Fourier transform and Sobolev spaces

The Fourier transform is one of the most powerful operators in analysis. Its scope and applications have been extended to areas as different as harmonic analysis, partial differential equations, signal theory, probabilities, and algebraic number theory.

Its virtues depend on the use of the functions $e^{i\alpha t} = \cos(\alpha t) + i \sin(\alpha t)$, which are the homomorphisms of the additive group $\mathbf{R}$ to the multiplicative group $\mathbf{T}$, and on the translation-invariance of the Lebesgue measure.

These facts are intimately linked to the fundamental properties of converting convolution and linear differential operators into multiplication operators, changing convolution and partial differential equations into algebraic equations, and yielding explicit solutions in basic equations such as Laplace, heat, and wave equations.

With the extension to distributions, the scope of the Fourier transform increased substantially. By considering the Sobolev spaces of functions with distributional derivatives in L^2 up to a certain order, a control on the smoothness properties of these functions is obtained. The reason is that the Fourier transform, which changes differentiation into multiplication, is an L^2 isometry and L^2 is a Hilbert space.

With the fundamental properties of the Fourier transform of distributions, we present an introduction to the theory of Sobolev spaces.[1] For

[1] Around 1930 the Russian mathematician Sergei L'vovich Sobolev introduced his space $W^{1,2}(\Omega)$, or $H^1(\Omega)$, with the use of weak derivatives as the natural Hilbert space for solving the Laplace or Poisson equation $-\triangle u = f$ with boundary conditions. A little later, in France Jean Leray considered a similar method to find weak solutions for the Navier-Stokes equation.

completeness, we give the basic definitions in the L^p setting but, in fact, we only use the Hilbert space case, $p = 2$.

The estimates given by the Sobolev norms are a standard tool to prove the existence and regularity of solutions for partial differential equations. We illustrate this by means of an application to the Dirichlet problem and by studying the eigenfunctions of the Laplacian, and we include Rellich's compactness theorem, a result which is of great importance in the applications.

7.1. The Fourier integral

For each $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$, e_ξ is the complex sinusoidal defined on $\mathbf{R}^n$ by

$$e_\xi(x) := \exp(2\pi i x \cdot \xi) = \exp\Big(2\pi i \sum_{k=1}^{n} x_k \xi_k\Big).$$

The **Fourier integral** $\widehat{f}$ of a function $f \in L^1(\mathbf{R}^n)$ is defined by

$$\widehat{f}(\xi) := \int_{\mathbf{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx = \langle f, e_{-\xi}\rangle.$$

The **Fourier transform** $\mathcal{F} : L^1(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n) \cap \mathcal{C}(\mathbf{R}^n)$, such that $\mathcal{F}f = \widehat{f}$, is a linear mapping and $\|\widehat{f}\|_\infty \le \|f\|_1$ (i.e., $\|\mathcal{F}\| \le 1$).

An application of Fubini's theorem, derivation under the integral, partial integration and elementary changes of variables show that the following useful properties of the Fourier transform hold on $L^1(\mathbf{R}^n)$:

(a) $\widehat{\tau_a f} = e_{-a}\widehat{f}$ and $\widehat{e_a f} = \tau_a \widehat{f}$.

(b) $\big[h^{-n} f(h^{-1}x)\big]\widehat{\ }(\xi) = \widehat{f}(h\xi)$ and $[f(hx)]\widehat{\ }(\xi) = h^{-n}\widehat{f}(h^{-1}\xi)$.

(c) $\widehat{f * g} = \widehat{f}\cdot\widehat{g}$ and $\int_{\mathbf{R}^n} f(y)\widehat{g}(y)\, dy = \int_{\mathbf{R}^n} \widehat{f}(y) g(y)\, dy$.

(d) $\partial_1 \widehat{f}(\xi) = (-2\pi i)[\widehat{x_1 f(x)}](\xi)$, if $x_1 f(x)$ is also integrable.

(e) $\widehat{\partial_1 f}(\xi) = 2\pi i \xi_1 \widehat{f}(\xi)$, if f is of class C^1 and $\partial_1 f$ is also integrable.

Note that to check (e) we can suppose $n = 1$. Then $f(t) = \int_0^t f'(x)\, dx + f(0)$ and $\lim_{t\to\pm\infty} f(t) = \int_0^t f'(x)\, dx + f(0)$ exists and is finite, since f' is assumed to be integrable, and this limit has to be 0, since f is also integrable. Integration by parts gives

$$\int_{-\infty}^{+\infty} f'(t) e^{-2\pi i\xi t}\, dt = f(t)e^{-2\pi i \xi t}\Big]_{t=-\infty}^{t=+\infty} + 2\pi i\xi \int_{-\infty}^{+\infty} f(t) e^{-2\pi i \xi t}\, dt$$

with $f(t)e^{-2\pi i\xi t}]_{t=-\infty}^{t=+\infty} = 0$.

The **inverse Fourier transform** is the mapping $\widetilde{\mathcal{F}}$ such that

$$(\widetilde{\mathcal{F}}f)(\xi) := \int_{\mathbf{R}^n} f(x)e^{2\pi i x\cdot\xi}\,dx = \langle f, e_\xi\rangle = \widehat{f}(-\xi).$$

Example 7.1. The Fourier integral of **the square wave**, $\chi_{(-1/2,1/2)}$, is the function sinc defined as

$$\operatorname{sinc}(\xi) := \frac{\sin(\pi\xi)}{\pi\xi}$$

since

$$\int_{-1/2}^{1/2} e^{-2\pi i\xi x}\,dx = \int_{-1/2}^{1/2} \cos(2\pi\xi x)\,dx - i\int_{-1/2}^{1/2} \sin(2\pi\xi x)\,dx = \operatorname{sinc}(\xi).$$

This function plays an important role in signal analysis and will appear again in Theorem 7.18.

Example 7.2. If W is the function defined on $\mathbf{R}^n$ by $W(x) = e^{-\pi|x|^2}$, then $\widehat{W} = W$ and

$$\int_{\mathbf{R}^n} e^{-\pi a|x|^2}e^{-2\pi i x\cdot\xi}\,dx = \frac{1}{a^{n/2}}e^{-\pi|\xi|^2/a} \tag{7.1}$$

for every $a > 0$.

If $n = 1$, from $W'(t) = -2\pi t W(t)$, the Fourier transform gives

$$2\pi\xi\widehat{W}(\xi) + (\widehat{W})'(\xi) = 0$$

and then

$$(e^{\pi\xi^2}\widehat{W}(\xi))' = 2\pi\xi\widehat{W}(\xi)e^{\pi\xi^2} + (\widehat{W})'(\xi)e^{\pi\xi^2} = 0.$$

Thus $e^{\pi\xi^2}\widehat{W}(\xi) = K$, a constant. The value of this constant is obtained from the Euler-Gauss integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ with the substitution $x = \sqrt{\pi}\,t$, which gives

$$K = \widehat{W}(0) = \int_{-\infty}^{\infty} e^{-\pi t^2}\,dt = 1,$$

so that $\widehat{W}(\xi) = W(\xi)$.

For n variables, $W(x) := e^{-\pi|x|^2} = W(x_1)\cdots W(x_n)$, and also $\widehat{W} = W$.

Another simple change of variables or an application of property (b) of the Fourier transform yields (7.1).

The operators $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ will be extended to certain distributions, known as temperate distributions, and their extensions will essentially keep properties (a)–(e).

To show why $\mathcal{F}$ is a valuable tool for solving some partial differential equations, consider the example of an initial value problem for the heat equation on $[0,\infty)\times\mathbf{R}^n$,

$$\partial_t u(t,x) - \triangle u(t,x) = 0, \qquad u(0,x) = f(x), \tag{7.2}$$

with $\triangle = \sum_{j=1}^n \partial_{x_j}^2$, and formally apply $\mathcal{F}$ with respect to the variable $x \in \mathbf{R}^n$. By property (d),

$$\partial_t \widehat{u}(t,\xi) + 4\pi^2|\xi|^2\widehat{u}(t,\xi) = 0, \qquad \widehat{u}(0,\xi) = \widehat{f}(\xi),$$

and, taking ξ as a parameter, we note that this is a very simple initial value problem for an ordinary linear differential equation whose solution is

$$\widehat{u}(t,\xi) = \widehat{f}(\xi)e^{-4\pi^2|\xi|^2 t}.$$

According to Example 7.2, if $a = 1/(4\pi t)$,

$$e^{-4\pi^2|\xi|^2 t} = (4\pi t)^{-n/2}\int_{\mathbf{R}^n} e^{-|x|^2/4t}e^{-2\pi i x\cdot\xi}\,dx$$

so that, by property (c),

$$\widehat{u}(t,\xi) = \widehat{f}(\xi)\widehat{W_t}(\xi) = \widehat{f * W_t}(\xi)$$

if $W_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Hence, the function

$$u(t,x) = (f * W_t)(x) = \frac{1}{(4\pi t)^{n/2}}\int_{\mathbf{R}^n} e^{-|y|^2/4t}f(x-y)\,dy \tag{7.3}$$

is a candidate for a solution of (7.2).

Note that

$$W_t(x) = \frac{1}{(\sqrt{4\pi t})^n}W\big(\frac{x}{\sqrt{4\pi t}}\big)$$

is a summability kernel, known as the **Gauss-Weierstrass kernel**, associated to the positive integrable function W, so that the initial value condition $\lim_{t\downarrow 0} f * W_t = f$ will hold.[2]

Moreover, equation (7.3) suggests $\Gamma(t,x) = W_t(x)Y(t)$ as a possible fundamental solution of the heat operator, which is the case as we have seen in Theorem 6.29.[3]

The following **Poisson theorem** relates Fourier integrals with Fourier series:

[2] See the details in Exercise 7.2.

[3] This was how Poisson constructed solutions for the heat equation in the work contained in his “Théorie mathématique de la chaleur” (1835).

Theorem 7.3. *Suppose that $f \in L^1(\mathbf{R})$ satisfies the condition*

$$\sum_{k=-\infty}^{\infty} \left|\widehat{f}\Big(\frac{k}{T}\Big)\right| < \infty. \tag{7.4}$$

Then there exists a continuous T-periodic function f_T on $\mathbf{R}$ such that

$$f_T(t) = \sum_{k=-\infty}^{\infty} f(t-kT) \ \text{a.e.}$$

and

$$f_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}\widehat{f}\Big(\frac{k}{T}\Big)e^{2\pi ikt/T},$$

a series which is uniformly convergent on $\mathbf{R}$, that is, in $\mathcal{C}_T(\mathbf{R})$.

Proof. Denote $L = T/2$. On $[-L, L)$ (and on every $[kT - L, kT + L)$), we can define a.e. the periodic function

$$f_T(t) := \sum_{k=-\infty}^{\infty} f(t-kT),$$

with convergence in $L^1(-L, L)$, since

$$\int_{-L}^{L} \sum_{k=-\infty}^{\infty} |f(t-kT)|\, dt = \|f\|_1 < \infty$$

and then $\sum_{k=-\infty}^{\infty} |f(t-kT)| < \infty$ a.e.

The Fourier coefficients of $f_T \in L^1_T(\mathbf{R})$ are

$$c_k(f_T) = \frac{1}{T}\widehat{f}\Big(\frac{k}{T}\Big),$$

since

$$\begin{aligned} c_k(f_T) &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-L}^{L} f(t-kT)e^{-2k\pi it/T}\, dt \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-L-kT}^{L-kT} f(t)e^{-2k\pi it/T}\, dt \\ &= \frac{1}{T}\int_{\mathbf{R}} f(t)e^{-2k\pi it/T}\, dt. \end{aligned}$$

From condition (7.4) and by the M-test of Weierstrass, it follows that the Fourier series is absolutely and uniformly convergent to a continuous function which coincides with f_T a.e. □

If the support of $\widehat{f}$ is compact, or if there are two constants $A, \delta > 0$ such that

$$|\widehat{f}(\xi)| \le A(1+|\xi|)^{-1-\delta},$$

then the integrable function f satisfies condition (7.4).

The function f_T is called the periodized extension of f. If $\operatorname{supp} f \subset [-T/2, T/2]$, f_T is the T-periodic extension of the restriction of f to the interval $[-T/2, T/2]$.

7.2. The Schwartz class $\mathcal{S}$

To define the Fourier transform of distributions, instead of $\mathcal{D}(\mathbf{R}^n)$ we need to consider a class of C^∞ functions that is invariant under the Fourier transform. Properties (c) and (d) of the Fourier transform suggest that we consider the complex vector space

$$\mathcal{S}(\mathbf{R}^n) := \big\{\varphi \in \mathcal{E}(\mathbf{R}^n);\ q_N(\varphi) < \infty \ \text{ for } N = 0, 1, 2, \dots\big\},$$

where

$$q_N(\varphi) := \sup_{x\in\mathbf{R}^n;\, |\alpha|\le N} (1+|x|^2)^N |D^\alpha \varphi(x)|.$$

Note that $|x^\alpha| \le (1+|x|^2)^N$ if $|\alpha| \le N$, so that $\varphi \in \mathcal{E}(\mathbf{R}^n)$ is in $\mathcal{S}(\mathbf{R}^n)$ if and only if, for every couple P, Q of polynomials, the function $P(x)Q(D)\varphi(x)$ is bounded.

The topology of $\mathcal{S}(\mathbf{R}^n)$ is defined by the sequence $q_0 \le q_1 \le q_2 \le \cdots$ of norms, so that the convergence $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbf{R}^n)$ is the uniform convergence on $\mathbf{R}^n$

$$x^\beta D^\alpha \varphi_k(x) \to x^\beta D^\alpha \varphi(x)$$

for all $\alpha, \beta \in \mathbf{N}^n$, which is equivalent to the uniform convergence

$$P \cdot Q(D)\varphi_k \to P \cdot Q(D)\varphi$$

for every couple P, Q of polynomials.

The following theorem collects some basic properties of this new space.

Theorem 7.4. *$\mathcal{S}(\mathbf{R}^n)$, with the topology defined by the increasing sequence of norms $\{q_N\}_{N=0}^\infty$, is a Fréchet space.*

The inclusions $\mathcal{D}(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n) \subset L^1(\mathbf{R}^n)$ are continuous (that is, for every compact set $K \subset \mathbf{R}^n$, the mappings $\mathcal{D}_K(\mathbf{R}^n) \hookrightarrow \mathcal{S}(\mathbf{R}^n) \hookrightarrow L^1(\mathbf{R}^n)$ are continuous), and $\mathcal{D}(\mathbf{R}^n)$ is dense in $\mathcal{S}(\mathbf{R}^n)$.

The differential operators $P(D)$, the multiplication by polynomials, translations, dilations, the symmetry, and every modulation or multiplication by a complex sinusoidal e_a are also continuous linear mappings of $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}(\mathbf{R}^n)$.

Proof. If $\{\varphi_k\} \subset \mathcal{S}(\mathbf{R}^n)$ is such that every $\{(1+|x|^2)^N D^\alpha \varphi_k(x)\}_{k=1}^\infty$ is a uniformly Cauchy sequence, then $(1+|x|^2)^N D^\alpha \varphi_k(x) \to \varphi_{N,\alpha}(x)$ uniformly as $k \to \infty$, so that $\varphi = \varphi_{0,0} \in \mathcal{E}(\mathbf{R}^n)$, since $D^\alpha \varphi_k(x) \to \varphi_{0,\alpha}(x)$ uniformly and then $\varphi \in \mathcal{E}(\mathbf{R}^n)$ and $\varphi_{0,\alpha}(x) = D^\alpha \varphi(x)$.

Hence $(1+|x|^2)^N D^\alpha \varphi_k(x) \to (1+|x|^2)^N D^\alpha \varphi(x)$ uniformly, which means that $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbf{R}^n)$.

Note that

$$\int_{\mathbf{R}^n} \frac{1}{(1+|x|^2)^N}\, dx = \omega_n \int_0^\infty \frac{r^{n-1}}{(1+r^2)^N}\, dr < \infty$$

if $n-1-2N < -1$.

Then, if $N > n/2$ and $\varphi \in \mathcal{S}(\mathbf{R}^n)$,

$$\|\varphi\|_1 \leq \int_{\mathbf{R}^n} q_N(\varphi)(1+|x|^2)^{-N}\, dx = C q_N(\varphi)$$

and $\mathcal{S}(\mathbf{R}^n) \hookrightarrow L^1(\mathbf{R}^n)$ is continuous.

Recall that the topology of $\mathcal{D}_K(\mathbf{R}^n)$ is defined by the increasing sequence of norms

$$p_N(\varphi) := \sup_{|\alpha|\leq N} \|D^\alpha \varphi\|_K = \sup_{|\alpha|\leq N} \|D^\alpha \varphi\|_{\mathbf{R}^n},$$

so that, if $\varphi \in \mathcal{D}_K(\mathbf{R}^n)$,

$$q_N(\varphi) \leq \sup_{x\in K}(1+|x|^2)^N p_N(\varphi),$$

and $\mathcal{D}_K(\mathbf{R}^n) \hookrightarrow \mathcal{S}(\mathbf{R}^n)$ is continuous.

To prove that $\mathcal{D}(\mathbf{R}^n)$ is dense in $\mathcal{S}(\mathbf{R}^n)$, let $\varphi \in \mathcal{S}(\mathbf{R}^n)$. To find functions $\varphi_N \in \mathcal{D}(\mathbf{R}^n)$ such that $\varphi_N \to \varphi$ in $\mathcal{S}(\mathbf{R}^n)$, choose $\bar{B}(0,1) \prec \varrho \prec \mathbf{R}^n$ and define $\varrho_N(x) = \varrho(N^{-1}x)$ $(N \in \mathbf{N})$, so that $\bar{B}(0,N) \prec \varrho_N \prec \mathbf{R}^n$.

