Chapter 1

Introduction

This chapter serves as an introduction of the entire book.

In Section 1.1, we first list several notations we will use throughout this book. Then, we introduce the concept of partial differential equations.

In Section 1.2, we discuss briefly well-posed problems for partial differential equations. We also introduce several function spaces whose associated norms are used frequently in this book.

In Section 1.3, we present an overview of this book.

1.1. Notation

In general, we denote by x points in $\mathbb{R}^n$ and write $x = (x_1, \cdots, x_n)$ in terms of its coordinates. For any $x \in \mathbb{R}^n$, we denote by $|x|$ the standard *Euclidean norm*, unless otherwise stated. Namely, for any $x = (x_1, \cdots, x_n)$, we have

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}.$$

Sometimes, we need to distinguish one particular direction as the time direction and write points in $\mathbb{R}^{n+1}$ as (x, t) for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In this case, we call $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ the *space variable* and $t \in \mathbb{R}$ the *time variable*. In $\mathbb{R}^2$, we also denote points by (x, y).

Let Ω be a domain in $\mathbb{R}^n$, that is, an open and connected subset in $\mathbb{R}^n$. We denote by $C(\Omega)$ the collection of all continuous functions in Ω, by $C^m(\Omega)$ the collection of all functions with continuous derivatives up to order m, for any integer $m \geq 1$, and by $C^\infty(\Omega)$ the collection of all functions with continuous derivatives of arbitrary order. For any $u \in C^m(\Omega)$, we denote by

$\nabla^m u$ the collection of all partial derivatives of u of order m. For $m = 1$ and $m = 2$, we usually write $\nabla^m u$ in special forms. For first-order derivatives, we write ∇u as a vector of the form

$$\nabla u = (u_{x_1}, \cdots, u_{x_n}).$$

This is the *gradient vector* of u. For second-order derivatives, we write $\nabla^2 u$ in the matrix form

$$\nabla^2 u = \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} & \cdots & u_{x_1x_n} \\ u_{x_2x_1} & u_{x_2x_2} & \cdots & u_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_nx_1} & u_{x_nx_2} & \cdots & u_{x_nx_n} \end{pmatrix}.$$

This is a symmetric matrix, called the *Hessian matrix* of u. For derivatives of order higher than two, we need to use multi-indices. A multi-index $\alpha \in \mathbb{Z}^n_+$ is given by $\alpha = (\alpha_1, \cdots, \alpha_n)$ with nonnegative integers $\alpha_1, \cdots, \alpha_n$. We write

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$

For any vector $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$, we denote

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

The partial derivative $\partial^\alpha u$ is defined by

$$\partial^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u,$$

and its order is $|\alpha|$. For any positive integer m, we define

$$|\nabla^m u| = \left(\sum_{|\alpha|=m} |\partial^\alpha u|^2 \right)^{\frac{1}{2}}.$$

In particular,

$$|\nabla u| = \left(\sum_{i=1}^n u_{x_i}^2 \right)^{\frac{1}{2}},$$

and

$$|\nabla^2 u| = \left(\sum_{i,j=1}^n u_{x_ix_j}^2 \right)^{\frac{1}{2}}.$$

A hypersurface in $\mathbb{R}^n$ is a surface of dimension $n-1$. Locally, a C^m-hypersurface can be expressed by $\{\varphi = 0\}$ for a C^m-function φ with $\nabla\varphi \neq 0$. Alternatively, by a rotation, we may take $\varphi(x) = x_n - \psi(x_1, \cdots, x_{n-1})$ for a C^m-function ψ of $n-1$ variables. A domain $\Omega \subset \mathbb{R}^n$ is C^m if its boundary $\partial\Omega$ is a C^m-hypersurface.

