Preface

Is it really necessary to classify partial differential equations (PDEs) and to employ different methods to discuss different types of equations? Why is it important to derive *a priori* estimates of solutions before even proving the existence of solutions? These are only a few questions any students who just start studying PDEs might ask. Students may find answers to these questions only at the end of a one-semester course in basic PDEs, sometimes after they have already lost interest in the subject. In this book, we attempt to address these issues *at the beginning*. There are several notable features in this book.

First, the importance of *a priori estimates* is addressed at the beginning and emphasized throughout this book. This is well illustrated by the chapter on first-order PDEs. Although first-order linear PDEs can be solved by the method of characteristics, we provide a detailed analysis of *a priori* estimates of solutions in sup-norms and in integral norms. To emphasize the importance of these estimates, we demonstrate how to prove the existence of weak solutions with the help of basic results from functional analysis. The setting here is easy, since L^2-spaces are needed only. Meanwhile, all important ideas are in full display. In this book, we do attempt to derive explicit expressions for solutions whenever possible. However, these explicit expressions of solutions of special equations usually serve mostly to suggest the correct form of estimates for solutions of general equations.

The second feature is the illustration of the necessity to classify second-order PDEs at the beginning. In the chapter on general second-order linear PDEs, immediately after classifying second-order PDEs into elliptic, parabolic and hyperbolic type, we discuss various boundary-value problems and initial/boundary-value problems for the Laplace equation, the heat equation and the wave equation. We discuss energy methods for proving uniqueness and find solutions in the plane by separation of variables. The explicit expressions of solutions demonstrate different properties of solutions of different types of PDEs. Such differences clearly indicate that there is unlikely to be a unified approach to studying PDEs.

Third, we focus on simple models of PDEs and study these equations in detail. We have chapters devoted to the Laplace equation, the heat equation and the wave equation, and use several methods to study each equation. For example, for the Laplace equation, we use three different methods to study its solutions: the fundamental solution, the mean-value property and the maximum principle. For each method, we indicate its advantages and its shortcomings. General equations are not forgotten. We also discuss maximum principles for general elliptic and parabolic equations and energy estimates for general hyperbolic equations.

The book is designed for a one-semester course at the graduate level. Attempts have been made to give a balanced coverage of different classes of partial differential equations. The choice of topics is influenced by the personal tastes of the author. Some topics may not be viewed as *basic* by others. Among those not found in PDE textbooks at a comparable level are estimates in L^∞-norms and L^2-norms of solutions of the initial-value problem for the first-order linear differential equations, interior gradient estimates and differential Harnack inequality for the Laplace equation and the heat equation by the maximum principle, and decay estimates for solutions of the wave equation. Inclusions of these topics reflect the emphasis on estimates in this book.

This book is based on one-semester courses the author taught at the University of Notre Dame in the falls of 2007, 2008 and 2009. During the writing of the book, the author benefitted greatly from comments and suggestions of many of his friends, colleagues and students in his classes. Tiancong Chen, Yen-Chang Huang, Gang Li, Yuanwei Qi and Wei Zhu read the manuscript at various stages. Minchun Hong, Marcus Khuri, Ronghua Pan, Xiaodong Wang and Xiao Zhang helped the author write part of Chapter 8. Hairong Liu did a wonderful job of typing an early version of the manuscript. Special thanks go to Charles Stanton for reading the entire manuscript carefully and for many suggested improvements.

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