Then, if $\varphi_N = \varrho_N \varphi$,

$$\begin{aligned} |D^\alpha(\varphi - \varphi_N)(x)| &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta}\varphi(x)||D^\beta(1-\varrho_N)(x)| \\ &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta}\varphi(x)| N^{-|\beta|} \sup_{x\in\mathbf{R}^n} |D^\beta(1-\varrho_N)(x/N)|, \end{aligned}$$

where we can select N so that $\sup_{|x|\geq N} |D^{\alpha-\beta}\varphi(x)| \leq \varepsilon$ for all $\beta \leq \alpha$, since $D^{\alpha-\beta}\varphi \in \mathcal{S}(\mathbf{R}^n)$. Note that $D^\beta(1-\varrho_N)(x) = 0$ if $|x| \leq N$, and we obtain

$$\sup_{x\in\mathbf{R}^n} |D^\alpha(\varphi - \varphi_N)(x)| \leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \varepsilon N^{-|\beta|} \sup_{|x|\geq N} |D^\beta(1-\varrho_N)(x/N)| \leq C_{N,\alpha}\varepsilon.$$

A similar estimate holds for every $\sup_{x\in\mathbf{R}^n}(1+|x|^2)^m|D^\alpha(\varphi-\varphi_N)(x)|$, since also

$$\sup_{|x|\geq N}(1+|x|^2)^m|D^{\alpha-\beta}\varphi(x)|\leq\varepsilon.$$

Thus, $q_m(\varphi-\varphi_N)\to 0$ if $N\to\infty$.

We leave the remaining part of the proof as an easy exercise. □

Theorem 7.5 (Inversion Theorem)**.** *The Fourier transform is a continuous bijective linear operator $\mathcal{F}:\mathcal{S}(\mathbf{R}^n)\to\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{F}^{-1}=\widetilde{\mathcal{F}}$.*

Proof. For every $\varphi\in\mathcal{S}(\mathbf{R}^n)$, the functions $D^\beta x^\alpha\varphi(x)$ are also in $\mathcal{S}(\mathbf{R}^n)$ and it follows from the properties of $\mathcal{F}$ that $\xi^\beta D^\alpha\widehat{\varphi}(\xi)$ are bounded and $\widehat{\varphi}\in\mathcal{S}(\mathbf{R}^n)$.

To prove that $\mathcal{F}:\mathcal{S}(\mathbf{R}^n)\to\mathcal{S}(\mathbf{R}^n)$ is continuous, note that

$$|\widehat{\varphi}(\xi)|\leq\int_{\mathbf{R}^n}|\varphi(x)|(1+|x|^2)^N(1+|x|^2)^{-N}\,dx\leq Cq_N(\varphi)\qquad(N>n/2),$$

so that, if $\varphi_k\to 0$ and $\widehat{\varphi}_k\to\psi$ in $\mathcal{S}(\mathbf{R}^n)$, then $\widehat{\varphi}_k(\xi)\to 0$ and $\psi(\xi)=0$, and the continuity now follows from the closed graph theorem.

To show that $\widetilde{\mathcal{F}}\mathcal{F}=I$, if we try a direct calculation of $\widetilde{\mathcal{F}}(\mathcal{F}\varphi)$, the integral

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\widehat{f}(x)e^{2\pi i(z-x)\cdot\xi}\,d\xi\,dx$$

is not absolutely convergent on $\mathbf{R}^{2n}$.

We avoid this problem by using properties (b) and (c) of the Fourier transform to obtain

$$\int_{\mathbf{R}^n}f(y)\widehat{g}(hy)\,dy=\int_{\mathbf{R}^n}f(y)[h^{-n}g(h^{-1}x)]\widehat{\ }(y)\,dy=\int_{\mathbf{R}^n}\widehat{f}(y)h^{-n}g(h^{-1}y)\,dy$$

which, with the substitution $hy=x$, becomes

$$\frac{1}{h^n}\int_{\mathbf{R}^n}f(h^{-1}x)\widehat{g}(x)\,dx=\int_{\mathbf{R}^n}\widehat{f}(y)\frac{1}{h^n}g(h^{-1}y)\,dy;$$

that is, $\int_{\mathbf{R}^n}f(h^{-1}x)\widehat{g}(x)\,dx=\int_{\mathbf{R}^n}\widehat{f}(y)g(h^{-1}y)\,dy$. By letting $h\to\infty$, the dominated convergence theorem gives

$$\int_{\mathbf{R}^n}f(0)\widehat{g}(x)\,dx=\int_{\mathbf{R}^n}\widehat{f}(y)g(0)\,dy.$$

If we choose $g=W$, since $\int_{\mathbf{R}^n}\widehat{W}(x)\,dx=\int_{\mathbf{R}^n}W(x)\,dx=1$ and $W(0)=1$,

$$f(0)=\int_{\mathbf{R}^n}\widehat{f}(y)\,dy.$$

An application of this identity to $f(x) = (\tau_{-x} f)(0)$ combined with property (a) gives

$$f(x) = \int_{\mathbf{R}^n} \widehat{\tau_{-x} f}(y)\, dy = \int_{\mathbf{R}^n} e_x(y) \widehat{f}(y)\, dy,$$

which is $f = \widetilde{\mathcal{F}}\mathcal{F} f$. The identity $f = \mathcal{F}\widetilde{\mathcal{F}} f$ is similar. □

As an application, let us present a new proof of the Riemann-Lebesgue lemma.

Corollary 7.6 (Riemann-Lebesgue)**.** If $f \in L^1(\mathbf{R}^n)$, then $\widehat{f} \in \mathcal{C}_0(\mathbf{R}^n)$; that is,

$$\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0.$$

Proof. We know that $\mathcal{D}(\mathbf{R}^n)$ is dense in $L^1(\mathbf{R}^n)$ (see Exercise 6.2) and, if $\varphi_k \to f$ in $L^1(\mathbf{R}^n)$, then $\widehat{\varphi}_k \to \widehat{f}$ uniformly with $\widehat{\varphi}_k \in \mathcal{S}(\mathbf{R}^n)$ so that they are null at infinity. □

Not only multiplication by a polynomial and by e_a is continuous on $\mathcal{S}(\mathbf{R}^n)$:

Theorem 7.7. *If $\psi \in \mathcal{S}(\mathbf{R}^n)$ and $\omega_s(x) := (1 + |x|^2)^{s/2}$ $(s \in \mathbf{R})$, the pointwise multiplications $\psi\cdot$ and $\omega_s\cdot$ and the convolution $\psi *$ are continuous linear operators of $\mathcal{S}(\mathbf{R}^n)$.*

Proof. Let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ and note that, if $|\alpha| \le N$,

$$(1+|x|^2)^N |D^\alpha(\psi\varphi)| \le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (1+|x|^2)^N |D^{\alpha-\beta}\psi(x)||D^\beta\varphi(x)| \le C q_N(\varphi),$$

since every $|D^{\alpha-\beta}\psi(x)|$ is bounded. The case of $\omega_s\cdot$ is similar, since $|\omega_s\varphi| \le (1+|x|^2)^N|\varphi(x)|$ and $|\partial_j\omega_s(x)| \le |sx_j\omega_{s-2}(t)|$.

Finally, $\mathcal{F}(\psi * \varphi) = \mathcal{F}(\psi)\mathcal{F}(\varphi)$, so that $\psi * \varphi = \widetilde{\mathcal{F}}(\mathcal{F}(\psi)\mathcal{F}(\varphi))$ and $\psi *$ is the product of continuous operators. □

7.3. Tempered distributions

If $\mathcal{S}'(\mathbf{R}^n)$ is the topological dual space of $\mathcal{S}(\mathbf{R}^n)$, it follows from Theorem 7.4 that the mapping $u \in \mathcal{S}'(\mathbf{R}^n) \mapsto v = u_{|\mathcal{D}(\mathbf{R}^n)} \in \mathcal{D}'(\mathbf{R}^n)$ is linear and one-to-one.

A distribution $v \in \mathcal{D}'(\mathbf{R}^n)$ is the restriction of an element $u \in \mathcal{S}'(\mathbf{R}^n)$ if and only if there exist $N \in \mathbf{N}$ and a constant $C_N > 0$ such that

$$|u(\varphi)| \le C_N q_N(\varphi) = C_N \sup_{x\in\mathbf{R}^n;\, |\alpha|\le N} (1+|x|^2)^N |D^\alpha\varphi(x)| \qquad (\varphi \in \mathcal{D}(\mathbf{R}^n)).$$

The elements of $\mathcal{S}'(\mathbf{R}^n)$, or their restrictions to $\mathcal{D}(\mathbf{R}^n)$, are called **tempered distributions**. On $\mathcal{S}'(\mathbf{R}^n)$ we consider the topology w^*, so that $u_k \to u$ in $\mathcal{S}'(\mathbf{R}^n)$ means that $\langle \varphi, u_k\rangle \to \langle \varphi, u\rangle$ for every $\varphi \in \mathcal{S}(\mathbf{R}^n)$.

It is customary to identify every $u \in \mathcal{S}'(\mathbf{R}^n)$ with its restriction $v \in \mathcal{D}'(\mathbf{R}^n)$, so that

$$\mathcal{S}'(\mathbf{R}^n) \subset \mathcal{D}'(\mathbf{R}^n).$$

Example 7.8. Suppose $1 \le p < \infty$ and $N \in \mathbf{N}$. If $(1+|x|^2)^{-N} f(x)$ is in $L^p(\mathbf{R}^n)$, then, as a distribution, $f \in \mathcal{S}'(\mathbf{R}^n)$. In particular,

$$L^p(\mathbf{R}^n) \hookrightarrow \mathcal{S}'(\mathbf{R}^n) \qquad (1 \le p \le \infty),$$

and this inclusion is continuous.

If $p = 1$,

$$\begin{aligned} |\langle \varphi, f\rangle| &= \int_{\mathbf{R}^n} (1+|x|^2)^{-N} f(x)(1+|x|^2)^N \varphi(x)\,dx \\ &\le \int_{\mathbf{R}^n} (1+|x|^2)^{-N} f(x) q_N(\varphi)\,dx \\ &= C_N q_N(\varphi). \end{aligned}$$

If $p > 1$, using Hölder's inequality, we obtain

$$|\langle \varphi, f\rangle| \le C_N^{1/p} \Big(\int_{\mathbf{R}^n} \big|(1+|x|^2)^N \varphi(x)\big|^{p'}\,dx \Big)^{1/p'}$$

and, if M is such that $\int_{\mathbf{R}^n} (1+|x|^2)^{N-M}\,dx = C < \infty$, a division and multiplication by $(1+|x|^2)^M$ give

$$|\langle \varphi, f\rangle| \le C_N^{1/p} C^{1/p'} \sup_{x\in\mathbf{R}^n} (1+|x|^2)^M |\varphi(x)| \le K q_M(\varphi).$$

Example 7.9. Every distribution with compact support, $u \in \mathcal{E}'(\mathbf{R}^n)$, is a tempered distribution.

The inclusion $\mathcal{S}(\mathbf{R}^n) \hookrightarrow \mathcal{E}(\mathbf{R}^n)$ is continuous, since the topology of $\mathcal{E}(\mathbf{R}^n)$ is defined by the seminorms

$$p_{K,N}(\varphi) = \sup_{|\alpha|\le N} \|D^\alpha \varphi\|_K$$

and $p_{K,N}(\varphi) \le q_N(\varphi)$ if $\varphi \in \mathcal{S}(\mathbf{R}^n)$.

Moreover $\mathcal{S}(\mathbf{R}^n)$ is dense in $\mathcal{E}(\mathbf{R}^n)$, since $\mathcal{D}(\mathbf{R}^n)$ is dense, and the restriction of $u \in \mathcal{E}'(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$ is a tempered distribution.

Let P be a polynomial with constant coefficients, $\psi \in \mathcal{S}(\mathbf{R}^n)$, and $u \in \mathcal{S}'(\mathbf{R}^n)$. As in the case of general distributions, we define Pu, $P(D)u$, and ψu by

$$\langle \varphi, P(D)u\rangle = \langle P(-D)\varphi, u\rangle, \quad \langle \varphi, Pu\rangle = \langle P\varphi, u\rangle, \quad \langle \varphi, \psi u\rangle = \langle \psi\varphi, u\rangle,$$

where $P(-D) = \sum_{|\alpha|\le N} c_\alpha (-1)^{|\alpha|} D^\alpha$ if $P(x) = \sum_{|\alpha|\le N} c_\alpha x^\alpha$ (the substitution of every $x_j^{\alpha_j}$ by $(-\partial_j)^{\alpha_j}$, and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$).

They belong to $\mathcal{S}'(\mathbf{R}^n)$, since they are the composition of continuous linear mappings. For instance, $P(D)u : \varphi \mapsto P(-D)\varphi \mapsto u(P(-D)\varphi)$, where $P(-D)$, $P\cdot$, and $\psi\cdot$ are continuous on $\mathcal{S}(\mathbf{R}^n)$, their transposes are $P(D)$, $P\cdot$, and $\psi\cdot$, and they are continuous linear operators of $\mathcal{S}'(\mathbf{R}^n)$.

Also translations, modulations, dilations, and the symmetry defined, respectively, by

$$\langle \varphi, \tau_a u\rangle = \langle \tau_{-a}\varphi, u\rangle, \quad \langle \varphi, e_a u\rangle = \langle e_a\varphi, u\rangle,$$
$$\langle \varphi(x), u(hx)\rangle = \langle h^{-n}\varphi(h^{-1}x), u(x)\rangle, \quad \langle \varphi, \tilde{u}\rangle = \langle \tilde{\varphi}, u\rangle,$$

are continuous linear mappings of $\mathcal{S}'(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^n)$.

7.3.1. Fourier transform of tempered distributions. Property (c) of the Fourier integral on $L^1(\mathbf{R}^n)$ suggests that we may also define the Fourier transform $\widehat{u}$ of any tempered distribution u by

$$\langle \varphi, \widehat{u}\rangle := \langle \widehat{\varphi}, u\rangle \qquad (\varphi \in \mathcal{S}(\mathbf{R}^n)).$$

Since $\mathcal{F}$ is continuous on $\mathcal{S}(\mathbf{R}^n)$, $\widehat{u} = u \circ \mathcal{F} \in \mathcal{S}'(\mathbf{R}^n)$ for any $u \in \mathcal{S}'(\mathbf{R}^n)$. We still write $\mathcal{F}u = \widehat{u}$.

Similarly, $\widetilde{\mathcal{F}}$ is defined on $\mathcal{S}'(\mathbf{R}^n)$ by $\langle \varphi, \widetilde{\mathcal{F}}u\rangle := \langle \widetilde{\mathcal{F}}\varphi, u\rangle$.

Theorem 7.10. *The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ is a bijective continuous linear extension of $\mathcal{F} : L^1(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$. The inverse of $\mathcal{F}$ on $\mathcal{S}'(\mathbf{R}^n)$ is $\widetilde{\mathcal{F}}$.*

The behavior of $\mathcal{F}$ on $L^1(\mathbf{R}^n)$ and on $\mathcal{S}(\mathbf{R}^n)$ with respect to derivatives, translations, modulations, dilations, and symmetry extends to the Fourier transform on $\mathcal{S}'(\mathbf{R}^n)$, and $\widetilde{\mathcal{F}}u = \widetilde{\mathcal{F}u}$.

Proof. If $f \in L^1(\mathbf{R}^n)$ and $u_f = \langle \cdot, f\rangle$, then $\mathcal{F}u_f = \langle \cdot, \widehat{f}\rangle$.

As the transpose of $\mathcal{F} : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$, $\mathcal{F} : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ is weakly continuous. By property (c) of the Fourier integral, if $f \in L^1(\mathbf{R}^n)$, then $\mathcal{F}u_f = u_{\widehat{f}}$ and $\mathcal{F} : L^1(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$ is the restriction of $\mathcal{F} : \mathcal{S}'(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$.

Also, $\widetilde{\mathcal{F}}\mathcal{F} = \mathrm{Id}$ and $\mathcal{F}\widetilde{\mathcal{F}} = \mathrm{Id}$, since

$$\langle \varphi, \widetilde{\mathcal{F}}\mathcal{F}u\rangle = \langle \mathcal{F}\widetilde{\mathcal{F}}\varphi, u\rangle = \langle \varphi, u\rangle.$$

Let us consider the behavior of $\mathcal{F}$ with respect to dilations:

$$\begin{aligned}
\langle \varphi(y), \mathcal{F}[u(hx)](y)\rangle &= \langle (\mathcal{F}\varphi)(x), u(hx)\rangle = \langle h^{-n}(\mathcal{F}\varphi)(h^{-1}x), u(x)\rangle \\
&= \langle \mathcal{F}[\varphi(hy)](x), u(x)\rangle = \langle \varphi(hy), (\mathcal{F}u)(y)\rangle \\
&= \langle \varphi(y), h^{-n}(\mathcal{F}u)(h^{-1}y)\rangle,
\end{aligned}$$

and $\widehat{u(hx)}$ is the distribution $h^{-n}\widehat{u}(h^{-1}y)$. We leave it to the reader to check the remaining statements.

For differential operators, $P(D) = \sum_{|\alpha|\le m} c_\alpha D^\alpha$ ($c_\alpha \in \mathbf{C}$),

$$\langle \varphi, \mathcal{F}P(D)u\rangle = \langle P(-D)\mathcal{F}\varphi, u\rangle = \langle P(2\pi i\xi)\mathcal{F}\varphi(\xi), u(\xi)\rangle$$

and $\mathcal{F}P(D)u$ is the distribution $P(2\pi i\xi)\mathcal{F}u(\xi)$. □

Example 7.11. $\widehat{1} = \delta$, since $\langle \widehat{\varphi}, 1\rangle = \int_{\mathbf{R}^n} \widehat{\varphi}(\xi)\, d\xi = \varphi(0) = \langle \varphi, \delta\rangle$. As a modulation of 1, the Fourier transform in $\mathcal{S}'(\mathbf{R}^n)$ of the function $e^{2k\pi i a\cdot x}$ is $\tau_a\delta = \delta_a$.