A *partial differential equation* (henceforth abbreviated as PDE) in a domain $\Omega \subset \mathbb{R}^n$ is a relation of independent variables $x \in \Omega$, an unknown function u defined in Ω, and a finite number of its partial derivatives. To solve a PDE is to find this unknown function. The *order* of a PDE is the order of the highest derivative in the relation. Hence for a positive integer m, the general form of an mth-order PDE in a domain $\Omega \subset \mathbb{R}^n$ is given by

$$F\big(x, u, \nabla u(x), \nabla^2 u(x), \cdots, \nabla^m u(x)\big) = 0 \quad \text{for } x \in \Omega.$$

Here F is a function which is continuous in all its arguments, and u is a C^m-function in Ω. A C^m-solution u satisfying the above equation in the pointwise sense in Ω is often called a *classical solution*. Sometimes, we need to relax regularity requirements for solutions when classical solutions are not known to exist. Instead of going into details, we only mention that it is an important method to establish first the existence of *weak solutions*, functions with less regularity than C^m and satisfying the equation in some weak sense, and then to prove that these weak solutions actually possess the required regularity to be classical solutions.

A PDE is *linear* if it is linear in the unknown functions and their derivatives, with coefficients depending on independent variables x. A general mth-order linear PDE in Ω is given by

$$\sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha u = f(x) \quad \text{for } x \in \Omega.$$

Here a_α is the coefficient of $\partial^\alpha u$ and f is the *nonhomogeneous term* of the equation. A PDE of order m is *quasilinear* if it is linear in the derivatives of solutions of order m, with coefficients depending on independent variables x and the derivatives of solutions of order $< m$. In general, an mth-order quasilinear PDE in Ω is given by

$$\sum_{|\alpha| = m} a_\alpha(x, u, \cdots, \nabla^{m-1} u) \partial^\alpha u = f(x, u, \cdots, \nabla^{m-1} u) \quad \text{for } x \in \Omega.$$

Several PDEs involving one or more unknown functions and their derivatives form a *partial differential system*. We define linear and quasilinear partial differential systems accordingly.

In this book, we will focus on first-order and second-order linear PDEs and first-order linear differential systems. On a few occasions, we will diverge to nonlinear PDEs.

1.2. Well-Posed Problems

What is the meaning of *solving* partial differential equations? Ideally, we obtain explicit solutions in terms of elementary functions. In practice this is only possible for very simple PDEs or very simple solutions of more general

PDEs. In general, it is impossible to find explicit expressions of all solutions of all PDEs. In the absence of explicit solutions, we need to seek methods to prove existence of solutions of PDEs and discuss properties of these solutions. In many PDE problems, this is all we need to do.

A given PDE may not have solutions at all or may have many solutions. When it has many solutions, we intend to assign *extra conditions* to pick up the most relevant solutions. Those extra conditions usually are in the form of boundary values or initial values. For example, when we consider a PDE in a domain, we can require that solutions, when restricted to the boundary, have prescribed values. This is the so-called *boundary-value problems*. When one variable is identified as the time and a part of the boundary is identified as an initial hypersurface, values prescribed there are called *initial values*. We use *data* to refer to boundary values or initial values and certain known functions in the equation, such as the nonhomogeneous term if the equation is linear.

Hadamard introduced the notion of *well-posed problems*. A given problem for a partial differential equation is *well-posed* if

(i) there is a solution;

(ii) this solution is unique;

(iii) the solution depends continuously in some suitable sense on the data given in the problem, i.e., the solution changes by a small amount if the data change by a small amount.

We usually refer to (i), (ii) and (iii) as the *existence*, *uniqueness* and *continuous dependence*, respectively. We need to emphasize that the well-posedness goes beyond the existence and uniqueness of solutions. The continuous dependence is particularly important when PDEs are used to model phenomena in the natural world. This is because measurements are always associated with errors. The model can make useful predictions only if solutions depend on data in a controllable way.

