Preface

The study of toric varieties is a wonderful part of algebraic geometry. There are elegant theorems and deep connections with polytopes, polyhedra, combinatorics, commutative algebra, symplectic geometry, and topology. Toric varieties also have unexpected applications in areas as diverse as physics, coding theory, algebraic statistics, and geometric modeling. Moreover, as noted by Fulton [**105**], "toric varieties have provided a remarkably fertile testing ground for general theories." At the same time, the concreteness of toric varieties provides an excellent context for someone encountering the powerful techniques of modern algebraic geometry for the first time. Our book is an introduction to this rich subject that assumes only a modest background yet leads to the frontier of this active area of research.

Brief Summary. The text covers standard material on toric varieties, including:

(a) Convex polyhedral cones, polytopes, and fans.

(b) Affine, projective, and abstract toric varieties.

(c) Complete toric varieties and proper toric morphisms.

(d) Weil and Cartier divisors on toric varieties.

(e) Cohomology of sheaves on toric varieties.

(f) The classical theory of toric surfaces.

(g) The topology of toric varieties.

(h) Intersection theory on toric varieties.

These topics are discussed in earlier texts on the subject, such as [**93**], [**105**] and [**219**]. One difference is that we provide more details, with numerous examples, figures, and exercises to illustrate the concepts being discussed. We also provide background material when needed. In addition, we cover a large number of topics previously available only in the research literature.

The Fifteen Chapters. To give you a better idea of what is in the book, we now highlight a few topics from each chapter.

Chapters 1, 2 and 3 cover the basic material mentioned in items (a)–(c) above. The toric varieties encountered include:

- The affine toric variety $Y_{\mathscr{A}}$ of a finite set $\mathscr{A} \subseteq M \simeq \mathbb{Z}^n$ (Chapter 1).
- The affine toric variety U_σ of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ (Chapter 1).
- The projective toric variety $X_{\mathscr{A}}$ of a finite set $\mathscr{A} \subseteq M \simeq \mathbb{Z}^n$ (Chapter 2).
- The projective toric variety X_P of a lattice polytope $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ (Chapter 2).
- The abstract toric variety X_Σ of a fan Σ in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ (Chapter 3).

The general definition of toric variety given in Chapter 3 does not assume that the variety is normal. This differs from other standard texts, which deal exclusively with normal toric varieties. Chapters 1 and 2 include numerous examples of non-normal toric varieties. Nevertheless, the vast majority of toric varieties in the book are normal, and whenever we say "the toric variety X_Σ", we are in the normal case since the toric variety of a fan is normal. When nonnormal toric varieties arise in later chapters, we always warn the reader that normality may fail.

Chapter 4 introduces Weil and Cartier divisors on toric varieties. We compute the class group and Picard group of a toric variety and define the sheaf $\mathscr{O}_{X_\Sigma}(D)$ associated to a Weil divisor D on a toric variety X_Σ.

Chapter 5 shows that the classical construction $\mathbb{P}^n = \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^*$ can be generalized to any toric variety X_Σ. The homogeneous coordinate ring $\mathbb{C}[x_0,\ldots,x_n]$ of $\mathbb{P}^n$ also has a toric generalization, called the total coordinate ring of X_Σ.

Chapter 6 relates Cartier divisors to invertible sheaves on X_Σ. We introduce ample, basepoint free, and nef divisors and discuss their relation to convexity. The structure of the nef cone and its dual, the Mori cone, are described in detail, as is the intersection pairing between divisors and curves.

Chapter 7 extends the relation between polytopes and projective toric varieties to a relation between polyhedra and projective toric morphisms $\phi : X_\Sigma \to U_\sigma$. We also discuss projective bundles over a toric variety and use these to classify smooth projective toric varieties of Picard number 2.

Chapter 8 relates Weil divisors to reflexive sheaves of rank one and defines Zariski p-forms. For $p = \dim X$, this gives the canonical sheaf ω_X and canonical divisor K_X. In the toric case we describe these explicitly and study the relation between reflexive polytopes and Gorenstein Fano toric varieties, meaning that $-K_{X_\Sigma}$ is ample. We find the 16 reflexive polygons in $\mathbb{R}^2$ (up to equivalence) and note the relation $|\partial P \cap M| + |\partial P^\circ \cap N| = 12$ for a reflexive polygon P and its dual P°.

Chapter 9 is about sheaf cohomology. We give two methods for computing sheaf cohomology on a toric variety and prove a dizzying array of cohomology vanishing theorems. Applications range from showing that normal toric varieties are Cohen-Macaulay to the Dehn-Sommerville equations for a simple polytope and counting lattice points in multiples of a polytope via the Ehrhart polynomial.

Chapter 10 studies toric surfaces, where we add a few twists to this classical subject. After using Hirzebruch-Jung continued fractions to compute the minimal resolution of a toric surface singularity, we discuss the toric meaning of ordinary continued fractions. We then describe unexpected connections with Gröbner fans and the McKay correspondence. Finally, we use the Riemann-Roch theorem on a smooth complete toric surface to explain the mysterious appearance of the number 12 in Chapter 8 when counting lattice points in reflexive polygons.

Chapter 11 proves the existence of toric resolutions of singularities for toric varieties of all dimensions. This is more complicated than for surfaces because of the existence of toric flips and flops. We consider simple normal crossing, crepant, log, and embedded resolutions and study how Rees algebras and multiplier ideals can be applied in the resolution problem. We also discuss toric singularities and show that a fan Σ is simplicial if and only if X_Σ has at worst finite quotient singularities and hence is rationally smooth. We also explain what canonical and terminal singularities mean in the toric context.

