Preface to Second Edition

Algebra is used by virtually all mathematicians, be they analysts, combinatorists, computer scientists, geometers, logicians, number theorists, or topologists. Nowadays, everyone agrees that some knowledge of Linear Algebra, Group Theory, and Commutative Algebra is necessary, and these topics are introduced in undergraduate courses. We continue their study.

This book can be used as a text for the first year of graduate Algebra, but it is much more than that. It can also serve more advanced graduate students wishing to learn topics on their own. While not reaching the frontiers, the book does provide a sense of the successes and methods arising in an area. In addition, this is a reference containing many of the standard theorems and definitions that users of Algebra need to know. Thus, this book is not merely an appetizer; it is a hearty meal as well.

When I was a student, Birkhoff–Mac Lane, *A Survey of Modern Algebra*, was the text for my first Algebra course, and van der Waerden, *Modern Algebra*, was the text for my second course. Both are excellent books (I have called this book *Advanced Modern Algebra* in homage to them), but times have changed since their first publication: Birkhoff and Mac Lane's book appeared in 1941; van der Waerden's book appeared in 1930. There are today major directions that either did not exist 75 years ago, or were not then recognized as being so important, or were not so well developed. These new areas involve Algebraic Geometry, Category Theory,[1] Computer Science, Homological Algebra, and Representation Theory.

Let me now address readers and instructors for whom this book is a text in a beginning graduate course. Instead of devoting the first chapters to a review of more elementary material (as I did in the first edition), here I usually refer to FCAA (my book, *A First Course in Abstract Algebra*, 3rd ed.). I have reorganized and rewritten this text, but here are some other major differences with the first edition.

[1] *A Survey of Modern Algebra* was rewritten, introducing categories, as Mac Lane–Birkhoff, *Algebra*, Macmillan, New York, 1967.

Noncommutative rings are now discussed earlier, so that left and right modules can appear at the same time. The existence of free groups is shown more simply (using only deletions instead of deletions and insertions). In fact, much of the section on presentations has been rewritten. Division rings and Brauer groups are introduced in the same chapter as the Wedderburn–Artin Theorems; their discussion then continues after cohomology has been studied. The section on Grothendieck groups has been completely rewritten. Some other items appearing here that were not treated in the first edition are Galois Theory for infinite extensions, the Normal Basis Theorem, abelian categories, and module categories.

The Table of Contents enumerates the highlights of each chapter; here are more details. Chapters 1 and 2 present the elements of Group Theory and of Commutative Algebra, along with Linear Algebra over arbitrary fields.

Chapter 3 discusses Galois Theory for finite field extensions (Galois Theory for infinite field extensions is discussed in Chapter 6, after inverse limits have been introduced). We prove the insolvability of the general polynomial of degree 5 and the Fundamental Theorem of Galois Theory. Among the applications are the Fundamental Theorem of Algebra and Galois's Theorem that a polynomial over a field of characteristic 0 is solvable by radicals if and only if its Galois group is a solvable group. The chapter ends by showing how to compute Galois groups of polynomials of degree ≤ 4.

Chapter 4 continues the study of groups, beginning with the Basis Theorem and the Fundamental Theorem for finite abelian groups (finitely generated abelian groups are discussed in Chapter 8). We then prove the Sylow Theorems (which generalize the Primary Decomposition to nonabelian groups), discuss solvable groups, simplicity of the linear groups $\mathrm{PSL}(2, k)$, unitriangular groups, free groups, presentations, and the Nielsen–Schreier Theorem (subgroups of free groups are free).

Chapter 5 continues the study of commutative rings, with an eye to discussing polynomial rings in several variables; unique factorization domains; Hilbert's Basis Theorem; applications of Zorn's Lemma (including existence and uniqueness of algebraic closures, transcendence bases, and inseparability); Lüroth's Theorem; affine varieties; Nullstellensatz over $\mathbb{C}$ (the full Nullstellensatz, for varieties over arbitrary algebraically closed fields, is proved in Chapter 10); primary decomposition of ideals; the Division Algorithm for polynomials in several variables; Buchberger's algorithm and Gröbner bases.

Chapter 6 introduces noncommutative rings, left and right R-modules; categories, functors, natural transformations, and categorical constructions; free modules, projectives, and injectives; tensor products; adjoint functors; flat modules; inverse and direct limits; infinite Galois Theory.

Chapter 7 continues the study of noncommutative rings, aiming toward Representation Theory of finite groups: chain conditions; Wedderburn's Theorem on finite division rings; Jacobson radical; Wedderburn–Artin Theorems classifying semisimple rings. These results are applied, using character theory, to prove Burnside's Theorem (finite groups of order $p^m q^n$ are solvable). After discussing multiply transitive groups, we prove Frobenius's Theorem that Frobenius kernels are normal subgroups of Frobenius groups. Since division rings have arisen naturally, we introduce Brauer groups. Chapter 9 views Brauer groups as cohomology groups, allowing us to prove existence of division rings of positive characteristic. This chapter ends with abelian categories and the characterization of categories of modules, enabling us to see why a ring R is intimately related to the matrix rings $\mathrm{Mat}_n(R)$ for $n \geq 1$.

Chapter 8 considers finitely generated modules over principal ideal domains (generalizing earlier theorems about finite abelian groups), and then goes on to apply these results to rational, Jordan, and Smith canonical forms for matrices over a field (the Smith normal form yields algorithms that compute elementary divisors of matrices). We also classify projective, injective, and flat modules over PIDs. Bilinear forms are introduced, along with orthogonal and symplectic groups. We then discuss some multilinear algebra, ending with an introduction to Lie Algebra.

Chapter 9 introduces homological methods, beginning with abstract simplicial complexes and group extensions, which motivate homology and cohomology groups, and Tor and Ext (existence of these functors is proved with derived functors). Applications are made to modules, cohomology of groups, and division rings. A descriptive account of spectral sequences is given, sufficient to indicate why Ext and Tor are independent of the variable resolved. We then pass from Homological Algebra to Homotopical Algebra: Algebraic K-Theory is introduced with a discussion of Grothendieck groups.

Chapter 10 returns to Commutative Algebra: localization, the general Nullstellensatz (using Jacobson rings), Dedekind rings and some Algebraic Number Theory, and Krull's Principal Ideal Theorem. We end with more Homological Algebra, proving the Serre–Auslander–Buchsbaum Theorem characterizing regular local rings as those noetherian local rings of finite global dimension and the Auslander–Buchsbaum Theorem that regular local rings are UFDs.

Each generation should survey Algebra to make it serve the present time.

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Joseph Rotman

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