Similarly, $\widehat{\delta} = 1$, and $\widehat{\delta}_a(\xi) = e^{2k\pi i a\cdot\xi}$.

Example 7.12. Every $f \in L^2_T(\mathbf{R})$ is a tempered distribution such that, in $\mathcal{S}'(\mathbf{R})$,

$$f = \sum_{k=-\infty}^{\infty} c_k(f) e^{2k\pi i t/T} \quad \text{and} \quad \widehat{f} = \sum_{k=-\infty}^{+\infty} c_k(f)\delta_{k/T}.$$

Here

$$c_k(f) = \frac{1}{T}\int_a^{a+T} f(t)e^{-2\pi i k t/T}\, dt \qquad (k \in \mathbf{Z})$$

are the Fourier coefficients of f.

Since

$$\int_{\mathbf{R}} \frac{|f(t)|^2}{1+t^2}\, dt \le \sum_{k=-\infty}^{+\infty} \int_{kT}^{(k+1)T} \frac{|f(t)|^2}{1+k^2}\, dt = C\|f\|_2^2,$$

it follows from Example 7.8 that $L^2_T(\mathbf{R}) \hookrightarrow \mathcal{S}'(\mathbf{R})$, continuously. Then

$$f = \sum_{k=-\infty}^{\infty} c_k(f) e^{2k\pi i t/T}$$

in $\mathcal{S}'(\mathbf{R})$, since this is true in $L^2_T(\mathbf{R})$.

But the Fourier transform is linear and continuous in $\mathcal{S}'(\mathbf{R})$, and maps $e^{2k\pi i t/T}$ into $\delta_{k/T}$, so that $f = \sum_{k=-\infty}^{\infty} c_k(f) e^{2k\pi i t/T}$ in $\mathcal{S}'(\mathbf{R})$.[4]

7.3.2. Plancherel Theorem.

We also have $L^p(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$, and the action of $\mathcal{F}$ on $L^2(\mathbf{R}^n)$ is especially important.

Theorem 7.13 (Plancherel[5])**.** *The restrictions of $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ to $L^2(\mathbf{R}^n)$ are linear bijective isometries such that $\widetilde{\mathcal{F}} = \mathcal{F}^{-1}$.*

[4] These results are also true for $f \in L^p_T(\mathbf{R})$ ($p > 1$), since $f = \sum_{k=-\infty}^{\infty} c_k(f)e^{2k\pi i t/T}$ in $L^p_T(\mathbf{R})$.

[5] Named after the Swiss mathematician Michel Plancherel, who in 1910 established conditions under which the theorem holds. It was first used in 1889 by Lord Rayleigh (John William Strutt) in the investigation of blackbody radiation.

Proof. Since $\mathcal{D}(\mathbf{R}^n)$ is dense in $L^2(\mathbf{R}^n)$ ($\langle\varphi, f\rangle = 0$ for every φ implies $f = 0$, so that $\mathcal{D}(\mathbf{R}^n)^\perp = \{0\}$ in $L^2(\mathbf{R}^n)$), the larger subspace $\mathcal{S}(\mathbf{R}^n)$ is also dense.

The identity $\int_{\mathbf{R}^n} \varphi(y)\widehat{g}(y)\, dy = \int_{\mathbf{R}^n} \widehat{\varphi}(y)g(y)\, dy$ holds for all $\varphi, g \in \mathcal{S}(\mathbf{R}^n)$. If $g = \overline{\widehat{\psi}}$, then $\widehat{g} = \bar{\psi}$ and $\int_{\mathbf{R}^n} \varphi(y)\overline{\psi(y)}\, dy = \int_{\mathbf{R}^n} \widehat{\varphi}(y)\overline{\widehat{\psi}(y)}\, dy$, i.e., $(\varphi, \psi)_2 = (\widehat{\varphi}, \widehat{\psi})_2$. This shows that $\mathcal{F}$ is an L^2-isometry on $\mathcal{S}(\mathbf{R}^n)$ that by continuity extends to a unique isometry $\mathcal{F}_2$ of $L^2(\mathbf{R}^n)$.

The restriction of $\mathcal{F}$ on $\mathcal{S}'(\mathbf{R}^n)$ to $\mathcal{F} : L^2(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ is continuous, since the inclusion $L^2(\mathbf{R}^n) \hookrightarrow \mathcal{S}'(\mathbf{R}^n)$ is continuous, and it coincides with $\mathcal{F}_2$ on the dense subspace $\mathcal{S}(\mathbf{R}^n)$; hence $\mathcal{F}_2 = \mathcal{F}$ on $L^2(\mathbf{R}^n)$.

The operator $\widetilde{\mathcal{F}}$, such that $\widetilde{\mathcal{F}}u = \widetilde{\mathcal{F}u}$, has a similar behavior. To show that $\widetilde{\mathcal{F}} = \mathcal{F}^{-1}$ on $L^2(\mathbf{R}^n)$, note that $\widetilde{\mathcal{F}}\mathcal{F} = \mathrm{Id} = \mathcal{F}\widetilde{\mathcal{F}}$ on $\mathcal{S}'(\mathbf{R}^n)$, so that also $\widetilde{\mathcal{F}}\mathcal{F} = \mathrm{Id} = \mathcal{F}\widetilde{\mathcal{F}}$ on $L^2(\mathbf{R}^n)$. $\square$

Remark 7.14. If $f, g \in L^2(\mathbf{R}^n)$, $\int_{\mathbf{R}^n} f(y)\overline{g(y)}\, dy = \int_{\mathbf{R}^n} \widehat{f}(y)\overline{\widehat{g}(y)}\, dy$, but on $L^2(\mathbf{R}^n)$ the Fourier transform can be seen as an improper integral for the convergence in $L^2(\mathbf{R}^n)$,

$$\widehat{f}(\xi) = \lim_{R\uparrow\infty} \int_{B(0,R)} f(x)e^{-2\pi i x\xi}\, dx.$$

Note that $\chi_{B(0,R)}f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ and

$$f = \lim_{R\uparrow\infty} \chi_{B(0,R)}f$$

in $L^2(\mathbf{R}^n)$, so that

$$\widehat{f} = \lim_{R\uparrow\infty} \mathcal{F}(\chi_{B(0,R)}f), \ \text{ with } \mathcal{F}(\chi_{B(0,R)}f)(\xi) = \int_{B(0,R)} f(x)e^{-2\pi i x\xi}\, dx.$$

Obviously, instead of $\chi_{B(0,R)}$ we can use more regular functions, such as $\varrho(R^{-1}x)$ with $\bar{B}(0,1) \prec \varrho \prec \mathbf{R}^n$.[6]

Example 7.15. If $\Omega > 0$ and $h \in \mathbf{R}$, then $\operatorname{sinc}(2\Omega t + h)$ is in $L^2(\mathbf{R})$, and

$$2\Omega[\operatorname{sinc}(2\Omega t + h)]\widehat{\ }(\xi) = e^{\pi i h\xi/\Omega}\chi_{[-\Omega,\Omega]}(\xi).$$

Indeed, since sinc is the Fourier transform of the square wave of Example 7.1, it belongs to $L^2(\mathbf{R})$, and the same happens with $\operatorname{sinc}(2\Omega t + h) = \operatorname{sinc}(2\Omega(t + h/2\Omega))$. By the Plancherel theorem $\widehat{\operatorname{sinc}} = \chi_{[-1/2,1/2]}$ and, using the properties of the Fourier transform,

$$\Big[\operatorname{sinc}\Big(2\Omega(t + \frac{h}{2\Omega})\Big)\Big]\widehat{\ }(\xi) = e^{2\pi i \frac{h}{2\Omega}\xi}[\operatorname{sinc}(2\Omega t)]\widehat{\ }(\xi) = \frac{1}{2\Omega}e^{\pi i \frac{h}{\Omega}\xi}\,\widehat{\operatorname{sinc}}(\xi/2\Omega)$$

[6]If $n = 1$, $\widehat{f}(\xi) = \lim_{M\to\infty} \int_{-M}^{M} f(x)e^{-2\pi i x\xi}\, dx$ for almost all $\xi \in \mathbf{R}$ holds if $f \in L^2$. This result is equivalent to the Carleson theorem on the almost everywhere convergence of Fourier series, one of the most celebrated theorems in Fourier analysis.

with $\widehat{\operatorname{sinc}}(\xi/2\Omega) = \chi_{[-1/2,1/2]}(\xi/2\Omega) = \chi_{[-\Omega,\Omega]}(\xi)$.

We also know from Example 7.9 that the distributions with compact support are tempered, so that we can consider the restriction of $\mathcal{F}$ to the space $\mathcal{E}'(\mathbf{R}^n)$. Let us denote

$$e_z(\xi) := e^{2\pi i z\cdot\xi} \qquad (\xi \in \mathbf{R}^n,\ z \in \mathbf{C}^n).$$

Theorem 7.16. *If $u \in \mathcal{E}'(\mathbf{R}^n)$, then $\widehat{u}$ is the restriction to $\mathbf{R}^n$ of the entire function*

$$F(z) := \langle e_{-z}, u\rangle.$$

For every $\alpha \in \mathbf{N}^n$ there is an integer N such that the function

$$(1+|x|^2)^{-N/2} D^\alpha \widehat{u}(x)$$

is bounded.

Proof. The function F is continuous on $\mathbf{C}$, since $e_z \to e_{z_0}$ in $\mathcal{E}(\mathbf{R}^n)$ as $z \to z_0$. Indeed, $D^\alpha e_z \to D^\alpha e_{z_0}$ uniformly on every compact subset of $\mathbf{R}$, since for one variable

$$e^{2\pi i t z} - e^{2\pi i t z_0} = \int_{[z_0,z]} 2\pi i t e^{2\pi i t\zeta}\, d\zeta.$$

Let us show that F is holomorphic in every variable z_j by an application of Morera's theorem; that is, $\int_\gamma F(z)\,dz_j = 0$ if γ is the oriented boundary of a rectangle in $\mathbf{C}$. By writing the integral as a limit of Riemann sums,

$$\int_\gamma F(z)\,dz_j = \int_\gamma u(e_{-z})\,dz_j = \Big\langle \int_\gamma e_{-z}(t)\,dz_j, u(t)\Big\rangle = 0,$$

since $\int_\gamma e^{-2\pi i z_j t_j}\,dz_j = 0$ for every $t_j \in \mathbf{R}$.

For every $\varphi \in \mathcal{D}_{[a,b]^n}(\mathbf{R}^n)$,

$$\langle \varphi(t), u(e_{-t})\rangle = \int_{[a,b]^n} \langle \varphi(t) e_{-t}, u\rangle\,dt = \Big\langle \int_{[a,b]^n} \varphi(t) e_{-t}(x)\,dt, u(x)\Big\rangle = \langle \widehat{\varphi}, u\rangle$$

and $\widehat{u} = F$ on $\mathbf{R}^n$.

Note that by the continuity of u on $\mathcal{E}(\mathbf{R}^n)$,

$$|D^\alpha \widehat{u}(\xi)| = |(-2\pi i)^{|\alpha|}\langle x^\alpha e_{-\xi}(x), u(x)\rangle| \le C \sup_{|\beta|\le N,\,|x|\le N} |D^\beta(x^\alpha e_{-\xi}(x))|$$

and it follows that $|D^\alpha \widehat{u}(\xi)| \le C'(1+|\xi|^2)^{N/2}$. □

7.4. Fourier transform and signal theory

A first main topic in the digital processing of signals is the analog-to-digital conversion by means of sampling, which changes a continuous time signal $f(t)$ into a discrete time signal $x = \{x[k]\}_{k=-\infty}^{\infty} \subset \mathbf{C}$, $x[k] := f(kT)$.

The band of an analog signal f is the smallest interval $[-\Omega, \Omega]$ which supports its Fourier transform $\widehat{f}$ and we shall see that for a band-limited signal, that is, with $\Omega < \infty$, sampling can be done in an efficient manner. It is worth observing that in this case f is analytic:

We know from Theorem 7.16 that, if $u \in \mathcal{S}'(\mathbf{R})$ has a Fourier transform with compact support $\widehat{u} \in \mathcal{E}'(\mathbf{R})$, then u is the restriction to $\mathbf{R}$ of the entire function,

$$F(z) := \langle e^{2\pi i \xi z}, \widehat{u}(\xi)\rangle.$$

This shows that a signal cannot be simultaneously band-limited and time-limited.[7] Usually, analog signals are of finite time, so they are not of limited band, but they are almost band-limited in the sense that $\widehat{u} \simeq 0$ outside of some finite interval $[-\Omega, \Omega]$. Sometimes, filtering of the analogical signal is convenient in order to reduce it to a band-limited signal.

We will suppose that $f \in L^2(\mathbf{R})$ and $\operatorname{supp} \widehat{f} \subset [-\Omega, \Omega]$, so f is analytic. The minimal value Ω_N of Ω is called the **Nyquist frequency**[8] of f.

Let us consider the T-periodized extension of $\widehat{f}$ with $T = 2\Omega$,

$$\widehat{f}_T(\xi) = \sum_{k=-\infty}^{\infty} \widehat{f}(\xi - kT) \qquad (P = 2\Omega),$$

which is in $L^2_T(\mathbf{R})$. Then

$$c_k(\widehat{f}_T) = \frac{1}{2\Omega} \widehat{\widehat{f}} \left(\frac{k}{2\Omega}\right) = \frac{1}{2\Omega} f\left(\frac{-k}{2\Omega}\right) \tag{7.5}$$

and

$$\widehat{f}_T(\xi) = \sum_{k=-\infty}^{\infty} \frac{1}{2\Omega} f\left(\frac{-k}{2\Omega}\right) e^{\pi i k \xi/\Omega},$$

with $L^2(-\Omega, \Omega)$-convergence of this Fourier series.

According to the inversion theorem, $f(t) = \int_{-\Omega}^{\Omega} \widehat{f}(\xi) e^{2\pi i t \xi}\, d\xi$ for every $x \in \mathbf{R}$, and the scalar product by $e^{-2\pi i t \xi/\Omega}$ on $[-\Omega, \Omega]$ gives the pointwise identity

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\Omega} f\left(\frac{-k}{2\Omega}\right) \int_{-\Omega}^{\Omega} e^{2\pi i (t + k/2\Omega)\xi}\, d\xi$$

[7] This is a version of the uncertainty principle

[8] Named after the Swedish engineer Harry Nyquist, who in 1927 determined that the number of independent pulses that could be put through a telegraph channel per unit of time is limited to twice the bandwidth of the channel.

and, by Example 7.15,

$$\frac{1}{2\Omega}\int_{-\Omega}^{\Omega} e^{2\pi i(t+k/2\Omega)\xi}\,d\xi = \operatorname{sinc}(2\Omega t + k). \tag{7.6}$$

This shows that we obtain a complete reconstruction of $f(t)$,

$$f(t) = \sum_{k=-\infty}^{\infty} f\Big(\frac{k}{2\Omega}\Big)\operatorname{sinc}(2\Omega t - k), \tag{7.7}$$

with pointwise convergence, by sampling with the sampling period $T_m = 1/2\Omega$.

Our aim is to prove that in fact we have uniform and L^2 convergence.

Lemma 7.17. *The functions*

$$\sqrt{2\Omega}\,\operatorname{sinc}(2\Omega t - k) \qquad (k \in \mathbf{Z}) \tag{7.8}$$

form an orthonormal system in $L^2(\mathbf{R})$.

Proof. If

$$\varphi_k := [\operatorname{sinc}(2\Omega t - k)]\widehat{\ } = [\operatorname{sinc}(2\Omega(t - k/2\Omega))]\widehat{\ },$$

according to Example 7.15

$$\varphi_k(\xi) = e^{\pi i k\xi/\Omega}[\operatorname{sinc}(2\Omega t)]\widehat{\ }(\xi) = e^{\pi i k\xi/\Omega}\frac{1}{2\Omega}\chi_{[-\Omega,\Omega]}(\xi).$$

From the Plancherel theorem,

$$([\operatorname{sinc}(2\Omega t - m)], [\operatorname{sinc}(2\Omega t - n)])_2 = (\varphi_m, \varphi_n)_2,$$

and then

$$(2\Omega)^2(\varphi_m, \varphi_n) = \int_{-\Omega}^{\Omega} e^{\pi i(m-n)\xi/\Omega}\,d\xi = 0$$

if $m \neq n$, and $(2\Omega)^2\|\varphi_m\|_2^2 = 2\Omega$, so that

$$\|[\operatorname{sinc}(2\Omega t - m)]\|_2 = 1/\sqrt{2\Omega}.$$

□

The family of functions (7.8) is called the **Shannon system**.

Theorem 7.18 (Shannon[9]). *Suppose* $f \in L^2(\mathbf{R})$ *and* $\operatorname{supp}\widehat{f} \subset [-\Omega, \Omega]$, *so that we can assume that* f *is continuous. Then*

$$f(t) = \sum_{k=-\infty}^{\infty} f\Big(\frac{k}{2\Omega}\Big)\operatorname{sinc}(2\Omega t - k)$$

in $L^2(\mathbf{R})$ *and uniformly.*

[9] Named after the electrical engineer and mathematician Claude Elwood Shannon, the founder of information theory in 1947.