In practice, both the uniqueness and the continuous dependence are proved by *a priori estimates*. Namely, we assume solutions already exist and then derive certain norms of solutions in terms of data in the problem. It is important to note that establishing *a priori* estimates is in fact the first step in proving the existence of solutions. A closely related issue here is the regularity of solutions such as continuity and differentiability. Solutions of a particular PDE can only be obtained if the right kind of regularity, or the right kind of norms, are employed. Two classes of norms are used often, sup-norms and L^2-norms.

Let Ω be a domain in $\mathbb{R}^n$. For any bounded function u in Ω, we define the sup-norm of u in Ω by

$$|u|_{L^\infty(\Omega)} = \sup_\Omega |u|.$$

For a bounded continuous function u in Ω, we may also write $|u|_{C(\Omega)}$ instead of $|u|_{L^\infty(\Omega)}$. Let m be a positive integer. For any function u in Ω with bounded derivatives up to order m, we define the C^m-norm of u in Ω by

$$|u|_{C^m(\Omega)} = \sum_{|\alpha|\le m} |\partial^\alpha u|_{L^\infty(\Omega)}.$$

If Ω is a bounded C^m-domain in $\mathbb{R}^n$, then $C^m(\bar{\Omega})$, the collection of functions which are C^m in $\bar{\Omega}$, is a Banach space equipped with the C^m-norm.

Next, for any Lebesgue measurable function u in Ω, we define the L^2-norm of u in Ω by

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega u^2\, dx \right)^{\frac{1}{2}},$$

where integration is in the Lebesgue sense. The L^2-space in Ω is the collection of all Lebesgue measurable functions in Ω with finite L^2-norms and is denoted by $L^2(\Omega)$. We learned from real analysis that $L^2(\Omega)$ is a Banach space equipped with the L^2-norm.

Other norms will also be used. We will introduce them as needed.

The basic formula for integration is the formula of *integration by parts*. Let Ω be a piecewise C^1-domain in $\mathbb{R}^n$ and $\nu = (\nu_1, \cdots, \nu_n)$ be the unit exterior normal vector to $\partial\Omega$. Then for any $u, v \in C^1(\Omega) \cap C(\bar{\Omega})$,

$$\int_\Omega u_{x_i} v\, dx = -\int_\Omega u v_{x_i}\, dx + \int_{\partial\Omega} u v \nu_i\, dS,$$

for $i = 1, \cdots, n$. Such a formula is the basis for L^2-estimates.

In deriving a priori estimates, we follow a common practice and use the “variable constant” convention. The same letter C is used to denote constants which may change from line to line, as long as it is clear from the context on what quantities the constants depend. In most cases, we are not interested in the value of the constant, but only in its existence.

1.3. Overview

There are eight chapters in this book.

The main topic in Chapter 2 is first-order PDEs. In Section 2.1, we introduce the basic notion of noncharacteristic hypersurfaces for initial-value problems for first-order PDEs. We discuss first-order linear PDEs, quasilinear PDEs and general nonlinear PDEs. In Section 2.2, we solve initial-value problems by the method of characteristics if initial values are prescribed on noncharacteristic hypersurfaces. We demonstrate that solutions of a system of ordinary differential equations (ODEs) yield solutions of the initial-value problems for first-order PDEs. In Section 2.3, we derive estimates of solutions of initial-value problems for first-order linear PDEs. The L^∞-norms and the L^2-norms of solutions are estimated in terms of those of initial values and nonhomogeneous terms. In doing so, we only assume the existence of solutions and do not use any explicit expressions of solutions. These estimates provide quantitative properties of solutions.

Chapter 3 should be considered as an introduction to the theory of second-order linear PDEs. In Section 3.1, we introduce the Laplace equation, the heat equation and the wave equation. We also introduce their general forms, elliptic equations, parabolic equations and hyperbolic equations, which will be studied in detail in subsequent chapters. In Section 3.2, we derive energy estimates of solutions of certain boundary-value problems. Consequences of such energy estimates are the uniqueness of solutions and the continuous dependence of solutions on boundary values and nonhomogeneous terms. In Section 3.3, we solve these boundary-value problems in the plane by separation of variables. Our main focus is to demonstrate different regularity patterns for solutions of different differential equations, the Laplace equation, the heat equation and the wave equation.

