Preface

Students are often surprised when they first hear the following definition: A stochastic process is a collection of random variables indexed by time. There seems to be no content here. There is no structure. How can anyone say anything of value about a stochastic process? The content and structure are in fact provided by the definitions of the various classes of stochastic processes that are so important for both theory and applications. There are processes in discrete or continuous time. There are processes on countable or general state spaces. There are Markov processes, random walks, Gaussian processes, diffusion processes, martingales, stable processes, infinitely divisible processes, stationary processes, and many more. There are entire books written about each of these types of stochastic process.

The purpose of this book is to provide an introduction to a particularly important class of stochastic processes — continuous time Markov processes. My intention is that it be used as a text for the second half of a year-long course on measure-theoretic probability theory. The first half of such a course typically deals with the classical limit theorems for sums of independent random variables (laws of large numbers, central limit theorems, random infinite series), and with some of the basic discrete time stochastic processes (martingales, random walks, stationary sequences). Alternatively, the book can be used in a semester-long special topics course for students who have completed the basic year-long course. In this case, students will probably already be familiar with the material in Chapter 1, so the course would start with Chapter 2.

The present book stresses the new issues that appear in continuous time. A difference that arises immediately is in the definition of the process. A discrete time Markov process is defined by specifying the law that leads from the state at one time to that at the next time. This approach is not possible in continuous time. In most cases, it is necessary to describe the transition law infinitesimally in time, and then prove under appropriate conditions that this description leads to a well-defined process for all time.

We begin with an introduction to Brownian motion, which is certainly the most important continuous time stochastic process. It is a special case of many of the types listed above — it is Markov, Gaussian, a diffusion, a martingale, stable, and infinitely divisible. It plays a fundamental role in stochastic calculus, and hence in financial mathematics. Through Donsker's theorem, it provides a framework for far reaching generalizations of the classical central limit theorem. While we will concentrate on this one process in Chapter 1, we will also discuss there the extent to which results and techniques apply (or do not apply) more generally. The infinitesimal definition mentioned in the previous paragraph is not necessary in the case of Brownian motion. However, our discussion of Brownian motion sets the stage for the setup that is required for processes that are defined in that way.

Next we discuss the construction problem for continuous time Markov chains. (The word "chain" here refers to the countability of the state space.) The main issue is to determine when the infinitesimal description of the process (given by the Q-matrix) uniquely determines the process via Kolmogorov's backward equations.

With an understanding of these two examples — Brownian motion and continuous time Markov chains — we will be in a position to consider the issue of defining the process in greater generality. Key here is the Hille-Yosida theorem, which links the infinitesimal description of the process (the generator) to the evolution of the process over time (the semigroup). Since usually only the generator is known explicitly, we will discuss how one deduces properties of the process from information about the generator. The main examples at this point are variants of Brownian motion, in which the relative speed of the particle varies spatially, and/or there is a special behavior at the boundary of the state space.

As an application of the theory of semigroups and generators, we then provide an introduction to a somewhat more recently developed area of probability theory — interacting particle systems. This is a class of probabilistic models that come up in many areas of application — physics, biology, computer science, and even a bit in economics and sociology. Infinitely many agents evolve in time according to certain probabilistic rules that involve interactions among the agents. The nature of these rules is dictated by the area of application. The main issue here is the nature of the long time behavior of the process.

Next we give an introduction to stochastic integration with respect to Brownian motion and other continuous (semi)martingales. Not only is this an important probabilistic tool, but in recent years, it has become an essential part of financial mathematics. We define the Itô integral and study its properties, which are quite different from those of ordinary integrals, as a consequence of the lack of smoothness of Brownian paths. Then we use it to construct local time for Brownian motion, and apply it to give a new perspective on some of the Brownian relatives from Chapter 3.

In the final chapter, we return to Brownian motion, now in higher dimensions, and describe one of its great successes in analysis — that of providing a probabilistic solution to the classical Dirichlet problem. This problem asks for harmonic functions (those satisfying $\Delta h = 0$) in a domain in R^n with prescribed boundary values. Then we discuss the Poisson equation $\frac{1}{2}\Delta h = -f$. Solutions to the Dirichlet problem and Poisson equation provide concrete answers to many problems involving Brownian motion in R^n. Examples are exit distributions from domains, and expected occupation times of subsets prior to exiting a domain.

The prerequisite for reading this book is a semester course in measure-theoretic probability that includes the material in the first four chapters of [**18**], for example. In particular, students should be familiar with laws of large numbers, central limit theorems, random walks, the basics of discrete time Markov chains, and discrete time martingales. To facilitate referring to this material, I have included the main definitions and results (mostly without proofs) in the Appendix. Approximately 200 exercises are placed within the sections as the relevant material is covered.

Chapters 1 and 2 are largely independent of one another, but should be read before Chapter 3. They provide motivation for the more abstract treatment of Feller processes there. The main places where Chapter 2 relies on material from Chapter 1 are in the discussions of the Markov and strong Markov properties. Rather than prove these in some generality, our approach is to prove them in the concrete context of Brownian motion. By making explicit the properties of Brownian motion that are used in the proofs, we are able simply to refer back to those proofs when these properties are discussed in Chapters 2 and 3.

The hearts of Chapters 2 and 3 are Sections 2.5 and 3.3 respectively. The prior sections in these chapters are intended to provide motivation for the transition from infinitesimal description to time evolution that is explained in those sectons. Therefore, the earlier sections need not be covered in full detail. In my classes, I often state the main results from the earlier sections without proving many of them, in order to allow ample time for the more important material in Sections 2.5 and 3.3.

The last three chapters can be covered in any order. Chapters 5 and 6 rely only slightly on Chapters 2 and 3, so one can easily create a short course based on Chapters 1, 5 and 6.

This book is based on courses I have taught at UCLA over many years. Unlike many universities, UCLA operates on the quarter system. I have typically covered most of the material in Chapters 1–3 and 6 in the third quarter of the graduate probability course, and Chapters 4 and 5 in special topics courses. There is more than enough material here for a semester course, even if Chapter 1 is skipped because students are already familiar with one-dimensional Brownian motion.

Despite my best efforts, some errors have probably made their way into the text. I will maintain a list of corrections at

http://www.math.ucla.edu/~tml/

Readers are encouraged to send me corrections at tml@math.ucla.edu.

As is usually the case with a text of this type, I have benefitted greatly from the work of previous authors, including those of [**12**], [**18**], [**21**], [**22**], [**39**], and [**40**]. I appreciate the comments and corrections provided by P. Caputo, S. Roch, and A. Vandenberg-Rodes, and especially T. Richthammer and F. Zhang, who read much of this book very carefully.

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