Proof. We have seen in (7.7) that the sequence

$$\begin{aligned} s_N(f,t) &= \sum_{k=-N}^{N} f\Big(\frac{k}{2\Omega}\Big) \operatorname{sinc}(2\Omega t - k) \\ &= \sum_{k=-N}^{N} f\Big(\frac{k}{2\Omega}\Big) \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} e^{2\pi i t\xi} e^{-2\pi i k\xi/2\Omega} \end{aligned}$$

is pointwise convergent to $f(t)$ and that

$$f(t) = \int_{-\Omega}^{\Omega} \widehat{f}(\xi) e^{2\pi i \xi t}\, d\xi.$$

Hence,

$$|f(t) - s_N(f,x)| = \Big| \int_{-\Omega}^{\Omega} \Big\{ \widehat{f}(\xi) - \sum_{k=-N}^{N} \frac{1}{2\Omega} f\Big(\frac{-k}{2\Omega}\Big) e^{2\pi i k\xi/2\Omega} \Big\} e^{2\pi i \xi t} d\xi \Big|$$

and, by Schwarz inequality,

$$|f(t) - s_N(f,x)| \le \Big(\int_{-\Omega}^{\Omega} \Big| \widehat{f}(\xi) - \sum_{k=-N}^{N} \frac{1}{2\Omega} f\Big(\frac{-k}{2\Omega}\Big) e^{2\pi i k\xi/2\Omega} \Big|^2 d\xi \Big)^{1/2} (2\Omega)^{1/2}.$$

We know from (7.5) that the Fourier coefficients of $\widehat{f}_T$ with respect to the trigonometric system are

$$c_k(\widehat{f}_T) = \frac{1}{2\Omega} f\Big(\frac{-k}{2\Omega}\Big).$$

Hence,

$$\sum_{k=-N}^{N} \frac{1}{2\Omega} f\Big(\frac{-k}{2\Omega}\Big) e^{2\pi i k\xi/2\Omega} = S_N(\widehat{f}_T, t),$$

and $S_N(\widehat{f}_T) \to \widehat{f}_T$ in $L^2_T(\mathbf{R})$.

Then,

$$\sup_{t\in\mathbf{R}} |f(t) - s_N(f,x)| \le (2\Omega)^{1/2} \|\widehat{f} - S_N(\widehat{f}_T)\|_{L^2(-\Omega,\Omega)},$$

which yields the uniform convergence.

Since $c(\widehat{f}_T) = \{c_k(\widehat{f}_T)\} \in \ell^2$ and the Shannon system is orthonormal,

$$g(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\Omega} f\Big(\frac{k}{2\Omega}\Big) \sqrt{2\Omega}\, \operatorname{sinc}(2\Omega t - k) \tag{7.9}$$

in $L^2(\mathbf{R})$. But some subsequence of the partial sums is a.e. convergent to g and uniformly convergent to f, so that $g = f$ as elements of $L^2(\mathbf{R})$. □

The Shannon theorem shows that the sampling rate of $\Omega_s = 2\Omega_N$ samples/seg is optimal, and it is called the **Nyquist rate**. If $\Omega_s > 2\Omega_N$, we are considering $\operatorname{supp} \widehat{f} \subset [-\Omega_s/2, \Omega_s/2]$, wider than the symmetric interval which supports $\widehat{f}$ and an unnecessary oversampling if Ω_s is much greater than $2\Omega_N$. If $\Omega_s > 2\Omega_N$, the function g obtained in the sum (7.9) differs from the original signal f and is called an **alias** of f.

In digital processing, the discrete time signals $x = \{x[k]\}_{k=-\infty}^{\infty}$ obtained by sampling from analogical signals are usually of finite time, so that $x[k] = 0$ if $|k| > N$ for some N, but for technical reasons it is convenient to consider more general signals. We say that x is **slowly increasing**, and write $x \in \ell$, if there exist two constants N and C, such that

$$|x[k]| \leq C|k|^N \qquad (k \neq 0).$$

The class ℓ is a vector space with the usual operations and slowly increasing sequences can be considered tempered distributions:

Theorem 7.19. *If $x \in \ell$, then*

$$u_x := \sum_{k=-\infty}^{+\infty} x[k]\delta_k$$

defines a tempered distribution, and the correspondence $x \in \ell \mapsto u_x \in \mathcal{S}'(\mathbf{R})$ is an injective linear mapping which shows that we can consider $\ell \subset \mathcal{S}'(\mathbf{R})$.

The Fourier transform of x as a tempered distribution is

$$\widehat{x} = \sum_{k=-\infty}^{+\infty} x[k]e^{-2\pi ik\xi}.$$

Proof. If $u = \sum_{k=-\infty}^{+\infty} x[k]\delta_k = 0$, for every $n \in \mathbf{Z}$ we can choose $\varphi \in \mathcal{S}$ with support in $(n-1, n+1)$ such that $u(\varphi) = x[n]$. Since

$$\sum_{k=-\infty}^{+\infty} |x[k]\varphi(k)| \leq C \sum_{k \neq 0} |k|^{-2N}|k|^{4N}|\varphi(k)| \leq Kq_{2N}(\varphi), \tag{7.10}$$

$u : \varphi \mapsto \sum_{k=-\infty}^{+\infty} x[k]\varphi(k)$ is linear and continuous on $\mathcal{S}(\mathbf{R})$. It is the limit in $\mathcal{S}'$ of the partial sums $u_N = \sum_{k=-N}^{N} x[k]\delta_k$, $u_N(\varphi) \to u(\varphi)$ if $\varphi \in \mathcal{S}$.

Moreover, $\widehat{u} = \sum_{k=-\infty}^{+\infty} x[k]\widehat{\delta}_k = \sum_{k=-\infty}^{+\infty} x[k]e^{-2\pi ik\xi}$. □

This Fourier transform (7.10) is called the **spectrum** of x.

If $\{x[k]\} \in \ell^2$, we have convergence in $L^2_1(\mathbf{R})$ and $x \in \ell^2 \mapsto \widehat{x} \in L^2_1(\mathbf{R})$ is a bijective isometry, such that $x[-k] = c_k(\widehat{x})$ $(k \in \mathbf{Z})$.

If $\{x[k]\} \in \ell^1$, then $\widehat{x} \in \mathcal{C}_1(\mathbf{R})$. Of course, $\ell^1 \subset \ell^2 \subset \ell$ (see Exercise 7.13).

Example 7.20. The Fourier transform of $x[j] = c\,\mathrm{sinc}\,(cj)$ $(0 < c < 1/2)$ is the 1-periodic square wave such that $\chi_{[-c/2,c/2]}(\xi)$ on $[-1/2, 1/2]$.

In this example, $x \in \ell^2$, since $\widehat{x} \in L^2_1(\mathbf{R})$, but $\widehat{x}$ is not continuous, so that $x \notin \ell^1$.

The Fourier transform of a signal and that of its samples are related as follows:

Theorem 7.21. *Let $f \in L^2(\mathbf{R})$ be a band-limited signal and let $x[j] := f(j/2\Omega)$, with $\Omega \geq \Omega_N$. Then*

$$\widehat{f}(\xi) = \frac{1}{2\Omega}\widehat{x}\Big(\frac{\xi}{2\Omega}\Big) \qquad (|\xi| \leq \Omega).$$

Proof. According to the inversion theorem,

$$x[j] = \int_{-1/2}^{1/2} \widehat{x}(\xi)e^{2\pi i\xi j}\,d\xi = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \widehat{x}\Big(\frac{\xi}{2\Omega}\Big)e^{2\pi i\xi j/2\Omega}\,d\xi$$

so, by the 2Ω-periodicity of $e^{2\pi i\xi j/2\Omega}$,

$$\begin{aligned} x[j] &= f\Big(\frac{j}{2\Omega}\Big) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i\xi j/2\Omega}\,d\xi = \sum_{k=-\infty}^{\infty}\int_{-\Omega}^{\Omega} \widehat{f}(\xi - 2\Omega k)e^{2\pi i\xi j/2\Omega}\,d\xi \\ &= \int_{-\Omega}^{\Omega}\Big(\sum_{k=-\infty}^{\infty} \widehat{f}(\xi - 2\Omega k)\Big)e^{2\pi i\xi j/2\Omega}\,d\xi \end{aligned}$$

and, from the uniqueness property of the Fourier coefficients,

$$\frac{1}{2\Omega}\widehat{x}\Big(\frac{\xi}{2\Omega}\Big) = \sum_{k=-\infty}^{\infty} \widehat{f}(\xi - 2\Omega k) = \widehat{f}_{2\Omega}.$$

This relation means that

$$\widehat{x}(\xi) = 2\Omega\sum_{k=-\infty}^{\infty} \widehat{f}(2\Omega(\xi - k)),$$

where the right side is $\widehat{f}$ scaled by the factor 2Ω. Since $\Omega \geq \Omega_N$, we have $\widehat{f}_{2\Omega}(\xi) = \widehat{f}(\xi)$ if $|\xi| \leq 1/2$ and then $\widehat{x}(\xi) = 2\Omega\widehat{f}(2\Omega\xi)$. □

7.5. The Dirichlet problem in the half-space

In Theorem 6.33 we have obtained the solution $u(x) = \int_S P(x,y)f(y)\,dy$ of the homogeneous Dirichlet problem[10] for the ball with the inhomogeneous boundary condition $u_{|S} = f$ by means of its Poisson kernel P.

Here, as in (7.2) for the heat equation, we will use the Fourier transform as a tool to solve the homogeneous Dirichlet problem in the half-space $\mathbf{R}^{1+n}_+ := \{(t,x) \in \mathbf{R} \times \mathbf{R}^n;\, t > 0\}$,

$$\partial_t^2 u + \triangle u = 0, \qquad u(0,x) = f(x), \tag{7.11}$$

where $\triangle = \sum_{j=1}^n \partial_{x_j}^2$.

We will be looking for bounded solutions,[11] that is, for bounded harmonic functions u on the half-space $t > 0$ such that, in some sense, $u(t,x) \to f(x)$ as $t \downarrow 0$.

7.5.1. The Poisson integral in the half-space. The Fourier transform changes a linear differential equation with constant coefficients

$$P(D)u = f,$$

with $P(D) = \sum_{|\alpha|\le m} c_\alpha D^\alpha$, into the algebraic equation

$$P(2\pi i\xi)\widehat{u}(\xi) = \widehat{f}(\xi),$$

where $P(x) = \sum_{|\alpha|\le m} c_\alpha x^\alpha = \sum_{|\alpha|\le m} c_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n}$.

For this reason, to find an integral kernel for the Dirichlet problem (7.11) similar to the Poisson kernel for the ball, we apply the Fourier transform in x to convert the partial differential equation in (7.11) into an ordinary differential equation.

Assuming for the moment that $f \in \mathcal{S}(\mathbf{R}^n)$, for every $t > 0$ we obtain

$$\partial_t^2 \widehat{u}(t,\xi) - 4\pi^2|\xi|^2\widehat{u}(t,\xi) = 0, \qquad (\widehat{u}(0,\xi) = \widehat{f}(\xi)),$$

and we are led to solve an ordinary differential equation in t for every $\xi \in \mathbf{R}^n$.

The general solution of this equation is

$$\widehat{u}(t,\xi) = A(\xi)e^{2\pi|\xi|t} + B(\xi)e^{-2\pi|\xi|t}, \quad A(\xi) + B(\xi) = \widehat{f}(\xi).$$

If we want to apply the inverse Fourier transform, and also because of the boundedness condition, we must have $A(\xi) = 0$. Then $B(\xi) = \widehat{f}(\xi)$ and

[10]The work of Johann Peter Gustav Lejeune Dirichlet included potential theory, integration of hydrodynamic equations, convergence of trigonometric series and Fourier series, and the foundation of analytic number theory and algebraic number theory. In 1837 he proposed what is today the modern definition of a function. After Gauss's death, Dirichlet took over his post in Göttingen.

[11]This boundedness requirement is imposed to obtain uniqueness; see Exercise 7.12.

with this election

$$u(t,x) = (P_t * f)(x), \qquad \widehat{P_t}(\xi) = e^{-2\pi|\xi|t},$$

where $P(t,x) := P_t(x)$ will be the **Poisson kernel for the half-space**.

If $n = 1$, an easy computation will show that the Fourier transform of $g(x) = e^{-2\pi|x|}$ on $\mathbf{R}$ is

$$\widehat{g}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}. \tag{7.12}$$

Note that the family of functions

$$P_t(x) = \frac{1}{\pi}\frac{x}{t^2+x^2} \qquad (t > 0)$$

is the Poisson summability kernel of Example 2.42, obtained from P_1 by letting $P_t(x) = (1/t)P_1(x/t)$.

To check (7.12), a double partial integration in

$$\int_{\mathbf{R}} e^{-2\pi|x|}e^{-2\pi i\xi x}\,dx = 2\int_0^\infty e^{-2\pi x}\cos(2\pi\xi x)\,dx$$

shows that

$$\begin{aligned}\widehat{g}(\xi) &= -\frac{1}{\pi}\Big[e^{-2\pi x}\cos(2\pi\xi x)\Big]_0^\infty - 2\pi\int_0^\infty e^{-2\pi x}\sin(2\pi\xi x)\,dx \\ &= \frac{1}{\pi} - 2\pi\int_0^\infty e^{-2\pi x}\sin(2\pi\xi x)\,dx = \frac{1}{\pi} - \xi^2\widehat{g}(\xi).\end{aligned}$$

Thus

$$\widehat{g}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}.$$

Obviously $P_1 = \widehat{g} > 0$ and $\int_{\mathbf{R}} P_1(x)\,dx = g(0) = 1$.

This result is extended for $n > 1$, but the calculation is somewhat more involved and it will be obtained from (7.1) and from

$$\int_0^\infty \frac{e^{-t}}{\sqrt{t}}e^{-a^2/4t}\,dt = \sqrt{\pi}e^{-a} \tag{7.13}$$

for $a > 0$.

To prove (7.13), we will use the obvious identity

$$\int_0^\infty e^{-(1+s^2)t}\,dt = 1/(1+s^2)$$

and also

$$\int_{\mathbf{R}} \frac{e^{iat}}{1+t^2}\,dt = \pi e^{-a},$$

which follows from (7.12) by a change of variables.

Indeed,

$$\begin{aligned} e^{-a} &= \frac{1}{\pi}\int_{\mathbf{R}} e^{iat}\int_0^\infty e^{-(1+t^2)s}\,ds\,dt = \frac{1}{\pi}\int_0^\infty e^{-s}\int_{\mathbf{R}} e^{iat}e^{-st^2}\,dt\,ds \\ &= 2\int_0^\infty e^{-s}\int_{\mathbf{R}} e^{2\pi iax}e^{-4\pi^2 sx^2}\,dx\,ds = \int_0^\infty e^{-s}e^{-a^2/4s}\frac{1}{\sqrt{s\pi}}\,ds \end{aligned}$$

and (7.13) follows.

Now we are ready to prove that

$$P(t,x) = c_n\frac{t}{(t^2+|x|^2)^{(n+1)/2}} \qquad \left(c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}\right); \tag{7.14}$$

that is,

$$\int_{\mathbf{R}^n} e^{-2\pi|\xi|t}e^{-2\pi ix\cdot\xi}\,d\xi = c_n\frac{t}{(t^2+|x|^2)^{(n+1)/2}}.$$

A change of variables allows us to suppose $t=1$ and, using (7.1) and (7.13),

$$\begin{aligned} P(1,x) &= \int_{\mathbf{R}^n} e^{-2\pi|\xi|}e^{-2\pi ix\cdot\xi}\,d\xi \\ &= \int_{\mathbf{R}^n}\left(\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-t}}{\sqrt{t}}e^{-4\pi^2|\xi|^2/4t}\,dt\right)e^{-2\pi ix\cdot\xi}\,d\xi \\ &= \int_0^\infty \frac{e^{-t}}{\sqrt{t\pi}}\int_{\mathbf{R}^n} e^{-\pi^2|\xi|^2/t}e^{-2\pi ix\cdot\xi}\,d\xi\,dt \\ &= \pi^{-(n+1)/2}(1+|x|^2)^{-(n+1)/2}\int_0^\infty e^{-x}x^{(n+1)/2}\,dx \\ &= \frac{c_n}{(1+|x|^2)^{(n+1)/2}}. \end{aligned}$$

According to Theorem 2.41, since $P_t = P(t,\cdot)$ is a summability kernel, if $f\in\mathcal{C}(\mathbf{R}^n)$ tends to 0 at infinity, then $u(t,x) = (P_t * f)\to f(x)$ uniformly as $t\downarrow 0$. If $f\in L^p(\mathbf{R}^n)$ $(1\le p<\infty)$, then $u(t,\cdot)\to f$ in L^p when $t\downarrow 0$. Thus, in both cases, f can be considered as the boundary value of u, defined on $t>0$.

Moreover, a direct calculation shows that $(\partial_t^2+\triangle)P(t,x)=0$, and then u is harmonic on the half-space $t>0$, since

$$(\partial_t^2+\triangle)\int_{\mathbf{R}^n} P(t,x-y)f(y)\,dy = 0.$$

Note also that if $|f|\le C$, then $|u(t,x)|\le C\int_{-\infty}^{+\infty} P(t,x-y)\,dt = C$.