In Chapter 4, we discuss the Laplace equation and the Poisson equation. The Laplace equation is probably the most important PDE with the widest range of applications. In the first three sections, we study harmonic functions (i.e., solutions of the Laplace equation), by three different methods: the fundamental solution, the mean-value property and the maximum principle. These three sections are relatively independent of each other. In Section 4.1, we solve the Dirichlet problem for the Laplace equation in balls and derive Poisson integral formula. Then we discuss regularity of harmonic functions using the fundamental solution. In Section 4.2, we study the mean-value property of harmonic functions and its consequences. In Section 4.3, we discuss the maximum principle for harmonic functions and its applications. In particular, we use the maximum principle to derive interior gradient estimates for harmonic functions and the Harnack inequality for positive harmonic functions. We also solve the Dirichlet problem for the Laplace equation in a large class of bounded domains by Perron's method. Last in Section 4.4, we briefly discuss classical solutions and weak solutions of the Poisson equation.

In Chapter 5, we study the heat equation, which describes the temperature of a body conducting heat, when the density is constant. In Section 5.1, we introduce Fourier transforms briefly and derive formally an explicit expression for solutions of the initial-value problem for the heat equation. In Section 5.2, we prove that such an expression indeed yields a classical solution under appropriate assumptions on initial values. We also discuss regularity of arbitrary solutions of the heat equation by the fundamental solution. In Section 5.3, we discuss the maximum principle for the heat equation and its applications. In particular, we use the maximum principle to derive interior gradient estimates for solutions of the heat equation and the Harnack inequality for positive solutions of the heat equation.

In Chapter 6, we study the n-dimensional wave equation, which represents vibrations of strings or propagation of sound waves in tubes for $n = 1$, waves on the surface of shallow water for $n = 2$, and acoustic or light waves for $n = 3$. In Section 6.1, we discuss initial-value problems and various initial/boundary-value problems for the one-dimensional wave equation. In Section 6.2, we study initial-value problems for the wave equation in higher-dimensional spaces. We derive explicit expressions of solutions in odd dimensions by the method of spherical average and in even dimensions by the method of descent. We also discuss global behaviors of solutions. Then in Section 6.3, we derive energy estimates for solutions of initial-value problems. Chapter 6 is relatively independent of Chapter 4 and Chapter 5 and can be taught after Chapter 3.

In Chapter 7, we discuss partial differential systems of first order and focus on existence of local solutions. In Section 7.1, we introduce noncharacteristic hypersurfaces for partial differential equations and systems of arbitrary order. We demonstrate that partial differential systems of arbitrary order can always be changed to those of first order. In Section 7.2, we discuss the Cauchy-Kovalevskaya theorem, which asserts the existence of analytic solutions of noncharacteristic initial-value problems for differential systems if all data are analytic. In Section 7.3, we construct a first-order linear differential system in $\mathbb{R}^3$ which does not admit smooth solutions in any subsets of $\mathbb{R}^3$. In this system, coefficient matrices are analytic and the nonhomogeneous term is a suitably chosen smooth function.

In Chapter 8, we discuss several differential equations we expect to study in more advanced PDE courses. Discussions in this chapter will be brief. In Section 8.1, we discuss basic second-order linear differential equations, including elliptic, parabolic and hyperbolic equations, and first-order linear symmetric hyperbolic differential systems. We will introduce appropriate boundary-value problems and initial-value problems and introduce appropriate function spaces to study these problems. In Section 8.2, we introduce several important nonlinear equations and focus on their background. This chapter is designed to be introductory.

Each chapter, except this introduction and the final chapter, ends with exercises. Level of difficulty varies considerably. Some exercises, at the most difficult level, may require long lasting efforts.