cohomology groups to the dual space of the homology groups. This will be explained in chapter 14.

Let M be a compact oriented smooth m-dimensional manifold. (Here we recall that if we use the term manifold, then it is automatically without boundary; in this book, manifolds with boundary are always called c-manifolds. Thus a compact manifold is what in the literature is often called a **closed manifold**, a compact manifold without boundary.) If M is compact and $g : \mathbf{S} \to M$ is a proper map, then $\mathbf{S}$ is actually compact. Thus we obtain a homomorphism

$$P : SH^k(M) \to SH_{m-k}(M)$$

which assigns to $[\mathbf{S}, g] \in SH^k(M)$ the class $[\mathbf{S}, g]$ considered as element of $SH_{m-k}(M)$. Here we only forget that the map g is smooth and consider it as a continuous map.

Theorem 12.2. (Poincaré duality) *For a* **closed** *smooth oriented m-dimensional manifold M the map*

$$P : SH^k(M) \to SH_{m-k}(M)$$

is an isomorphism

Proof: For the proof we apply the following useful approximation result for continuous maps from a stratifold to a smooth manifold. It is another nice application of partitions of unity.

Proposition 12.3. *Let $f : \mathbf{S} \to N$ be a continuous map, which is smooth in an open neighbourhood of a closed subset $A \subset \mathbf{S}$. Then there is a smooth map $g : \mathbf{S} \to N$ which agrees with f on A and which is homotopic to f rel. A.*

Proof: The proof is the same as for a map f from a smooth manifold M to N in [**B-J**, Theorem 14.8]. More precisely, there it is proved that if we embed N as a closed subspace into an Euclidean space $\mathbb{R}^n$ then we can find a smooth map g arbitrarily close to f. The proof only uses that M supports a smooth partition of unity. Finally, sufficiently close maps are homotopic by ([**B-J**] Satz 12.9).
q.e.d.

As a consequence we obtain a similar result for c-stratifolds.

Proposition 12.4. *Let $f : \mathbf{T} \to M$ be a continuous map from a smooth c-stratifold $\mathbf{T}$ to a smooth manifold M, whose restriction to $\partial\mathbf{T}$ is a smooth map. Then f is homotopic rel. boundary to a smooth map.*

The proof follows from 12.3 using an appropriate closed subset in the collar of $\overset{\circ}{\mathbf{T}}$ for the subset A.

We apply this result to finish our proof. If $g : \mathbf{S} \to M$ represents an element of $SH_{m-k}(M)$, we can apply Proposition 12.3 to replace g by a homotopic smooth map g' and so $[\mathbf{S}, g] = P([\mathbf{S}, g'])$. This gives surjectivity of P. Similarly we use the relative version 12.4 to show injectivity. Namely, if for $[\mathbf{S}_1, g_1]$ and $[\mathbf{S}_2, g_2]$ in $SH^k(M)$ we have $P([\mathbf{S}_1, g_1]) = P([\mathbf{S}_2, g_2])$, there is a bordism $(\mathbf{T}, G)$ between these two pairs, where G is a continuous map whose restriction to the boundary is smooth. We apply Proposition 12.4 to replace G by a smooth map G' which agrees with the restriction of G on the boundary. Thus $[\mathbf{S}_1, g_1] = [\mathbf{S}_2, g_2] \in SH^k(M)$ and P is injective.
q.e.d.

By considering bordism classes of proper maps on $\mathbb{Z}/2$-oriented regular stratifolds we can define $\mathbb{Z}/2$**-cohomology groups** for arbitrary (non-oriented) smooth manifolds as we did in the integral case. The only difference is that we replace oriented regular stratifolds by $\mathbb{Z}/2$-oriented regular stratifolds which means that $\mathbf{S}^{n-1} = \varnothing$ and that no condition is placed on the orientability of the top stratum. The corresponding cohomology groups are denoted by

$$SH^k(M;\mathbb{Z}/2).$$

The proof of Poincaré duality works the same way for $\mathbb{Z}/2$-(co)homology:

Theorem 12.5. (Poincaré duality for $\mathbb{Z}/2$-(co)homology) *For a* **closed** *smooth oriented manifold M the map*

$$P : SH^k(M;\mathbb{Z}/2) \to SH_{m-k}(M;\mathbb{Z}/2)$$

is an isomorphism.

As mentioned above, we want to provide other methods for computing the cohomology groups. They are based on the same ideas as used for computing homology groups, namely to show that the cohomology groups fulfill axioms similar to the axioms of homology groups. One of the applications of these axioms will be an isomorphism between $SH^k(M) \otimes \mathbb{Q}$ and $\mathrm{Hom}(SH_k(M), \mathbb{Q})$ and an isomorphism of $\mathbb{Z}/2$-vector spaces between $SH^k(M;\mathbb{Z}/2)$ and $\mathrm{Hom}(SH_k(M;\mathbb{Z}/2), \mathbb{Z}/2)$. The occurrence of the dual spaces $\mathrm{Hom}(SH_k(M), \mathbb{Q})$ and $\mathrm{Hom}(SH_k(M;\mathbb{Z}/2), \mathbb{Z}/2)$ indicates a difference between the fundamental properties of homology and cohomology. The induced maps occurring should reverse their directions. We will see that this is the case.