7.5.2. The Hilbert transform. In the two-variables case we write

$$P(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

We have seen that it is a harmonic function on the half-plane $y > 0$. This also follows from the fact that it is the real part of the holomorphic function

$$\frac{i}{\pi z} = \frac{1}{\pi}\frac{y+ix}{x^2+y^2} = P_y(x) + iQ_y(x).$$

The imaginary part,

$$Q_y(x) = Q(x,y) = \frac{1}{\pi}\frac{x}{x^2+y^2},$$

which is the conjugate function of $P(x,y)$, is the **conjugate Poisson kernel**, which for every $x \neq 0$ satisfies

$$\lim_{y\downarrow 0} Q_y(x) = \frac{1}{\pi}\frac{1}{x}.$$

This limit is not locally integrable and does not appear directly as a distribution, but we can consider its principal value as a regularization of $1/x$ as in Exercise 6.16. Let us describe it as a limit in $\mathcal{S}'(\mathbf{R})$ of Q_y:

By definition, for every $\varphi \in \mathcal{S}(\mathbf{R})$,

$$\langle \varphi(x), \mathrm{pv}\frac{1}{x}\rangle := \mathrm{pv}\int_{-\infty}^{+\infty} \frac{\varphi(x)}{x}\,dx = \lim_{\varepsilon\downarrow 0}\int_{\mathbf{R}} h_\varepsilon(x)\varphi(x)\,dx$$

if $h_\varepsilon(x) = x^{-1}\chi_{\{|x|>\varepsilon\}}(x)$. This limit exists since $\int_{\varepsilon<|x|<1} x^{-1}\varphi(0)\,dx = 0$ and then

$$\langle \varphi(x), \mathrm{pv}\frac{1}{x}\rangle = \int_{|x|<1}\frac{\varphi(x)-\varphi(0)}{x}\,dx + \int_{|x|>1}\frac{\varphi(x)}{x}\,dx.$$

Theorem 7.22. *In* $\mathcal{S}'(\mathbf{R})$,

$$\frac{1}{\pi}\,\mathrm{pv}\frac{1}{x} = \lim_{y\downarrow 0} Q_y$$

and

$$\mathcal{F}(\frac{1}{\pi}\,\mathrm{pv}\frac{1}{x}) = -i\,\mathrm{sgn}\,.$$

Proof. For the first equality, we only need to show that $F_\varepsilon := \pi Q_\varepsilon - h_\varepsilon \to 0$ in $\mathcal{S}'(\mathbf{R})$ as $\varepsilon \to 0$. But we note that, for every $\varphi \in \mathcal{S}(\mathbf{R})$,

$$\begin{aligned}\langle \varphi, F_\varepsilon\rangle &= \int_{\{|x|<\varepsilon\}}\frac{x\varphi(x)}{\varepsilon^2+x^2}\,dx + \int_{\{|x|>\varepsilon\}}\Big(\frac{x\varphi(x)}{\varepsilon^2+x^2} - \frac{\varphi(x)}{x}\Big)\,dx \\ &= \int_{\{|x|<1\}}\frac{x\varphi(\varepsilon x)}{1+x^2}\,dx - \int_{\{|x|>1\}}\frac{\varphi(\varepsilon x)}{x(1+x^2)}\,dx\end{aligned}$$

and both integrals tend to 0 as $\varepsilon \to 0$, by dominated convergence.

For the second formula, a direct computation of the Fourier transform of the function $-i(\operatorname{sgn}\xi)e^{-2\pi y|\xi|}$ shows that

$$\widehat{Q}_y(\xi) = -i(\operatorname{sgn}\xi)e^{-2\pi y|\xi|}$$

and then

$$\mathcal{F}(\frac{1}{\pi}\operatorname{pv}\frac{1}{x})(\xi) = \lim_{y\downarrow 0} -i(\operatorname{sgn}\xi)e^{-2\pi y|\xi|} = -i(\operatorname{sgn}\xi)$$

in $\mathcal{S}'(\mathbf{R})$, since

$$\lim_{y\downarrow 0}\int_{\mathbf{R}} -i(\operatorname{sgn}\xi)e^{-2\pi y|\xi|}\varphi(\xi)\,d\xi = \int_{\mathbf{R}} -i(\operatorname{sgn}\xi)\varphi(\xi)\,d\xi$$

for every $\varphi \in \mathcal{S}(\mathbf{R})$. □

The **Hilbert transform**[12] is the fundamental map of harmonic analysis and signal theory $H : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ defined by

$$\widehat{Hf}(\xi) = -i\operatorname{sgn}(\xi)\widehat{f}(\xi) \qquad (f \in L^2(\mathbf{R})).$$

Theorem 7.23. *The Hilbert transform is a bijective linear isometry such that*

$$H^2 = -I \ \text{ and } \ H^* = -H.$$

For every $\varphi \in \mathcal{S}(\mathbf{R})$,

$$H\varphi = \frac{1}{\pi}\operatorname{pv}\frac{1}{x} * \varphi = \lim_{y\downarrow 0}(Q_y * \varphi) \ \text{ in } \mathcal{S}'(\mathbf{R}).$$

Proof. If $m(\xi) := -i\operatorname{sgn}\xi$, it is clear that $M : f \mapsto mf$ is a bijective linear isometry of $L^2(\mathbf{R})$ and, by the Plancherel theorem, $H = \widetilde{\mathcal{F}}M\mathcal{F}$ is also a bijective linear isometry. Moreover $H^2 = -I$, since $m^2 = -1$, and

$$(Hf, g)_2 = (\widetilde{\mathcal{F}}M\mathcal{F}f, g)_2 = (M\mathcal{F}f, \mathcal{F}g)_2 = (\mathcal{F}f, -M\mathcal{F}g)_2 = (f, -Hg)_2.$$

With Theorem 7.22 in hand and from the properties of the convolution,

$$\mathcal{F}(H\varphi) = \lim_{y\downarrow 0}\widehat{Q}_y\widehat{\varphi} = \lim_{y\downarrow 0}\mathcal{F}(Q_y * \varphi),$$

so that $H\varphi = \lim_{y\downarrow 0}(Q_y * \varphi)$ in $\mathcal{S}'(\mathbf{R})$.

Also $H\varphi = \widetilde{\mathcal{F}}(M\mathcal{F}\varphi) = (\widetilde{\mathcal{F}}m) * \varphi = \frac{1}{\pi}\operatorname{pv}\frac{1}{x} * \varphi$. □

[12] The name was coined by the English mathematician G. H. Hardy after David Hilbert, who was the first to observe the conjugate functions in 1912. He also showed that the function $\sin(\omega t)$ is the Hilbert transform of $\cos(\omega t)$ and this gives the $\pm\pi/2$ phase-shift operator, which is a basic property of the Hilbert transform in signal theory.

If $\varphi \in \mathcal{S}(\mathbf{R})$, its bounded harmonic extension to the half-plane $y > 0$,

$$u(x, y) = (P_y * \varphi)(x)$$

has

$$v(x, y) = (Q_y * \varphi)(x)$$

as the conjugate function, so that

$$\begin{aligned} F(z) &= u(x, y) + iv(x, y) \\ &= \frac{1}{\pi} \int_{\mathbf{R}} \frac{y}{(x-t)^2 + y^2} \varphi(t)\, dt + \frac{i}{\pi} \int_{\mathbf{R}} \frac{x-t}{(x-t)^2 + y^2} \varphi(t)\, dt \\ &= \frac{1}{\pi i} \int_{\mathbf{R}} \frac{t - \bar{z}}{|t - z|^2} \varphi(t)\, dt \end{aligned}$$

is holomorphic on $\Im z = y > 0$, and it is continuous on $y \geq 0$ with

$$\lim_{y \downarrow 0} F(z) = \varphi(x) + i(H\varphi)(x).$$

Since $F^2(z) = u^2(x, y) - v^2(x, y) + i2u(x, y)v(x, y)$ is also holomorphic, $2uv$ is the conjugate function of $u^2 - v^2$, and $H(\varphi^2 - (H\varphi)^2) = 2\varphi H\varphi$, where $H^{-1} = -H$. Thus

$$(H\varphi)^2 = \varphi^2 + 2H(\varphi H\varphi). \tag{7.15}$$

We can write

$$H\varphi(x) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\varphi(x-y)}{y}\, dy = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\varphi(y)}{x-y}\, dy$$

in the sense of the principal value, and the integrals are called singular integrals. The kernel

$$K(x, y) = \frac{1}{x - y}$$

is far from satisfying the conditions of the Young inequalities (2.20), but H will still be an operator of type (p, p) if $1 < p < \infty$:

Theorem 7.24 (M. Riesz)**.** *For every $1 < p < \infty$, H is a bounded operator of $L^p(\mathbf{R})$.*

Proof. We claim that if $\|H\varphi\|_p \leq C_p\|\varphi\|_p$, then $\|H\varphi\|_{2p} \leq (2C_p + 1)\|\varphi\|_{2p}$.

Indeed, either $\|H\varphi\|_{2p} \leq \|\varphi\|_{2p}$ and there is nothing to prove, or $\|\varphi\|_{2p} \leq \|Hf\|_{2p}$. In this last case, by (7.15),

$$\begin{aligned} \|H\varphi\|_{2p}^2 &= \|(H\varphi)^2\|_p \leq \|\varphi^2\|_p + 2\|H(\varphi H\varphi)\|_p \\ &\leq \|\varphi\|_{2p}^2 + 2C_p\|\varphi H\varphi\|_p \\ &\leq \|\varphi\|_{2p}^2 + 2C_p\|\varphi\|_{2p}\|H\varphi\|_{2p} \\ &\leq \|\varphi\|_{2p}(1 + 2C_p)\|H\varphi\|_{2p} \end{aligned}$$

as claimed.

From this claim, starting from $\|H\varphi\|_2 = \|\varphi\|_2$, we obtain by induction

$$\|H\varphi\|_{2^n} \le (2^n - 1)\|\varphi\|_{2^n} \qquad (n = 1, 2, \ldots)$$

and an application of the Riesz-Thorin interpolation theorem[13] gives

$$\|H\varphi\|_p \le C_p\|\varphi\|_p \qquad (\varphi \in \mathcal{S}(\mathbf{R}))$$

for every $2 \le p < \infty$, so that $H(L^p(\mathbf{R})) \subset L^p(\mathbf{R})$ and H is of type (p, p) for these values of p.

Suppose now that $1 < p < 2$, so that $p' > 2$ and H is a bounded linear operator $L^{p'}(\mathbf{R}) \to L^{p'}(\mathbf{R})$.

Then

$$\begin{aligned}\|Hf\|_p &= \sup\left\{\left|\int_{\mathbf{R}} g(t)H\varphi(t)\,dt\right|;\ \|g\|_{p'} \le 1\right\} \\ &= \sup\left\{\left|\int_{\mathbf{R}} \varphi(t)Hg(t)\,dt\right|;\ \|g\|_{p'} \le 1\right\} \le \|\varphi\|_p C_{p'}.\end{aligned}$$

□

7.6. Sobolev spaces

7.6.1. The spaces $W^{m,p}$. Let Ω be a nonempty open subset of $\mathbf{R}^n$, $1 \le p \le \infty$ and $m \in \mathbf{N}$.

The **Sobolev space of order** $m \in \mathbf{N}$ on Ω is defined by

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega),\ |\alpha| \le m\},$$

where the $D^\alpha u$ represent the distributional derivatives of u. We endow $W^{m,p}(\Omega)$ with the topology of the norm

$$\|u\|_{(m,p)} := \sum_{|\alpha| \le m} \|D^\alpha u\|_p.$$

Note that the linear maps $W^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$, $W^{m+1,p}(\Omega) \hookrightarrow W^{m,p}(\Omega)$, and $D^\alpha : W^{m,p}(\Omega) \to W^{m-|\alpha|,p}(\Omega)$ are continuous, if $|\alpha| \le m$.

In $\mathbf{R}^N$ all the norms are equivalent, so that $\|\cdot\|_{(m,p)}$ is equivalent to the norm

$$\|u\| := \max_{|\alpha| \le m} \|D^\alpha u\|_p.$$

It is easy to show that $W^{m,p}(\Omega)$ is a Banach space by describing it as the subspace of $\prod_{|\alpha| \le m} L^p(\Omega)$ of all elements of the type $\{D^\alpha u\}_{|\alpha| \le m}$. It is closed, since, if $\{D^\alpha u_k\}_{|\alpha| \le m} \to \{u^{(\alpha)}\}_{|\alpha| \le m}$ in the product space, then $D^\alpha u_k \to u^{(\alpha)}$ ($|\alpha| \le m$) in $L^p(\Omega)$.

[13] Marcel Riesz proved his convexity theorem, the Riesz-Thorin Theorem 2.45 for $p(\vartheta) \le q(\vartheta)$, to use it in the proof of this fact and related results of harmonic analysis. See footnote 14 in Chapter 2.

Note that if $f_k \to f$ in $L^p(\Omega)$, then also $f_k \to f$ in $\mathcal{D}'(\Omega)$ since

$$|\langle \varphi, f - f_k \rangle \leq \|\varphi\|_{p'} \|f - f_k\|_p.$$

Hence $D^\alpha u_k \to D^\alpha u = u^{(\alpha)}$ in $\mathcal{D}'(\Omega)$, so that $\{D^\alpha u_k\}_{|\alpha| \leq m} \to \{D^\alpha u\}_{|\alpha| \leq m}$.

In the case $p = 2$, we can renorm the space with the equivalent norm

$$\|u\|_{m,2} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|_2^2 \Big)^{1/2},$$

so that $W^{m,2}(\Omega)$ becomes a Hilbert space with the scalar product

$$(u, v)_{m,2} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_2.$$

Every $u \in W^{m,p}(\Omega) \subset L^p(\Omega)$ is a class of functions, and it is said that it is a C^m function if it has a C^m representative.

In the case of one variable, we can consider $W^{m,p}(a, b) \subset \mathcal{C}[a, b]$ for every $m \geq 1$, since, if $u = v$ a.e. and both of them are continuous, then $u = v$. Moreover, the following regularity result holds:

Theorem 7.25. *If $u \in W^{1,p}(a, b)$ and $v(t) := \int_c^t u'(s)\, ds$, then $u(t) = v(t) + C$ a.e., so that u coincides a.e. with a continuous function, which we still denote by u, such that*

$$u(x) - u(y) = \int_y^x u'(s)\, ds \qquad (x, y \in (a, b)).$$

The distributional derivative u' is the a.e. derivative of u, and the inclusion $W^{1,p}(a, b) \hookrightarrow \mathcal{C}[a, b]$ is continuous.

Proof. Function v, as a primitive of $u' \in L^1_{\mathrm{loc}}(a, b)$, is absolutely continuous on $[a, b]$, and, by the Lebesgue differentiation theorem, u' is its a.e. derivative. The distributional derivatives v' and u' are the same, since, by partial integration and from $\varphi(a) = \varphi(b) = 0$ when $\varphi \in \mathcal{D}(a, b)$, we obtain

$$\langle -\varphi', v \rangle = -\int_a^b \varphi'(t) \int_c^t u'(s)\, ds\, dt = \int_a^b u'(t)\varphi(t)\, dt = \langle \varphi, u' \rangle.$$

But $(u - v)' = 0$ implies $u - v = C$, and u is continuous on $[a, b]$.

It follows from $u = v + C$ that $u(x) - u(y) = v(x) - v(y) = \int_y^x u'(s)\, ds$.

If $u_k \to u$ in $W^{1,p}(a, b)$ and $u_k \to v$ in $\mathcal{C}[a, b]$, then $v = u$, since there exists a subsequence of $\{u_k\}$ which is a.e. convergent to u. Hence, $W^{1,p}(a, b) \hookrightarrow \mathcal{C}[a, b]$ has a closed graph. $\square$

Remark 7.26. It can be shown that, if $m > n/p$ and $1 \leq p < \infty$, $W^{m,p}(\Omega) \subset \mathcal{E}^k(\Omega)$ whenever $k < m - (n/p)$.

We will prove this result in the Hilbert space case $p = 2$ (see Theorem 7.29).

7.6.2. The spaces $H^s(\mathbf{R}^n)$. There is a Fourier characterization of the space $W^{m,2}(\mathbf{R}^n)$ which will allow us to define the Sobolev spaces of fractional order $s \in \mathbf{R}$ on $\mathbf{R}^n$.

In Theorem 7.7 we saw that the pointwise multiplication by the function

$$\omega_s(\xi) := (1 + |\xi|^2)^{s/2}$$

is a continuous linear operator of $\mathcal{S}(\mathbf{R}^n)$, and it can be extended to $\mathcal{S}'(\mathbf{R}^n)$, with $\omega_s u \in \mathcal{S}'(\mathbf{R}^n)$ for every $u \in \mathcal{S}'(\mathbf{R}^n)$, defined as usual by

$$\langle \varphi, \omega_s u \rangle = \langle \omega_s \varphi, u \rangle.$$

We define the operator Λ^s on $\mathcal{S}'(\mathbf{R}^n)$ in terms of the Fourier transform

$$\Lambda^s u = \mathcal{F}^{-1}(\omega_s \widehat{u}).$$

It is a bijective continuous linear operator of $\mathcal{S}(\mathbf{R}^n)$ and of $\mathcal{S}'(\mathbf{R}^n)$, with $(\Lambda^s)^{-1} = \Lambda^{-s}$ and such that

$$\widehat{\Lambda^s u} = \omega_s \widehat{u}.$$

It is called the **Fourier multiplier with symbol** ω_s, since it is the result of the multiplication by ω_s "at the other side of the Fourier transform".

Since $\Lambda^{2m} = (\,\mathrm{Id}\, - (4\pi)^{-1}\triangle)^m$, we can also write

$$\Lambda^s = (\,\mathrm{Id}\, - (4\pi)^{-1}\triangle)^{s/2}.$$

We define the **Sobolev space of order** $s \in \mathbf{R}$,

$$H^s(\mathbf{R}^n) = \{u \in \mathcal{S}'(\mathbf{R}^n);\ \Lambda^s u \in L^2(\mathbf{R}^n)\},$$

i.e., $H^s(\mathbf{R}^n) = \Lambda^{-s}(L^2(\mathbf{R}^n))$, and we provide it with the norm

$$\|u\|_{(s)} = \|\Lambda^s u\|_2 = \|\widehat{\Lambda^s u}\|_2 = \|\omega_s \widehat{u}\|_2,$$

associated with the scalar product $(u, v)_{(s)} = (\Lambda^s u, \Lambda^s v)_2$.