Chapters 12 and 13 describe the singular and equivariant cohomology of a complete simplicial toric variety X_Σ and prove the Hirzebruch-Riemann-Roch and equivariant Riemann-Roch theorems when X_Σ is smooth. We compute the fundamental group of X_Σ and study the moment map, with a brief mention of topological models of toric varieties and connections with symplectic geometry. We describe the Chow ring and intersection cohomology of a complete simplicial toric variety. After proving Riemann-Roch, we give applications to the volume polynomial and lattice point enumeration in polytopes.

Chapters 14 and 15 explore the rich connections that link geometric invariant theory, the secondary fan, the nef and moving cones, Gale duality, triangulations, wall crossings, flips, extremal contractions, and the toric minimal model program.

Appendices. The book ends with three appendices:

- Appendix A: The History of Toric Varieties.
- Appendix B: Computational Methods.
- Appendix C: Spectral Sequences.

Appendix A surveys the history of toric geometry since its origins in the early 1970s. It is fun to see how the concepts and terminology evolved. Appendix B discusses some of the software packages for toric geometry and gives examples to illustrate what they can do. Appendix C gives a brief introduction to spectral sequences and describes the spectral sequences used in Chapters 9 and 12.

Prerequisites. We assume that the reader is familiar with the material covered in basic graduate courses in algebra and topology, and to a somewhat lesser degree, complex analysis. In addition, we assume that the reader has had some previous experience with algebraic geometry, at the level of any of the following texts:

- *Ideals, Varieties and Algorithms* by Cox, Little and O'Shea [**69**].
- *Introduction to Algebraic Geometry* by Hassett [**133**].
- *Elementary Algebraic Geometry* by Hulek [**151**].
- *Undergraduate Algebraic Geometry* by Reid [**239**].
- *Computational Algebraic Geometry* by Schenck [**247**].
- *An Invitation to Algebraic Geometry* by Smith, Kahanpää, Kekäläinen and Traves [**254**].

Chapters 9 and 12 assume knowledge of some basic algebraic topology. The books by Hatcher [**135**] and Munkres [**211**] are useful references here.

Readers who have studied more sophisticated algebraic geometry texts such as Harris [**130**], Hartshorne [**131**], or Shafarevich [**246**] certainly have the background needed to read our book. For readers with a more modest background, an important prerequisite is a willingness to absorb a lot of algebraic geometry.

Background Sections. Since we do not assume a complete knowledge of algebraic geometry, Chapters 1–9 each begin with a background section that introduces the definitions and theorems from algebraic geometry that are needed to understand the chapter. References where proofs can be found are provided. The remaining chapters do not have background sections. For some of those chapters, no further background is necessary, while for others, the material is more sophisticated and the requisite background is given by careful references to the literature.

What Is Omitted. We work exclusively with varieties defined over the complex numbers $\mathbb{C}$. The toric variety of a fan can be defined as a scheme over $\mathrm{Spec}(\mathbb{Z})$, and many properties of toric varieties hold in this generality (see [**82**]). Most results in the book are valid over an algebraically closed field (see, for example, [**172**]). When the field is not algebraically closed, a more sophisticated situation can occur where the torus is not split and the "toric variety" contains not the torus but rather a principal homogeneous space for the torus (see [**92**] for a treatment of this topic). None of this is in the book because of our focus on $\mathbb{C}$.

We also do not consider toric stacks (see [**39**] for an introduction). Moreover, our viewpoint is primarily algebro-geometric. Thus, while we hint at some of the connections with symplectic geometry and topology in Chapter 12, we do not do justice to this side of the story. Even within the algebraic geometry of toric varieties, there are many topics we have had to omit, though we provide some references that should help readers who want to explore these areas. We have also omitted any discussion of how toric varieties are used in physics and applied mathematics. Some pointers to the literature are given in our discussion of the recent history of toric varieties in §A.2 of Appendix A.

The Structure of the Text. We number theorems, propositions, and equations based on the chapter and the section. Thus §3.2 refers to Section 2 of Chapter 3, and Theorem 3.2.6, equation (3.2.6) and Exercise 3.2.6 all appear in this section. Definitions, theorems, propositions, lemmas, remarks, and examples are numbered together in one sequence within each section.

Some individual chapters have appendices. Within a chapter appendix the same numbering system is used, except that the section number is a capital A. This means that Theorem 3.A.3 is in the appendix to Chapter 3. On the other hand, the three appendices at the end of the book are treated in the numbering system as chapters A, B, and C. Thus Definition C.1.1 is in the first section of Appendix C.

The end (or absence) of a proof is indicated by $\square$, and the end of an example is indicated by $\Diamond$.

For the Instructor. There is much more material here than you can cover in any one-semester graduate course, probably more than you can cover in a full year in most cases. So choices will be necessary depending on the background and the interests of the student audience. We think it is reasonable to expect to cover most of Chapters 1–6, 8 and 9 in a one-semester course where the students have a minimal background in algebraic geometry. More material can be covered, of course, if the students know more algebraic geometry. If time permits, you can use toric surfaces (Chapter 10) to illustrate the power of the basic material and introduce more advanced topics such as the resolution of singularities (Chapter 11) and the Riemann-Roch theorem (Chapter 13).

Finally, we emphasize that the exercises are extremely important. We have found that when the students work in groups and present their solutions, their engagement with the material increases. We encourage instructors to consider using this strategy.

For the Student. The book assumes that you will be an active reader. This means in particular that you should do tons of exercises—this is the best way to learn about toric varieties. If you have a modest background in algebraic geometry, then reading the book requires a commitment to learn *both* toric varieties *and* algebraic geometry. It will be a lot of work but is worth the effort. This is a great subject.

Send Us Feedback. We greatly appreciate hearing from instructors, students, or general readers about what worked and what didn't. Please notify one or all of us about any typographical or mathematical errors you might find.

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