3. The Mayer-Vietoris sequence

One of the most powerful tools for computing cohomology groups is, as it is for homology, the Mayer-Vietoris sequence. To formulate it we have to define for an open subset U of a smooth oriented manifold M the map induced by the inclusion $i : U \to M$. We equip U with the orientation induced from M. If $g : \mathbf{S} \to M$ is a smooth proper map we consider the open subset $g^{-1}(U) \subset \mathbf{S}$ and restrict g to this open subset. It is again a proper map (why?) and thus we define

$$i^*[\mathbf{S}, g] := [g^{-1}(U), g|_{g^{-1}(U)}].$$

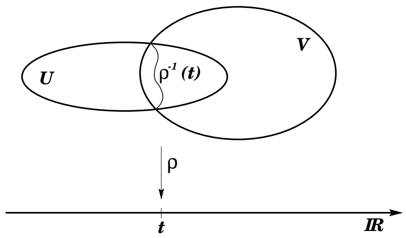
This is obviously well defined and gives a homomorphism $i^* : SH^k(M) \to SH^k(U)$. This map reverses the direction of the arrows, as was motivated above. If V is an open subset of U and $j : V \to U$ is the inclusion, then by construction

$$j^* i^* = (ij)^*.$$

The next ingredient for the formulation of the Mayer-Vietoris sequence is the coboundary operator. We consider open subsets U and V in a smooth oriented manifold M, denote $U \cup V$ by X and define the coboundary operator

$$\delta : SH^k(U \cap V) \to SH^{k+1}(U \cup V)$$

as follows. We introduce the disjoint closed subsets $A := X - V$ and $B := X - U$. We choose a smooth map $\rho : U \cup V \to \mathbb{R}$ mapping A to 1 and B to -1. Now we consider $[\mathbf{S}, f] \in SH^k(U \cap V)$. Let $s \in (-1, 1)$ be a regular value of ρf. The preimage $\mathbf{D} := (\rho f)^{-1}(s)$ is an oriented regular stratifold of dimension $n-1$ sitting in $\mathbf{S}$. We define $\delta([\mathbf{S}, f]) := [\mathbf{D}, f|_{\mathbf{D}}] \in SH^{k+1}(X)$. It is easy to check that $f|_{\mathbf{D}}$ is proper.



As with the definition of the boundary map for the Mayer-Vietoris sequence in homology, one shows that δ is well defined and that one obtains an exact sequence. For details we refer to Appendix B.

At first glance this definition of the coboundary operator looks strange since $f(\mathbf{D})$ is contained in $U \cap V$. But considered as a class in the cohomology of $U \cap V$ it is trivial. It is even zero in $SH^{k+1}(U)$ as well as in $SH^{k+1}(V)$. The reason is that in the construction of δ we can decompose $\mathbf{S}$ as $\mathbf{S}_+ \cup_{\mathbf{D}} \mathbf{S}_-$ with $\rho(\mathbf{S}_+) \geq s$ and $\rho(\mathbf{S}_-) \leq s$ (as for the boundary operator in homology we can assume up to bordism that there is a bicollar along $\mathbf{D}$). Then $(\mathbf{S}_-, f|_{\mathbf{S}_-})$ is a zero bordism of $(\mathbf{D}, f|_{\mathbf{D}})$ in U (note that $f|_{\mathbf{S}_-}$ is proper as a map into U and not into V). Similarly $(\mathbf{S}_+, f|_{\mathbf{S}_+})$ is a zero bordism of $(\mathbf{D}, f|_{\mathbf{D}})$ in V. But in $SH^{k+1}(U \cup V)$ it is in general non-trivial.

We summarize:

Theorem 12.6. (Mayer-Vietoris sequence for integral cohomology) *The following sequence is exact and commutes with induced maps:*

$$\cdots \to SH^n(U \cup V) \to SH^n(U) \oplus SH^n(V)$$
$$\to SH^n(U \cap V) \xrightarrow{\delta} SH^{n+1}(U \cup V) \to \cdots .$$

The map $SH^n(U \cup V) \to SH^n(U) \oplus SH^n(V)$ is given by $\alpha \mapsto (j_U^*(\alpha), j_V^*(\alpha))$, the map from $SH^n(U) \oplus SH^n(V)$ to $SH^n(U \cap V)$ by $(\alpha, \beta) \mapsto i_U^*(\alpha) - i_V^*(\beta)$.

4. Exercises

(1) Compute the cohomology groups $SH^k(\mathbb{R}^n)$ for $k \geq 0$. (Hint: For $SH^0(\mathbb{R}^n)$ construct a map to $\mathbb{Z}$ by counting points with orientation in the preimage of a regular value. For degree > 0 apply Sard's theorem.) What happens for $k < 0$?

(2) Let $f : M \to N$ be a submersion (i.e., the differential df_x at each point $x \in M$ is surjective). Let $[g : \mathbf{S} \to N]$ be a cohomology class in $SH^k(N)$. Show that the pull-back $\{(x, y) \in (M \times \mathbf{S}) \mid (f(x) = g(y)\}$ is a stratifold and that the projection to the first factor is a proper map. Show that this construction gives a well defined homomorphism $f^* : SH^k(N) \to SH^k(M)$. (This is a special case of the induced map which we will define later.)

(3) Let M be a smooth manifold. Show that the map $p^* : SH^k(M) \to SH^k(M \times \mathbb{R})$ is injective. (Hint: Construct a map $SH^k(M \times \mathbb{R}) \to SH^k(M)$ by considering for $[g : \mathbf{S} \to M \times \mathbb{R}]$ a regular value of $p_2 g$.) We will see later that p^* is an isomorphism; try to prove this directly.