It is a Hilbert space, since Λ^s is a linear bijective isometry between $H^s(\mathbf{R}^n)$ and $L^2(\mathbf{R}^n) = H^0(\mathbf{R}^n)$ and also from $H^r(\mathbf{R}^n)$ onto $H^{r-s}(\mathbf{R}^n)$, which corresponds to $\widehat{u} \mapsto \omega_s \widehat{u}$:

$$\|\Lambda^s u\|_{(r-s)} = \|\omega_{r-s}\widehat{\Lambda^s u}\|_2 = \|\omega_r \widehat{u}\|_2 = \|u\|_{(r)}.$$

If $t < s$, $H^s(\mathbf{R}^n) \hookrightarrow H^t(\mathbf{R}^n)$, and $\mathcal{S}(\mathbf{R}^n)$ is a dense subspace of every $H^s(\mathbf{R}^n)$, since it is dense in $L^2(\mathbf{R}^n)$ and $\Lambda^{-s} : L^2(\mathbf{R}^n) \to H^s(\mathbf{R}^n)$ and $\Lambda^{-s}(\mathcal{S}(\mathbf{R}^n)) = \mathcal{S}(\mathbf{R}^n)$.

Theorem 7.27. *If $k \in \mathbf{N}$ and $s \in \mathbf{R}$,*

$$H^s(\mathbf{R}^n) = \{u \in \mathcal{S}'(\mathbf{R}^n);\ D^\alpha u \in H^{s-k}(\mathbf{R}^n)\, \forall |\alpha| \le k\},$$

and $\|\cdot\|_{(s)}$ and $u \mapsto \sum_{|\alpha|\le k} \|D^\alpha u\|_{(s-k)}$ are two equivalent norms.

In particular, if $s = m$, then $H^s(\mathbf{R}^n) = W^{m,2}(\mathbf{R}^n)$ with equivalent norms.

Proof. If $u \in H^s(\mathbf{R}^n)$, note that every

$$\widehat{D^\alpha u}(\xi) = (2\pi i\xi)^\alpha \widehat{u}(\xi) = (2\pi i \xi)^\alpha \omega_{-s}(\xi)\widehat{\Lambda^s u}(\xi)$$

is a locally integrable function, since $\widehat{\Lambda^s u} \in L^2(\mathbf{R}^n)$ and $(2\pi i\xi)^\alpha \omega_{-s}(\xi)$ is continuous.

If $|\alpha| \le k$, then

$$|\xi^\alpha| = |\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}| \le |\xi|^{|\alpha|} = \Big(\sum_{j=1}^n \xi_j^2\Big)^{|\alpha|/2}$$

since, for every j, $|\xi_j|^{\alpha_j} \le (\sum_{i=1}^n \xi_i^2)^{\alpha_j/2} = |\xi|^{\alpha_j}$. So $|\xi^\alpha| \le (1+|\xi|^2)^{k/2} = \omega_k(\xi)$.

Moreover $(1+|\xi|^2)^{k/2} \le C\sum_{|\alpha|\le k} |\xi^\alpha|$, since $\sum_{|\alpha|\le k} |\xi^\alpha| > 0$ everywhere and

$$F(\xi) := \frac{(1+|\xi|^2)^{k/2}}{\sum_{|\alpha|\le k} |\xi^\alpha|}$$

is a continuous function on $\mathbf{R}^n$ such that, if $|\xi| \ge 1$,

$$F(\xi) \le \frac{(1+|\xi|^2)^{k/2}}{\sum_{j=1}^n |\xi_j|^k} \le 2^{k/2} \frac{|\xi|^k}{\sum_{j=1}^n |\xi_j|^k} \le C.$$

Thus,

$$(1+|\xi|^2)^{k/2} \simeq \sum_{|\alpha|\le k} (2\pi)^{|\alpha|} |\xi^\alpha| = \sum_{|\alpha|\le k} |(2\pi i \xi)^\alpha|.$$

From these estimates we obtain

$$\|u\|_{(s)} = \|\omega_{s-k}\omega_k \widehat{u}\|_2 \le C \sum_{|\alpha|\le k} \|\omega_{s-k} |(2\pi i\xi)^\alpha \widehat{u}|\|_2 = C \sum_{|\alpha|\le k} \|D^\alpha u(\xi)\|_{(s-k)}$$

and also, if $|\alpha| \le k$,

$$\|D^\alpha u\|_{(s-k)} = \|\omega_{s-k} |(2\pi i \xi)^\alpha \widehat{u}(\xi)|\|_2 \le C\|\omega_{s-k}\omega_k \widehat{u}\|_2 = C\|u\|_{(s)}.$$

□

Theorem 7.28 (Sobolev)**.** *If $s - k > n/2$, $H^s(\mathbf{R}^n) \subset \mathcal{E}^k(\mathbf{R}^n)$.*

Proof. By polar integration, $\int_{\mathbf{R}^n}(1+|\xi|^2)^{k-s}\,d\xi<\infty$ if and only if

$$\int_1^\infty (1+r^2)^{k-s} r^{n-1}\,dr \simeq \int_1^\infty r^{2k-2s} r^{n-1}\,dr<\infty,$$

i.e., when $2k-2s+n-1<-1$, which is equivalent to condition $s-k>n/2$.

Then, if $\varphi\in\mathcal{S}(\mathbf{R}^n)$, multiplication and division in

$$D^\alpha\varphi(x)=\int_{\mathbf{R}^n}\widehat{D^\alpha\varphi}(\xi)e^{2\pi i x\cdot\xi}\,d\xi=\int_{\mathbf{R}^n}(2\pi i\xi)^\alpha\widehat{\varphi}(\xi)e^{2\pi i x\cdot\xi}\,d\xi$$

by $\omega_{s-k}(\xi)=(1+|\xi|^2)^{(s-k)/2}$ followed by an application of the Schwarz inequality yields

$$\begin{aligned}|D^\alpha\varphi(x)| &\le \int_{\mathbf{R}^n}|(2\pi\xi)^\alpha|\omega_{s-k}(\xi)|\widehat{\varphi}(\xi)|\omega_{k-s}(\xi)\,d\xi\\ &\le \|D^\alpha\varphi\|_{(s-k)}\Big(\int_{\mathbf{R}^n}(1+|\xi|^2)^{k-s}\,d\xi\Big)^{1/2}\end{aligned}$$

and

$$\|D^\alpha\varphi\|_\infty\le C\|D^\alpha\varphi\|_{(s-k)}. \tag{7.16}$$

When $u\in H^s(\mathbf{R}^n)$, we can consider $\varphi_m\to u$ in $H^s(\mathbf{R}^n)$ with $\varphi_k\in\mathcal{S}(\mathbf{R}^n)$, the estimate (7.16) ensures that $\{D^\alpha\varphi_m\}_{m=1}^\infty\subset\mathcal{S}(\mathbf{R}^n)$ is uniformly Cauchy, and we obtain that $D^\alpha\varphi_m\to D^\alpha u$ uniformly, if $|\alpha|\le k$. This proves that $u\in\mathcal{E}^k(\mathbf{R}^n)$. □

7.6.3. The spaces $H^m(\Omega)$. If $m\in\mathbf{N}$ and for any open set $\Omega\subset\mathbf{R}^n$, we will use the notation

$$H^m(\Omega)=W^{m,2}(\Omega)$$

suggested by Theorem 7.27.

In this case, as a consequence of the Sobolev Theorem 7.28 we obtain

Theorem 7.29. *If $m-k>n/2$, $H^m(\Omega)\subset\mathcal{E}^k(\Omega)$.*

Proof. It is sufficient to prove that $u\in H^m(\Omega)\subset L^2(\Omega)$ is a C^k function on a neighborhood of every point. By multiplying u by a test function if necessary, we can suppose that as a distribution its support is a compact subset K of Ω.

If $K\prec\eta\prec\Omega$ and if $\bar u$ is the extension of $u\in L^2(\Omega)$ by zero to $\bar u\in L^2(\mathbf{R}^n)$, then $\bar u\in H^m(\mathbf{R}^n)$, since for every $|\alpha|\le m$ we can apply the Leibniz rule to

$$\langle\varphi,D^\alpha\bar u\rangle=\langle\varphi,D^\alpha(\eta\bar u)\rangle$$

to show that $\langle\cdot,D^\alpha\bar u\rangle$ is L^2-continuous on test functions, so $D^\alpha\bar u\in L^2(\mathbf{R}^n)$ by the Riesz representation theorem, and $\bar u\in H^m(\mathbf{R}^n)$. By Theorem 7.28 we know that $\bar u\in\mathcal{E}^k(\mathbf{R}^n)$ and then $u\in\mathcal{E}^k(\Omega)$. □

Remark 7.30. It is useful to approximate functions in $H^m(\Omega)$ by C^∞ functions. The space $H^m(\Omega)$ can be defined as the completion of $\mathcal{E}(\Omega) \cap H^m(\Omega)$ under the norm $\|\cdot\|_{m,2}$. In fact it can be proved that $\mathcal{E}(\Omega)\cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$ for any $1 \le p < \infty$ and $m \ge 1$ an integer.[14]

We content ourselves with the following easier approximation result known as the Friedrichs theorem.

Theorem 7.31. *If* $u \in H^1(\Omega)$*, then there exists a sequence* $\{\varphi_m\} \subset \mathcal{D}(\mathbf{R}^n)$ *such that* $\lim_m \|u-\varphi_m\|_{L^p(\Omega)} = 0$ *and* $\lim_m \|\partial_j u - \partial_j \varphi_m\|_{L^p(\omega)} = 0$ *for every* $1 \le j \le n$ *and for every open set* ω *such that* $\bar{\omega}$ *is a compact subset of* Ω.

Proof. Denote by u^o the extension of u by zero on $\mathbf{R}^n$ and choose a mollifier ϱ_ε. Then $\varrho_\varepsilon * u^o \to u^o$ in $L^2(\mathbf{R}^n)$ and so $\lim_{\varepsilon\to 0} \|u - \varrho_\varepsilon * u^o\|_{L^p(\Omega)} = 0$. Allow $0 < \varepsilon < d(\bar{\omega}, \Omega^c)$, so that $(\varrho_\varepsilon * u^o)(x) = (\varrho_\varepsilon * u)(x)$ and $\partial_j(\varrho_\varepsilon * u^o)(x) = (\varrho_\varepsilon * (\partial_j u)^o)(x)$ for every $x \in \omega$. So $\lim_{\varepsilon\to 0} \|\partial_j u - \partial_j(\varrho_\varepsilon * u^o)\|_{L^2(\omega)} = 0$, since $(\partial_j u)^o \in L^2(\mathbf{R}^n)$.

If $\varepsilon_m \downarrow 0$, let us multiply the functions $f_m = \varrho_{\varepsilon_m} * u^o \in \mathcal{E}(\mathbf{R}^n)$ by the cut-off functions χ_m such that $\bar{B}(0,m) \prec \chi_m \prec \mathbf{R}^n$. By the dominated convergence theorem, if $\varphi_m = \chi_m f_m$, then

$$\|\varphi_m - u^o\|_2 \le \|\chi_m(f_m - u^o)\|_2 + \|\chi_m u^o - u^o\|_2 \to 0 \text{ as } m \to \infty$$

and $\bar{\omega} \subset B(0,m)$ for large m. It follows that the test functions φ_m satisfy all the requirements. □

As an application, we can prove the following **chain rule** for functions $v \in H^1(\Omega)$ and any $\varrho \in \mathcal{E}(\mathbf{R})$ with bounded derivative and such that $\varrho(0) = 0$:

$$\partial_j(\varrho \circ v) = (\varrho' \circ v)\partial_j v \qquad (1 \le j \le n). \tag{7.17}$$

Indeed, since $|\varrho'| \le M$ and $\varrho(0) = 0$, by the mean value theorem $|\varrho(t)| \le M|t|$. Thus $|\varrho\circ v| \le M|v|$ and $\varrho\circ v \in L^2(\Omega)$. It is also clear that $(\varrho'\circ v)\partial_j v \in L^2(\Omega)$.

Pick $\varphi_m \in \mathcal{D}(\mathbf{R}^n)$ as in Theorem 7.31. Then, for every $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi \subset \omega$, from the usual chain rule

$$\int_\Omega (\varrho \circ \varphi_m)(x)\partial_j\varphi(x)\,dx = \int_\Omega (\varrho' \circ \varphi_m)(x)\partial_j\varphi_m(x)\varphi(x)\,dx.$$

Here $\varrho \circ \varphi_m \to \varrho \circ v$ in $L^2(\Omega)$ and $(\varrho' \circ \varphi_m)\partial_j\varphi_m \to (\varrho' \circ v)\partial_j v$ in $L^2(\omega)$ by dominated convergence, and (7.17) follows.

[14] This is the Meyer-Serrin theorem and a proof can be found in [**1**] or [**17**].

7.6.4. The spaces $H_0^m(\Omega)$. When looking for distributional solutions $u \in H^m(\Omega)$ in boundary value problems such as the Dirichlet problem with a homogeneous boundary condition, it does not make sense to consider the pointwise values $u(x)$ of u.

Vanishing on the boundary in the distributional sense is defined by considering u as an element of a convenient subspace of $H^m(\Omega)$.

The Sobolev space $H_0^m(\Omega)$ is defined as the closure of

$$\mathcal{D}^m(\Omega) = \bigcup_{K \in \mathcal{K}(\Omega)} \mathcal{D}_K^m(\Omega)$$

in $H^m(\Omega)$, and it is endowed with the restriction of the norm of $H^m(\Omega)$.

In this definition, $\mathcal{D}^m(\Omega)$ can be replaced by $\mathcal{D}(\Omega)$:

Theorem 7.32. *For every $m \in \mathbf{N}$, $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$.*

Proof. Let $\psi \in \mathcal{D}^m(\Omega) \subset \mathcal{D}^m(\mathbf{R}^n)$. If $\varrho \geq 0$ is a test function supported by $\bar{B}(0,1)$ such that $\|\varrho\|_1 = 1$, then $\varrho_k(x) := k^n \varrho(kx)$ is another test function such that $\|\varrho\|_1 = 1$, now supported by $\bar{B}(0,1/k)$. Moreover $\varrho_k * \psi \in \mathcal{D}(\mathbf{R}^n)$ and

$$D^\alpha(\varrho_k * \psi) = \varrho_k * D^\alpha \psi \to D^\alpha \psi$$

in $L^2(\mathbf{R}^n)$ for every $|\alpha| \leq m$ with

$$\operatorname{supp} \varrho_k * \psi \subset \operatorname{supp} \varrho_k + \operatorname{supp} \psi \subset \Omega$$

if k is large enough. Hence, $\mathcal{D}(\mathbf{R}^n) \ni \varrho_k * \psi \to \psi$ in $H^m(\Omega)$ and $\psi \in \overline{\mathcal{D}(\Omega)}$, closure in $H_0^m(\Omega)$. $\square$

The class $\mathcal{S}(\mathbf{R}^n)$ is dense in $H^m(\mathbf{R}^n)$, $\mathcal{D}(\mathbf{R}^n)$ is also dense in $\mathcal{S}(\mathbf{R}^n)$, and the inclusion $\mathcal{S}(\mathbf{R}^n) \hookrightarrow H^m(\mathbf{R}^n)$ is continuous, so that $\mathcal{D}(\mathbf{R}^n)$ is dense in $H^m(\mathbf{R}^n)$ and

$$H_0^m(\mathbf{R}^n) = H^m(\mathbf{R}^n).$$

The fact that the elements in $H_0^m(\Omega)$ can be considered as distributions that vanish on the boundary $\partial\Omega$ of Ω is explained by the following results, where for simplicity we restrict ourselves to the special and important case $m = 1$.

Theorem 7.33. *If $u \in H^1(\Omega)$ is compactly supported, then $u \in H_0^1(\Omega)$, and its extension $\bar{u}$ by zero on $\mathbf{R}^n$ belongs to $H^1(\mathbf{R}^n)$.*

Proof. If $u \in H^1(\Omega)$ has a compact support $K \subset \Omega$, it is shown as in Theorem 7.29 that $\bar{u} \in L^2(\mathbf{R}^n)$ belongs to $H^1(\mathbf{R}^n)$ by using η such that $K \prec \eta \prec \Omega$.

But $\mathcal{D}(\mathbf{R}^n)$ is dense in $H^1(\mathbf{R}^n)$ and it follows from $\varphi_m \to \bar{u}$ in $H^1(\mathbf{R}^n)$ that $\varphi_m \eta \to u$ in $H^1(\Omega)$ with $\varphi_m \eta \in \mathcal{D}(\Omega)$, so that $u \in H_0^1(\Omega)$. $\square$

Theorem 7.34. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set. If $u \in H^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and $u_{|\partial\Omega} = 0$, then $u \in H_0^1(\Omega)$.*

Proof. Assume that u is real and let $\varrho \in \mathcal{E}^1(\mathbf{R})$ be such that $|\varrho(t)| \leq |t|$ on $\mathbf{R}$, $\varrho = 0$ on $[-1,1]$, and $\varrho(t) = t$ on $(-2,2)^c$. If $[-1,1] \prec \varphi \prec (-2,2)$, just take $\varrho(t) = t(1-\varphi(t))$.

If $v \in H^1(\Omega)$, then $\varrho \circ v \in L^2(\Omega)$ and by the chain rule (7.17) also $\partial_j(\varrho \circ v) = (\varrho' \circ v) \cdot \partial_j v \in L^2(\Omega)$, so $\varrho \circ v \in H^1(\Omega)$.

