Preface

The motivation for these lecture notes on minimal surfaces is to have a treatment that begins with almost no prerequisites and ends up with current research topics. We touch upon some of the applications to other fields including low dimensional topology, general relativity, and materials science.

Minimal surfaces date back to Euler and Lagrange and the beginning of the calculus of variations. Many of the techniques developed have played key roles in geometry and partial differential equations. Examples include monotonicity and tangent cone analysis originating in the regularity theory for minimal surfaces, estimates for nonlinear equations based on the maximum principle arising in Bernstein's classical work, and even Lebesgue's definition of the integral that he developed in his thesis on the Plateau problem for minimal surfaces.

The only prerequisites needed for this book are a basic knowledge of Riemannian geometry and some familiarity with the maximum principle. Of the various ways of approaching minimal surfaces (from complex analysis, PDE, or geometric measure theory), we have chosen to focus on the PDE aspects of the theory.

In Chapter 1, we will first derive the minimal surface equation as the Euler-Lagrange equation for the area functional on graphs. Subsequently, we derive the parametric form of the minimal surface equation (the first variation formula). The focus of the first chapter is on the basic properties of minimal surfaces, including the monotonicity formula for area and the Bernstein theorem. We also mention some examples. In the next to last section of Chapter 1, we derive the second variation formula, the stability inequality, and define the Morse index of a minimal surface. In the last section, we introduce multi-valued minimal graphs which will play a major role later when we discuss results from [**CM3**]–[**CM7**]. We will also give a local example, from [**CM18**], of spiraling minimal surfaces (like the helicoid) that can be decomposed into multi-valued graphs but where the rate of spiraling is far from constant.

Chapter 2 deals with generalizations of the Bernstein theorem. We begin the chapter by deriving Simons' inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbb{R}^n$. In the later sections, we discuss various applications of this inequality. The first application is a theorem of Choi and Schoen giving curvature estimates for minimal surfaces with small total curvature. Using this estimate, we give a short proof of Heinz's curvature estimate for minimal graphs. Next, we discuss a priori estimates for stable minimal surfaces in three-manifolds, including estimates on area and total curvature of Colding and Minicozzi and the curvature estimate of Schoen. After that, we follow Schoen, Simon and Yau and combine Simons' inequality with the stability inequality to show higher L^p bounds for the square of the norm of the second fundamental form for stable minimal hypersurfaces. The higher L^p bounds are then used together with Simons' inequality to show curvature estimates for stable minimal hypersurfaces and to give a generalization due to De Giorgi, Almgren, and Simons of the Bernstein theorem proven in Chapter 1. We introduce a notion of "almost stabilility" that plays a crucial role in understanding embedded surfaces. Next, we return to multi-valued minimal graphs and prove an important result from [**CM3**] which states that the separation grows sublinearly if the multi-valued graph has enough sheets. We close the chapter with a discussion of minimal cones in Euclidean space and the relationship to the Bernstein theorem.

We start Chapter 3 by introducing stationary varifolds as a generalization of classical minimal surfaces. We next prove the Sobolev inequality of Michael and Simon. After that, we prove a generalization, due to Colding and Minicozzi, of the Bernstein theorem for minimal surfaces discussed in the preceding chapter. Namely, following [**CM6**], we will show in Chapter 3 that, in fact, a bound on the density gives an upper bound for the smallest affine subspace that the minimal surface lies in. We will deduce this theorem from the properties of the coordinate functions (in fact, more generally, properties of harmonic functions) on k-rectifiable stationary varifolds of arbitrary codimension in Euclidean space. Finally, in the last section, we introduce another notion of weak convergence (called bubble convergence) that was developed to explain the bubbling phenomenon that occurs in conformally invariant problems, including two-dimensional harmonic maps and J-holomorphic curves. We will show that bubble convergence implies varifold convergence.

Chapter 4 begins with the solution to the classical Plateau problem for maps from surfaces. There is a close connection between energy and area in dimension two and the main issue is to understand the lack of compactness, called "bubbling", for maps with bounded energy. The first three sections cover the basic existence results for the Dirichlet and Plateau problems for maps from disks, while the fourth section discusses branch points. After that, we turn to the existence of harmonic maps from the two-sphere, following the approach by Sacks and Uhlenbeck, [**SaUh**], of first minimizing a perturbed energy functional and then taking the limit as the perturbation goes to zero.

In Chapter 5, we use a very general argument, whose basic idea goes back to H.A. Schwarz in the 1870s and G.D. Birkhoff in the 1910s, to find minimal spheres on any sphere. The treatment will follow the papers [**CM27**] and [**CM28**] where some of the existence results were new. The idea of both Schwarz and Birkhoff was to use a min-max argument to show existence of critical point for variational problems. This allows us, in particular, to produce minimal surfaces that are not stable. In the min-max construction of minimal surfaces, one sweeps out the manifold by surfaces keeping track of the areas of the slices of the sweepout. One then tries to extract a convergent sequence of maximal slices for which the area of the maximal slice converges to the infimum of the maximal slices of all sweepouts.

Chapter 6 focuses on the regularity of classical solutions to the Plateau problem. After some general discussion of unique continuation and nodal sets, we study the local description of minimal surfaces in a neighborhood of either a branch point or a point of nontransverse intersection. Following Osserman and Gulliver, we rule out interior branch points for solutions of the Plateau problem. In the remainder of the chapter, we prove the embeddedness of the solution to the Plateau problem when the boundary is in the boundary of a mean convex domain. This last result is due to Meeks and Yau.

In Chapter 7, we discuss the theory of minimal surfaces in three-manifolds. We begin by explaining how to extend the earlier results to this case (in particular, monotonicity, the strong maximum principle, and some of the other basic estimates for minimal surfaces). We then prove the results of Hersch, and Choi and Wang. Next, we prove the compactness theorem of Choi and Schoen for embedded minimal surfaces in three-manifolds with positive Ricci curvature. An important point for this compactness result is that by results of Choi and Wang and Yang and Yau such minimal surfaces have uniform area bounds. We then prove the positive mass theorem of Schoen and Yau. In the last section, we prove the Colding-Minicozzi finite extinction theorem for Ricci flow on a homotopy three-sphere.

Finally, in Chapter 8, we will present some recent results on embedded minimal surfaces in $\mathbb{R}^3$. We begin with a local result from [**CM4**] which shows that an embedded minimal disk is either graphical or, on a slightly larger scale, contains a double-spiral staircase. We also state the one-sided curvature estimate from [**CM6**]. This theorem roughly asserts that an embedded minimal disk in $\mathbb{R}^3$, that lies on one side of a plane and comes close to it is a graph over the plane. The novel thing about this estimate is that it does not require any a priori bound unlike all of the classical results for minimal surfaces discussed in the previous chapters of this book. These results are the starting point for the structure of embedded minimal surfaces obtained in the series of papers [**CM3**], [**CM4**], [**CM5**], and [**CM6**]. We then describe an application (from [**CM10**]) of the one-sided curvature estimate to prove the Generalized Nitsche Conjecture (proven originally by P. Collin, [**Co**]). After this, we turn to the resolution of the Calabi-Yau conjectures for embedded surfaces from [**CM24**]. Finally, we describe the main structure theorems from [**CM7**] for embedded minimal surfaces with finite genus and several recent uniqueness results that have relied upon the structure theory of [**CM3**]–[**CM7**].

At the end of the book, problems and exercises are given.

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