# Preface

The aim of this book is to present the basic facts of linear functional analysis related to applications to some fundamental aspects of mathematical analysis.

If mathematics is supposed to show common general facts and structures of particular results, functional analysis does this while dealing with classical problems, many of them related to ordinary and partial differential equations, integral equations, harmonic analysis, function theory, and the calculus of variations.

In functional analysis, individual functions satisfying specific equations are replaced by classes of functions and transforms which are determined by each particular problem. The objects of functional analysis are spaces and operators acting between them which, after systematic studies intertwining linear and topological or metric structures, appear to be behind classical problems in a kind of cleaning process.

In order to make the scope of functional analysis clearer, I have chosen to sacrifice generality for the sake of an easier understanding of its methods, and to show how they clarify what is essential in analytical problems. I have tried to avoid the introduction of cold abstractions and unnecessary terminology in further developments and, when choosing the different topics, I have included some applications that connect functional analysis with other areas.

The text is based on a graduate course taught at the Universitat de Barcelona, with some additions, mainly to make it more self-contained. The material in the first chapters could be adapted as an introductory course on functional analysis, aiming to present the role of duality in analysis, and also the spectral theory of compact linear operators in the context of Hilbert and Banach spaces.

In this first part of the book, the mutual influence between functional analysis and other areas of analysis is shown when studying duality, with von Neumann's proof of the Radon-Nikodym theorem based on the Riesz representation theorem for the dual of a Hilbert space, followed by the representations of the duals of the $L^p$ spaces and of $\mathcal{C}(K)$, in this case by means of complex Borel measures.

The reader will also see how to deal with initial and boundary value problems in ordinary linear differential equations via the use of integral operators. Moreover examples are included that illustrate how functional analytic methods are useful in the study of Fourier series.

In the second part, distributions provide a natural framework extending some fundamental operations in analysis. Convolution and the Fourier transform are included as useful tools for dealing with partial differential operators, with basic notions such as fundamental solutions and Green's functions.

Distributions are also appropriate for the introduction of Sobolev spaces, which are very useful for the study of the solutions of partial differential equations. A clear example is provided by the resolution of the Dirichlet problem and the description of the eigenvalues of the Laplacian, in combination with Hilbert space techniques.

The last two chapters are essentially devoted to the spectral theory of bounded and unbounded self-adjoint operators, which is presented by using the Gelfand transform for Banach algebras. This spectral theory is illustrated with an introduction to the basic axioms of quantum mechanics, which motivated many studies in the Hilbert space theory.

Some very short historical comments have been included, mainly by means of footnotes. For a good overview of the evolution of functional analysis, J. Dieudonné's and A. F. Monna's books, [**10**] and [**31**], are two good references.

The limitation of space has forced us to leave out many other important topics that could, and probably should, have been included. Among them are the geometry of Banach spaces, a general theory of locally convex spaces and structure theory of Fréchet spaces, functional calculus of nonnormal operators, groups and semigroups of operators, invariant subspaces, index theory, von Neumann algebras, and scattering theory. Fortunately, many excellent texts dealing with these subjects are available and a few references have been selected for further study.

A small number of references have been gathered at the end of each chapter to focus the reader's attention on some appropriate items from a general bibliographical list of 44 items.

Almost 240 exercises are gathered at the end of the chapters and form an important part of the book. They are intended to help the reader to develop techniques and working knowledge of functional analysis. These exercises are highly nonuniform in difficulty. Some are very simple, to aid in better understanding of the concepts employed, whereas others are fairly challenging for the beginners. Hints and solutions are provided at the end of the book.

The prerequisites are very standard. Although it is assumed that the reader has some a priori knowledge of general topology, integral calculus with Lebesgue measure, and elementary aspects of normed or Hilbert spaces, a review of the basic aspects of these topics has been included in the first chapters.

I turn finally to the pleasant task of thanking those who helped me during the writing. Particular thanks are due to Javier Soria, who revised most of the manuscript and proposed important corrections and suggestions. I have also received valuable advice and criticism from María J. Carro and Joaquim Ortega-Cerdà. I have been very fortunate to have received their assistance.

Joan Cerdà
Universitat de Barcelona