Hence $u_m := m^{-1}\varrho(mu) \in H_0^1(\Omega)$ since $\operatorname{supp} u_m \subset \{|u| \geq 1/m\}$, which is a compact subset of the bounded open set Ω. By dominated convergence, $u_m \to u$ in $H^1(\Omega)$ and it follows that $u \in H_0^1(\Omega)$. $\square$

It can be shown that this result is also true for unbounded open sets and that the converse holds when Ω is of class C^1:

$$u \in \mathcal{C}(\bar{\Omega}) \cap H_0^1(\Omega) \Rightarrow u_{|\partial\Omega} = 0. \tag{7.18}$$

We only include here the proof in the easy case $n = 1$:

Theorem 7.35. *If $n = 1$ and $\Omega = (a,b)$, then $H_0^1(a,b)$ is the class of all functions $u \in H^1(a,b) \subset \mathcal{C}[a,b]$ such that $u(a) = u(b) = 0$.*

Proof. By Theorem 7.34, we only need to show that $u(a) = u(b) = 0$ for every $u \in H_0^1(a,b) \subset \mathcal{C}[a,b]$. But, if $\mathcal{D}(a,b) \ni \varphi_k \to u$ in $H_0^1(a,b)$, also $\varphi_k \to u$ uniformly, since $H_0^1(a,b) \hookrightarrow \mathcal{C}[a,b]$ is continuous by Theorem 7.25, and then $\varphi_k(a) = \varphi_k(b) = 0$. $\square$

7.7. Applications

To show how Sobolev spaces provide a good framework for the study of differential equations, let us start with a one-dimensional problem.

7.7.1. The Sturm-Liouville problem. We consider here the problem of solving

$$-(pu')' + qu = f, \;\; u(a) = u(b) = 0 \tag{7.19}$$

when $q \in \mathcal{C}[a,b]$, $p \in \mathcal{C}^1[a,b]$, and $p(t) \geq \delta > 0$.

If $f \in \mathcal{C}[a,b]$, a **classical solution** is a function $u \in \mathcal{C}^2[a,b]$ that satisfies (7.19) at every point.

If $f \in L^2(a,b)$, a **weak solution** is a function $u \in H_0^1(a,b)$ whose distributional derivatives satisfy $-(pu')' + qu = f$, i.e.,

$$\int_a^b p(t)u'(t)\varphi'(t)\,dt + \int_a^b q(t)u(t)\varphi(t)\,dt = \int_a^b f(t)\varphi(t)\,dt \qquad \big(\varphi \in \mathcal{D}(a,b)\big).$$

If $v \in H_0^1(a,b)$, by taking $\varphi_k \to v$ ($\varphi_k \in \mathcal{D}(a,b)$), the identity

$$\int_a^b p(t)u'(t)v'(t)\,dt + \int_a^b q(t)u(t)v(t)\,dt = \int_a^b f(t)v(t)\,dt$$

also holds.

To prove the existence and uniqueness of a weak solution for this Sturm-Liouville problem, with $f \in L^2(a,b)$, we define

$$B(u,v) := \int_a^b p(t)u'(t)\overline{v'(t)}\,dt + \int_a^b q(t)u(t)\overline{v(t)}\,dt.$$

Then we obtain a sesquilinear continuous form on $H_0^1(a,b) \times H_0^1(a,b)$ and $(\cdot, f)_2 \in H_0^1(a,b)'$, since

$$|B(u,v)| \leq \|p\|_\infty \|u'\|_2 \|v'\|_2 + \|q\|_\infty \|u\|_2 \|v\|_2 \leq c\|u\|_{(1,2)}\|v\|_{(1,2)}$$

and $|(u,f)_2| \leq \|f\|_2 \|u\|_{(1,2)}$.

If B is coercive, we can apply the Lax-Milgram theorem and, for a given $f \in L^2(a,b)$, there exists a unique $u \in H_0^1(a,b)$ such that $B(v,u) = (v,f)_2$, which means that u is the uniquely determined weak solution of problem (7.19).

For instance, if also $q(t) \geq \delta > 0$, then

$$B(u,u) = \int_a^b (p(t)|u'(t)|^2 + q(t)|u(t)|^2)\,dt \geq \delta \|u\|^2_{H_0^1(a,b)}$$

and B is coercive.

Finally, if $f \in \mathcal{C}[a,b]$, the weak solution u is a C^2 function, and then it is a classical solution. Indeed, $pu' \in L^2(a,b)$ satisfies $(pu')' = qu - f$, which is continuous; then $g := pu'$ and $u' = g/p$ are C^1 functions on $[a,b]$, so that $u \in \mathcal{C}^2[a,b]$, and $u(a) = u(b) = 0$ by Theorem 7.35.

7.7.2. The Dirichlet problem. Now let Ω be a nonempty bounded open domain in $\mathbf{R}^n$ with $n > 1$, and consider the Dirichlet problem

$$-\triangle u = f, \qquad u = 0 \text{ on } \partial\Omega \qquad (f \in L^2(\Omega)). \tag{7.20}$$

If f is continuous on $\bar{\Omega}$ and u is a classical solution, then $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and (7.20) holds in the pointwise sense.

Let us write $(\nabla u, \nabla v)_2 := \sum_{j=1}^n (\partial_j u, \partial_j v)_2$. When trying to obtain existence and uniqueness of such a solution, we again start by looking for solutions in a weak sense. After multiplying by test functions $\varphi \in \mathcal{D}(\Omega)$, by integration we are led to consider functions $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $u = 0$ on $\partial\Omega$ and

$$(\varphi, -\triangle u)_2 = (\varphi, f)_2 \qquad \big(\varphi \in \mathcal{D}(\Omega)\big),$$

or, equivalently, such that

$$(\nabla\varphi, \nabla u)_2 = (\varphi, f)_2 \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Then it follows from Theorem 7.34 that $u \in H_0^1(\Omega)$, and

$$(\nabla v, \nabla u)_2 = (v, f)_2 \qquad (v \in H_0^1(\Omega)) \tag{7.21}$$

since for every $v \in H_0^1(\Omega)$ we can take $\varphi_k \to v$ in $H^1(\Omega)$, so that $\varphi_k \to v$ and $\partial_j\varphi_k \to \partial_j u$ in $L^2(\Omega)$.

A **weak solution** of the Dirichlet problem is a function $u \in H_0^1(\Omega)$ such that $-\triangle u = f$ in the distributional sense or, equivalently, such that property (7.21) is satisfied.

Every classical solution is a weak solution, and we can look for weak solutions even for $f \in L^2(\Omega)$.

To prove the existence and uniqueness of such a weak solution, we will use the **Dirichlet norm** $\|\cdot\|_D$ on $H_0^1(\Omega)$, defined by

$$\|u\|_D^2 = \int_\Omega |\nabla u|^2 = \int_\Omega \sum_{j=1}^n |\partial_j u(x)|^2\, dx.$$

It is a true norm, associated to the scalar product

$$(u, v)_D := (\nabla u, \nabla v)_2,$$

and it is equivalent to the original one:

Lemma 7.36 (Poincaré)**.** *There is a constant C depending on the bounded domain Ω such that*

$$\|u\|_2 \le C\|u\|_D \qquad (u \in H_0^1(\Omega)), \tag{7.22}$$

and on $H_0^1(\Omega)$ the Dirichlet norm $\|\cdot\|_D$ and the Sobolev norm $\|\cdot\|_{(1,2)}$ are equivalent.

Proof. Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, we only need to prove (7.22) for test functions $\varphi \in \mathcal{D}(\Omega) \subset \mathcal{D}(\mathbf{R}^n)$.

If $\Omega \subset [a, b]^n$, let us consider any $x = (x_1, x') \in \Omega$ and write

$$\varphi(x) = \int_a^{x_1} \partial_1\varphi(t, x')\, dt.$$

By the Schwarz inequality,

$$|\varphi(x)| \le (b-a)^{1/2}\Big(\int_a^b |\partial_1\varphi(t, x')|^2\, dt\Big)^{1/2}$$

and then, by Fubini's theorem,

$$\|\varphi\|_2^2 \le (b-a)\int_{[a,b]^n} dx \int_a^b |\partial_1\varphi(t, x')|^2\, dt = (b-a)^2\|\partial_1\varphi\|_2^2$$

with $|\partial_1\varphi| \leq |\nabla\varphi|$, and (7.22) follows.

From this estimate,

$$\|u\|^2_{(1,2)} = \|u\|^2_2 + \||\nabla u|\|^2_2 \leq (C^2+1)\|u\|^2_D$$

and obviously also $\|u\|_D \leq \|u\|_{(1,2)}$. □

Theorem 7.37. *The Dirichlet problem (7.20) on the bounded domain Ω has a uniquely determined weak solution $u \in H^1_0(\Omega)$ for every $f \in L^2(\Omega)$, and the operator*

$$\triangle^{-1} : L^2(\Omega) \to H^1_0(\Omega)$$

is continuous.

Proof. If C is the constant that appears in the Poincaré lemma, then, by the Schwarz inequality,

$$|(v,f)_2| \leq \|f\|_2\|v\|_2 \leq C\|f\|_2\|v\|_D$$

and $(\cdot,f)_2 \in H^1_0(\Omega)'$ with $\|(\cdot,f)_2\| \leq C\|f\|_2$. By the Riesz representation theorem, there is a uniquely determined function $u \in H^1_0(\Omega)$ such that $(v,f)_2 = (v,u)_D$ for all $v \in H^1_0(\Omega)$, which is property (7.21).

The estimate $\|u\|_D = \|(\cdot,f)_2\|_{H^1_0(\Omega)'} \leq C\|f\|_2$ shows that $\|\triangle^{-1}\| \leq C$. □

An application of (7.18) shows that a weak solution of class C^2 is also a classical solution if Ω is C^1:

Theorem 7.38. *Let $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}(\bar\Omega)$ and $f \in \mathcal{C}(\Omega)$. If u is a weak solution of the Dirichlet problem (7.20) and Ω is a C^1 domain, then u is a classical solution; that is, $-\triangle u(x) = f(x)$ for every $x \in \Omega$ and $u(x) = 0$ for every $x \in \partial\Omega$.*

Proof. By (7.18), $u_{|\partial\Omega} = 0$. Since $u \in \mathcal{C}^2(\Omega)$, the distribution $\triangle u$ is the function $\triangle u(x)$ on Ω, and the distributional relation $-\triangle u = f$ is an identity of functions. □

We have not proved (7.18) if $n > 1$, and the proof of the regularity of the weak solutions is more delicate. For instance, if $f \in \mathcal{C}^\infty(\bar\Omega)$ and the boundary $\partial\Omega$ is C^∞, it can be shown that every weak solution u is also in $\mathcal{E}(\bar\Omega)$, so that it is also a classical solution. More precisely, the following result holds:

Theorem 7.39. *Let Ω be a bounded open set of $\mathbf{R}^n$ of class C^{m+2} with $m > n/2$ (or $\mathbf{R}^n$ or $\mathbf{R}^n_+ = \{x : x_n > 0\}$) and let $f \in H^m(\Omega)$. Then every weak solution u of the Dirichlet problem (7.20) belongs to $\mathcal{C}^2(\bar\Omega)$, and it is a classical solution.*

7.7.3. Eigenvalues and eigenfunctions of the Laplacian. We are going to apply the spectral theory for compact operators. The following result will be helpful:

Theorem 7.40. *Suppose that $\Phi \subset L^2(\mathbf{R}^n)$ satisfies the following three conditions:*

(a) *Φ is bounded in $L^2(\mathbf{R}^n)$,*

(b) *$\lim_{R\to\infty} \int_{|x|>R} |f(x)|^2\,dx = 0$ uniformly on $f \in \Phi$, and*

(c) *$\lim_{h\to 0} \|f - \tau_h f\|_2 = 0$ uniformly on $f \in \Phi$.*

Then the closure $\bar{\Phi}$ of Φ is compact in $L^2(\mathbf{R}^n)$.

Proof. Let $\varepsilon > 0$. By (b), we can choose $R > 0$ so that

$$\int_{|x|>R} |f(x)|^2\,dx \le \varepsilon^2 \qquad (f \in \Phi).$$

Choose $0 \le \varphi \in \mathcal{D}(B(0,1))$ with $\int \varphi = 1$, so that $\varphi_k(x) = k^n\varphi(kx)$ is a summability kernel on $\mathbf{R}^n$ such that $\operatorname{supp}\varphi_k \subset \bar{B}(0,1/k)$, and we know that $\lim_{k\to\infty} \|f * \varphi_k - f\|_2 = 0$ if $f \in L^2(\mathbf{R}^n)$. In fact, since $\varphi_k = 0$ on $|y| \ge 1/k$, it follows from the proof of Theorem 2.41 that

$$|(f*\varphi_k)(x) - f(x)| = \Big|\int_{|y|<1/k} [f(x-y) - f(y)]\varphi_k(y)\,dy\Big|$$

and then $\|f*\varphi_k - f\|_2 \le \sup_{|h|\le 1/k} \|\tau_h f - f\|_2$. Thus, by (c), we can choose N so that

$$\|f - f*\varphi_N\|_2 \le \varepsilon \qquad (f \in \Phi).$$

Moreover it follows very easily from the Schwarz inequality that

$$|(f*\varphi_N)(x) - (f*\varphi_N)(y)| \le \|\tau_{x-y}f - f\|_2\|\varphi_N\|_2$$

and also

$$|(f*\varphi_N)(x)| \le \|f\|_2\|\varphi_N\|_2.$$

These estimates, with conditions (a) and (c), allow us to apply the Ascoli-Arzelà theorem on $\bar{B}(0,R) \subset \mathbf{R}^n$ to the restrictions of the functions $f*\varphi_N$ with $f \in \Phi$, which can be covered by a finite family of balls in $\mathcal{C}(\bar{B}(0,R))$ with the centers in Φ,

$$B_{\mathcal{C}(\bar{B}(0,R))}(f_1,\delta), \dots, B_{\mathcal{C}(\bar{B}(0,R))}(f_m,\delta),$$

for every $\delta > 0$.

Note that at every point $x \in \mathbf{R}^n$

$$\begin{aligned} |f(x) - f_j(x)| \;\le\;& \chi_{\{|x|>R\}}(x)|f(x)| + \chi_{\{|x|>R\}}(x)|f_j(x)| \\ &+|f(x) - (f*\varphi_n)(x)| + |f_j(x) - (f_j*\varphi_N)(x)| \\ &+\chi_{\{|x|\le R\}}(x)|(f*\varphi_N)(x) - (f_j*\varphi_N)(x)| \end{aligned}$$

and the previous estimates yield

$$\|f - f_j\|_2 \le 4\varepsilon + |B(0,R)|^{1/2} \sup_{|x|\le R} |(f*\varphi_N)(x) - (f_j*\varphi_N)(x)|,$$

so that, by choosing $\delta = \varepsilon/|B(0,R)|^{1/2}$, it follows that $\|f - f_j\|_2 \le 5\varepsilon$ and, by Theorem 1.1, the closure of Φ in $L^2(\mathbf{R}^n)$ is compact. □

Remark 7.41. Obvious changes in the proof, such as applying Hölder's inequality instead of the Schwarz inequality, shows that the above theorem has an evident extension to $L^p(\mathbf{R}^n)$ if $1 \le p < \infty$.

Theorem 7.42 (Rellich[15]). *If Ω is a bounded open set of $\mathbf{R}^n$, the natural inclusion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Proof. The extension by zero mapping $L^2(\Omega) \hookrightarrow L^2(\mathbf{R}^n)$ is isometric, so that it is sufficient to prove the compactness of the extension by zero map of Theorem 7.33, i.e.,

$$f \in H_0^1(\Omega) \mapsto \tilde{f} \in H^1(\mathbf{R}^n) = H_0^1(\mathbf{R}^n) \subset L^2(\mathbf{R}^n) \qquad (\tilde{f}(x) = 0 \text{ if } x \in \Omega^c).$$

This follows as an application of Theorem 7.40 when Φ is $\tilde{B} = \{\tilde{f};\, f \in B\}$, if B is the closed unit ball in $H_0^1(\Omega)$.

Indeed, $\tilde{B}$ is contained in the unit ball of $L^2(\mathbf{R}^n)$ and, if $\Omega \subset B(0,R)$, then $\int_{|x|>R} |\tilde{f}|^2 = 0$, so that conditions (a) and (b) of Theorem 7.40 are satisfied. To prove (c), note that

$$\|\tau_h u - u\|_2 \le |h| \||\nabla u|\|_2 \qquad (u \in H^1(\mathbf{R}^n)) \tag{7.23}$$

since we can consider $\varphi_k \in \mathcal{D}(\mathbf{R}^n)$ so that $\varphi_k \to u$ in $H^1(\mathbf{R}^n)$ as $k \to \infty$ and for every test function φ we have

$$|\tau_h\varphi(x) - \varphi(x)|^2 = \Big|\int_0^1 h\cdot\nabla\varphi(x-th)\,dt\Big|^2 \le |h|^2 \int_0^1 |\nabla\varphi(x-th)|^2\,dt,$$

by the Schwarz inequality, and (7.23) follows for φ by integration.

Then $\|\tau_h\tilde{f} - \tilde{f}\|_2 \le |h|$ for every $f \in B$, which is property (c) for $\tilde{B}$. □

Theorem 7.43. *If Ω is a bounded open set of $\mathbf{R}^n$, then $(-\triangle)^{-1}$ is a compact and injective self-adjoint operator on $L^2(\Omega)$ and on $H_0^1(\Omega)$.*

Proof. The compactness of $(-\triangle)^{-1} : L^2(\Omega) \to L^2(\Omega)$ follows by considering the decomposition

$$(-\triangle)^{-1} : L^2(\Omega) \to H_0^1(\Omega) \hookrightarrow L^2(\Omega),$$

[15] The theorem, in this case $p = 2$, is attributed to the South-Tyrolian mathematician Franz Rellich (1930 in Göttingen) and to Vladimir Kondrachov (1945) for the more general case stating that $W_0^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q < np/(n-p)$ if $p < n$, and in $\mathcal{C}(\bar{\Omega})$ if $p > n$. For a proof we refer the reader to Gilbarg and Trudinger [**17**] and Brezis [**5**].

where $(-\triangle)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ is continuous by Theorem 7.37 and, by the Rellich Theorem 7.42, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

Similarly,

$$(-\triangle)^{-1} : H_0^1(\Omega) \hookrightarrow L^2(\Omega) \overset{(-\triangle)^{-1}}{\longrightarrow} H_0^1(\Omega)$$

is also compact.

Note that $(-\triangle)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ is bijective, by Theorem 7.37.

If $u = (-\triangle)^{-1}\varphi$ and $v = (-\triangle)^{-1}\psi$, with $\varphi, \psi \in \mathcal{D}(\Omega)$, then

$$(u, v)_D = (\nabla u, \nabla v)_2 = (-\triangle u, v)_2 = (\varphi, v)_2,$$

so that

$$((-\triangle)^{-1}\varphi, \psi)_D = (\varphi, (-\triangle)^{-1}\psi)_D$$

and

$$((-\triangle)^{-1}u, v)_D = (u, (-\triangle)^{-1}v)_D$$

for all $u, v \in H_0^1(\Omega)$ by continuity. Also

$$((-\triangle)^{-1}\varphi, \psi)_2 = (\nabla\varphi, \nabla\psi)_2 = (\varphi, \psi)_D = (\varphi, (-\triangle)^{-1}\psi)_2.$$

This shows that $(-\triangle)^{-1}$ is self-adjoint on $H_0^1(\Omega)$ and on $L^2(\Omega)$. $\square$

Note that

$$((-\triangle)^{-1}u, v)_D = (u, v)_2 \qquad (u, v \in H_0^1(\Omega)), \tag{7.24}$$

from the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$. Thus $(-\triangle)^{-1}$ is a positive operator on $H_0^1(\Omega)$ in the sense that

$$((-\triangle)^{-1}u, u)_D > 0 \tag{7.25}$$

if $0 \neq u \in H_0^1(\Omega)$.

An eigenfunction for the Laplacian on $H_0^1(\Omega)$ is an element $u \in H_0^1(\Omega)$ such that $\triangle u = \lambda u$ for some λ, which is said to be an eigenvalue of $\triangle$ if there exists some nonzero eigenfunction u such that $\triangle u = \lambda u$.

Hence 0 is not an eigenvalue, since $\triangle : H_0^1(\Omega) \to L^2(\Omega)$ is injective, and $u \in H_0^1(\Omega)$ is an eigenfunction for the eigenvalue λ of $\triangle$ if and only if

$$(-\triangle)^{-1}u = -\frac{1}{\lambda}u.$$

The solutions of this equation form the eigenspace for this eigenvalue λ. Note that $\lambda < 0$, as a consequence of the positivity property (7.25) of $(-\triangle)^{-1}$.

From the spectral theory of compact self-adjoint operators,

$$(-\triangle)^{-1} : H_0^1(\Omega) \to H_0^1(\Omega)$$

has a spectral representation

$$(7.26) \qquad (-\triangle)^{-1}v = \sum_{k=0}^{\infty} \mu_k (v, u_k)_D u_k \qquad (v \in H_0^1(\Omega)),$$

a convergent series in $H_0^1(\Omega)$, where $\mu_k = -1/\lambda_k \downarrow 0$ is the sequence of the eigenvalues of $(-\triangle)^{-1}$ and $\{u_k\}_{k=0}^{\infty}$ is an orthonormal system in $H_0^1(\Omega)$ with respect to $(\cdot,\cdot)_D$ such that $(-\triangle)^{-1}u_k = \mu_k u_k$. Since $(-\triangle)^{-1}$ is injective, $\{u_k\}_{k=0}^{\infty}$ is a basis in $H_0^1(\Omega)$. Moreover $(u_k, u_j)_2 = 0$ if $k \neq j$, by (7.24).

Theorem 7.44. *Suppose $f \in L^2(\Omega)$. The weak solution of the Dirichlet problem*

$$-\triangle u = f \quad (u \in H_0^1(\Omega))$$

is given by the sum

$$u = -\sum_{k=0}^{\infty} (f, u_k)_2 u_k$$

in $H_0^1(\Omega)$. The sequence of eigenfunctions $\sqrt{-\lambda_k}u_k$ is an orthonormal basis of $L^2(\Omega)$.

Proof. It follows from (7.24) applied to the elements $u_k \in H_0^1(\Omega)$ that

$$\|u_k\|_2^2 = \mu_k = -1/\lambda_k, \qquad (u_k, u_m)_2 = 0 \text{ if } m \neq k.$$

Moreover, since $H_0^1(\Omega)$ contains $\mathcal{D}(\Omega)$, it is densely and continuously included in $L^2(\Omega)$, $\{u_k\}_{k=0}^{\infty}$ is total in $L^2(\Omega)$, and the orthonormal system $\{\sqrt{-\lambda_k}u_k\}_{k=0}^{\infty}$ is complete in $L^2(\Omega)$.

For every $f \in L^2(\Omega)$,

$$(7.27) \qquad f = \sum_{k=0}^{\infty} (f, \sqrt{-\lambda_k}u_k)_2 \sqrt{-\lambda_k}u_k = -\sum_{k=0}^{\infty} \lambda_k (f, u_k)_2 u_k$$

in $L^2(\Omega)$ and $\{\sqrt{-\lambda_k}(f, u_k)_2\}_{k=0}^{\infty} \in \ell^2$. Also $\{(f, u_k)_2\}_{k=0}^{\infty} \in \ell^2$, since $\sqrt{-\lambda_k} \to \infty$.

We can define

$$u = \sum_{k=0}^{\infty} (f, u_k)_2 u_k$$

since the series converges in $H_0^1(\Omega)$. Then

$$\triangle u = \sum_{k=0}^{\infty} (f, u_k)_2 \triangle u_k = \sum_{k=0}^{\infty} \lambda_k (f, u_k)_2 u_k$$

in $\mathcal{D}'(\Omega)$. In (7.27) we have a sum in $\mathcal{D}'(\Omega)$, so that $-\triangle u = f$. $\square$

To complete our discussion, we want to show that the eigenfunctions u_k are in $\mathcal{E}(\Omega)$, so that they are classical solutions of $-\triangle u_k = \lambda_k u_k$. The method we are going to use is easily extended to any elliptic linear differential operator L with constant coefficients.

Theorem 7.45. *Suppose $L = \triangle + \lambda$ and $u \in \mathcal{D}'(\Omega)$, where Ω is a nonempty open set in $\mathbf{R}^n$. If $Lu \in \mathcal{E}(\Omega)$, then $u \in \mathcal{E}(\Omega)$.*

Proof. If $Lu \in \mathcal{E}(\Omega)$, then

$$\varphi Lu \in H^s(\mathbf{R}^n) \quad \forall \varphi \in \mathcal{D}(\Omega) \tag{7.28}$$

for every $s \in \mathbf{R}$. We claim that it follows from (7.28) that

$$\varphi u \in H^{s+2}(\mathbf{R}^n) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Then an application of Theorem 7.29 shows that $\varphi u \in \mathcal{E}(\Omega)$ for every $\varphi \in \mathcal{D}(\Omega)$ and then $u \in \mathcal{E}(\Omega)$.

To prove this claim, let $\operatorname{supp} \varphi \subset U$, U an open set with compact closure $\bar{U}$ in Ω, and choose $\bar{U} \prec \psi \prec \Omega$. Note that $\psi u \in \mathcal{E}'(\mathbf{R}^n)$ and it follows from Theorem 7.16 that $\psi u \in H^t(\mathbf{R}^n)$. By decreasing t if necessary, we can suppose that $s + 2 - t = k \in \mathbf{N}$.

Let $\psi_0 = \psi$, $\psi_k = \varphi$ and define $\psi_1, \ldots, \psi_{k-1}$ by recurrence so that

$$\operatorname{supp} \psi_{j+1} \prec \psi_j \prec U_j \subset \{\psi_{j-1} = 1\}.$$

It is sufficient to show that $\psi_j u \in H^{t+j}(\mathbf{R}^n)$, since then $\varphi u = \psi_k u \in H^{t+k}(\mathbf{R}^n) = H^{s+2}(\mathbf{R}^n)$ will complete the proof.

We only need to prove that if $\varphi, \psi \in \mathcal{D}(\Omega)$ are such that

$$\operatorname{supp} \varphi \prec \psi \quad \text{and} \quad \psi u \in H^t(\mathbf{R}^n),$$

then $\varphi u \in H^{t+1}(\mathbf{R}^n)$.

From the definition of L and from the condition $\operatorname{supp} \varphi \prec \psi$,

$$[L, \varphi]u = L(\varphi u) - \varphi L u = \sum_{j=1}^{n} \big((\partial_j^2 \varphi) u + 2(\partial_j \varphi)\partial_j u\big)$$

is a differential operator of order 1 with smooth coefficients and satisfies $[L, \varphi]u = [L, \varphi](\psi u)$. Hence $L(\varphi u) = [L, \varphi](\psi u) + \varphi L u$ with $[L, \varphi](\psi u) \in H^{t+1}(\mathbf{R}^n)$ and $\varphi L u \in \mathcal{D}(\mathbf{R}^n)$, and we conclude that

$$L(\varphi u) \in H^{t+1}(\mathbf{R}^n).$$

Therefore also

$$(\Delta - 1)(\varphi u) = L(\varphi u) - (\lambda - 1)\varphi u \in H^{t+1}(\mathbf{R}^n)$$

and, since $\Delta - 1$ is a bijective operator from $H^{t-1}(\mathbf{R}^n)$ to $H^{t+1}(\mathbf{R}^n)$, we conclude that $\varphi u \in H^{t-1}(\mathbf{R}^n)$. $\square$

For a more complete analysis concerning the Dirichlet problem we refer the reader to Brezis, "Analyse fonctionelle"[**5**], and Folland, "Introduction to Partial Differential Equations" [**14**].

7.8. Exercises

Exercise 7.1. Calculate the Fourier integral of the following functions on $\mathbf{R}$:

(a) $f_1(t) = te^{-t^2}$.
(b) $f_2(t) = \chi_{(a,b)}$.
(c) $f_3(t) = e^{-|t|}$.
(d) $f_4(t) = (1+t^2)^{-1}$.

Exercise 7.2. Assuming that $1 \le p \le \infty$ and $f \in L^p(\mathbf{R}^n)$, check that the Gauss-Weierstrass kernel

$$W_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \qquad (h > 0)$$

is a summability kernel in $\mathcal{S}(\mathbf{R}^n)$, and prove that

$$u(t,x) := \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} e^{-|y|^2/4t} f(x-y)\, dy \qquad (t > 0)$$

defines a solution of the heat equation

$$\partial_t u - \triangle u = 0$$

on $(0,\infty) \times \mathbf{R}^n$.

If $p < \infty$, show that $\lim_{t\downarrow 0} u(t,\cdot) = f$ in $L^p(\mathbf{R}^n)$. If f is bounded and continuous, prove that u has an extension to a continuous function on $[0,\infty) \times \mathbf{R}^n$ such that $u(0,x) = f(x)$ for all $x \in \mathbf{R}^n$.

Exercise 7.3. The heat flow in an infinitely long road, given an initial temperature f, is described as the solution of the problem

$$\partial_t u(x,t) = \partial_x^2 u(x,t), \quad u(x,0) = f(x).$$

Prove that if $f \in \mathcal{C}_0(\mathbf{R})$ is integrable, then the unique bounded classical solution is

$$u(x,t) = \int_{\mathbf{R}} \widehat{f}(\xi) e^{-4\pi^2\xi^2 t} e^{2\pi i \xi x}\, d\xi = (f * K_t)(x)$$

where

$$K_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{x^2}{4t}}.$$

Exercise 7.4. Find the norm of the Fourier transform $\mathcal{F} : L^1(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$.

Exercise 7.5. Is it true that $f, g \in L^1(\mathbf{R})$ and $f * g = 0$ imply either $f = 0$ or $g = 0$?

Exercise 7.6. Let $1 \le p \le \infty$. Prove that $\varphi \in \mathcal{E}(\mathbf{R}^n)$ is in $\mathcal{S}(\mathbf{R}^n)$ if and only if the functions $x^\beta D^\alpha \varphi(x)$ are all in $L^p(\mathbf{R}^n)$. Show that the inclusion $\mathcal{S}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$ is continuous.

Exercise 7.7. Show that if a rational function f belongs to $\mathcal{S}(\mathbf{R}^n)$, then $f = 0$.

Exercise 7.8. If f is a function on $\mathbf{R}^n$ such that $f\varphi \in \mathcal{S}(\mathbf{R}^n)$ for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$, prove that the pointwise multiplication $f\cdot$ is a continuous linear operator on $\mathcal{S}(\mathbf{R}^n)$.

Exercise 7.9. Suppose $f \in \mathcal{S}(\mathbf{R})$ and $\widehat{f} \in \mathcal{D}_{[-R,R]}$, and let $1 \le p < \infty$. Prove that there is a sequence of constants $C_n \ge 0$ $(n = 0, 1, 2, \ldots)$, which depend only on p, such that

$$\|f^{(n)}\|_p \le C_n R^n \|f\|_p \qquad (n \in \mathbf{N}).$$

Exercise 7.10 (Hausdorff-Young)**.** Show that, if $1 \le p \le 2$ and $f \in L^p(\mathbf{R}^n)$, then $\widehat{f} \in L^\infty(\mathbf{R}^n) + L^2(\mathbf{R}^n)$ and $\mathcal{F} : L^p(\mathbf{R}^n) \to L^{p'}(\mathbf{R}^n)$.

Exercise 7.11. Show that $\operatorname{sinc} \in L^2(\mathbf{R}) \backslash L^1(\mathbf{R})$, and find $\|\operatorname{sinc}\|_2$, $\mathcal{F}(\operatorname{sinc})$, and $\widetilde{\mathcal{F}}(\operatorname{sinc})$.

Exercise 7.12. If u is a solution of the Dirichlet problem (7.11) on a half-plane, find another solution by adding to u an appropriate harmonic function.

Exercise 7.13. Show that $\ell^p \subset \ell$ $(1 \le p \le \infty)$ and that the injective linear map $\ell^p \hookrightarrow \mathcal{S}'$ such that $x[k] \mapsto \sum_{k=-\infty}^{\infty} x[k]\delta_k(t)$ is continuous.

Exercise 7.14 (Hausdorff-Young)**.** Show that $(\sum_{k=-\infty}^{\infty} |c_k(f)|^{p'})^{1/p'} \le \|f\|_p$, with the usual change if $p' = \infty$, if $1 \le p \le 2$ and $f \in L^p(\mathbf{T})$.

Exercise 7.15 (Poisson summation formula)**.** Prove the following facts:

(a) If $\varphi \in \mathcal{S}(\mathbf{R})$, $\varphi_1(t) = \sum_{k=-\infty}^{+\infty} \varphi(t-k)$ is uniformly convergent.

(b) The Fourier series of φ_1 is also uniformly convergent.

(c) $\sum_{k=-\infty}^{+\infty} \varphi(k) = \sum_{k=-\infty}^{+\infty} \widehat{\varphi}(k)$, with absolute convergence.

(d) For the Dirac comb, $\widehat{\text{Ш}} = \text{Ш}$.

Exercise 7.16. If $1 \le q \le \infty$, prove that

$$\|f\| := \|\{\|D^\alpha f\|_p\}_{|\alpha| \le m}\|_q$$

defines on $W^{p,m}(\Omega)$ a norm which is equivalent to $\|\cdot\|_{(m,p)}$.

Exercise 7.17. Every $u \in W^{1,p}(a, \infty)$ is uniformly continuous.

Exercise 7.18. For a half-line (a, ∞) prove a similar result to Theorem 7.25, now about the continuity of $W^{1,p}(a,\infty) \hookrightarrow \mathcal{C}[a,\infty) \cap L^\infty(a,\infty)$.

Exercise 7.19. Every $u \in W^{1,p}(0,\infty)$ can be extended to $Ru \in W^{1,p}(\mathbf{R})$ so that $Ru(t) = u(-t)$ if $t < 0$ and $Ru(x) - u(0) = \int_0^x v(t)\,dt$ for all $x \in \mathbf{R}$. Moreover, $R : W^{1,p}(0,\infty) \to W^{1,p}(\mathbf{R})$ is linear and continuous.

Exercise 7.20. The extension by zero, $P : H_0^1(\Omega) \to H^1(\mathbf{R}^n)$, is a continuous operator.

Exercise 7.21. If $\partial\Omega$ has zero measure, the extension by zero operator, P, satisfies $\partial_j P u = P \partial_j u$ for every $u \in H_0^1(\Omega)$.

Exercise 7.22. If $u \in H^1(-1,1)$, its extension by zero, u^o, is not always in $H^1(\mathbf{R})$.

Exercise 7.23. If $s - k > n/2$ ($s \in \mathbf{R}$) and $m - k > n/2$ ($m \in \mathbf{N}$), prove that the inclusions $H^s(\mathbf{R}^n) \hookrightarrow \mathcal{E}^k(\mathbf{R}^n)$ and $H^m(\Omega) \hookrightarrow \mathcal{E}^k(\Omega)$ of the Sobolev Theorem 7.28 are continuous.

Exercise 7.24. Let $u(x) = e^{-2\pi|x|}$ as in Example 7.12. Prove the following facts:

(a) $u, u' \in L^2(\mathbf{R})$ (distributional derivative), and $u \in H^s(\mathbf{R})$ if $s < 3/2$.

(b) $u \notin H^{3/2}(\mathbf{R})$.

Exercise 7.25. Prove that the Dirichlet problem

$$-\triangle u + u = f, \quad u = 0 \text{ on } \partial\Omega \quad (f \in L^2(\Omega))$$

has a unique weak solution by applying the Lax-Milgram theorem to the sesquilinear form

$$B(u,v) := \int_\Omega \big(\nabla u(x) \cdot \nabla \bar{v}(x) + u(x)\bar{v}(x)\big)\,dx$$

on $H_0^1(\Omega) \times H_0^1(\Omega)$.

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