Chapter 10

Toric Surfaces

In this chapter, we will apply the theory developed so far to study the structure of 2-dimensional normal toric varieties (toric surfaces). We will describe their singularities, introduce the idea of a resolution of singularities, and also classify smooth complete toric surfaces. Along the way, we will encounter two types of continued fractions, Hilbert bases, the Gröbner fan, the McKay correspondence, the Riemann-Roch theorem, the sectional genus, and the number 12.

§10.1. Singularities of Toric Surfaces and Their Resolutions

Singular Points of Toric Surfaces. If X_Σ is the toric surface of a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^2$, then minimal generators of the rays $\rho \in \Sigma(1)$ are primitive and hence extend to a basis of N. Then Theorem 3.1.19 implies that the toric surface obtained by removing the fixed points of the torus action (i.e., the points corresponding to the 2-dimensional cones under the Orbit-Cone Correspondence) is smooth. There are only finitely many such points, so X_Σ has at most finitely many singular points. Moreover, 2-dimensional cones are always simplicial, so from Example 1.3.20, each of these singular points is a finite abelian quotient singularity (isomorphic to the image of the origin in the quotient $\mathbb{C}^2/G$ where G is a finite abelian group).

All cones are assumed to be rational and polyhedral. A 2-dimensional strongly convex cone in $N_\mathbb{R} \simeq \mathbb{R}^2$ has the following normal form that will facilitate our study of the singularities of toric surfaces.

Proposition 10.1.1. *Let $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^2$ be a 2-dimensional strongly convex cone. Then there exists a basis e_1, e_2 for N such that*

$$\sigma = \operatorname{Cone}(e_2, de_1 - ke_2),$$

where $d > 0$, $0 \le k < d$, and $\gcd(d,k) = 1$.

Proof. We will need the following modified division algorithm here and at several other points in this chapter (Exercise 10.1.1):

(10.1.1) Given integers l and $d > 0$, there are unique integers s and k such that $l = sd - k$ and $0 \leq k < d$.

Say $\sigma = \mathrm{Cone}(u_1, u_2)$, where u_i are primitive vectors. Since u_1 is primitive, we can take it as part of a basis of N, and we let $e_2 = u_1$. Since σ is strongly convex, for any basis e_1', e_2 for N, it will be true that

$$u_2 = de_1' + le_2$$

for some $d \neq 0$. By replacing e_1' by $-e_1'$ if necessary, we can assume $d > 0$. By (10.1.1), there are integers s, k such that $l = sd - k$, where $0 \leq k < d$. Using this integer s, let $e_1 = e_1' + se_2$. Then e_1, e_2 is also a basis for N and

$$u_2 = de_1 + (l - sd)e_2 = de_1 - ke_2.$$

Hence $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ as claimed, and $\gcd(d,k) = 1$ follows since u_2 is primitive. □

We will call the integers d, k in this statement the *parameters* of the cone σ, and $\{e_1, e_2\}$ is called a *normalized basis* for N relative to σ. The uniqueness of d, k will be studied in Proposition 10.1.3 below.

Using the normal form, we next describe the local structure of the point p_σ in the affine toric variety U_σ. Recall from Example 1.3.20 that if $N' \subseteq N$ is the sublattice generated by the ray generators of σ, then $U_\sigma \simeq \mathbb{C}^2/G$, where $G = N/N'$. In our situation, $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and

$$N' = \mathbb{Z}e_2 \oplus \mathbb{Z}(de_1 - ke_2) = d\mathbb{Z}e_1 \oplus \mathbb{Z}e_2,$$

so it follows easily that

$$G = N/N' \simeq \mathbb{Z}/d\mathbb{Z}. \tag{10.1.2}$$

In particular, for singularities of toric surfaces, the finite group G is always cyclic.

The action of G on $\mathbb{C}^2$ is determined by the integers d, k as follows. We write

$$\mu_d = \{\zeta \in \mathbb{C} \mid \zeta^d = 1\}$$

for the group of dth roots of unity in $\mathbb{C}$. Then a choice of a primitive dth root of unity defines an isomorphism of groups $\mu_d \simeq \mathbb{Z}/d\mathbb{Z}$.

Proposition 10.1.2. *Let M' be the dual lattice of N' and let $m_1, m_2 \in M'$ be dual to u_1, u_2 in N'. Using the coordinates $x = \chi^{m_1}$ and $y = \chi^{m_2}$ of $\mathbb{C}^2$, the action of $\zeta \in \mu_d \simeq N/N'$ on $\mathbb{C}^2$ is given by*

$$\zeta \cdot (x, y) = (\zeta x, \zeta^k y).$$

Furthermore, $U_\sigma \simeq \mathbb{C}^2/\mu_d$ with respect to this action.

Proof. The general discussion in §1.3 shows that the quotient $N/N' \simeq \mathbb{Z}/d\mathbb{Z}$ acts on the coordinate ring of $\mathbb{C}^2$ via

$$(u+N')\cdot \chi^{m'} = e^{2\pi i\langle m',u\rangle}\chi^{m'}, \tag{10.1.3}$$

where $m' \in \sigma^\vee \cap M'$ and $u = je_1$ for $0 \le j \le d-1$.

An easy calculation shows that $\langle m_1, e_1\rangle = 1/d$ and $\langle m_2, e_1\rangle = k/d$. Hence if we set up the isomorphism $\mu_d \simeq N/N'$ by mapping $e^{2\pi ij/d} \mapsto je_1 + N'$, then for all $\zeta = e^{2\pi ij/d} \in \mu_d$, we have

$$\zeta\cdot(x,y) = (e^{2\pi ij/d}x, e^{2\pi ijk/d}y) = (\zeta x, \zeta^k y)$$

by (10.1.3). This is what we wanted to show. □

We next describe the slight but manageable ambiguity in the normal form for 2-dimensional cones. Two cones are *lattice equivalent* if there is a bijective $\mathbb{Z}$-linear mapping $\varphi : N \to N$ taking one cone to the other. After a choice of basis for N, such mappings are defined by matrices in $\mathrm{GL}(2,\mathbb{Z})$.

Proposition 10.1.3. *Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ and $\widetilde{\sigma} = \mathrm{Cone}(e_2', \widetilde{d}e_1' - \widetilde{k}e_2')$ be cones in normal form that are lattice equivalent. Then $\widetilde{d} = d$ and either $\widetilde{k} = k$ or $k\widetilde{k} \equiv 1 \bmod d$.*

Proof. Since the cones are lattice equivalent, writing N' and $\widetilde{N}'$ for the sublattices as in (10.1.2), there is a bijective $\mathbb{Z}$-linear mapping $\varphi : N \to N$ such that $\varphi(N') = \widetilde{N}'$. Hence $N/\widetilde{N}' \simeq N/N'$, so $\widetilde{d} = d$. The statement about k and $\widetilde{k}$ is left to the reader in Exercise 10.1.2. □

Here are two examples to illustrate Proposition 10.1.2.

Example 10.1.4. First consider a cone

$$\sigma = \mathrm{Cone}(e_2, de_1 - e_2)$$

with parameters $d > 1$ (so the cone is not smooth) and $k = 1$. This is precisely the cone considered in Example 1.2.22. The corresponding toric surface U_σ is the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$. The quotient $\widehat{C}_d \simeq \mathbb{C}^2/\mu_d$ was studied in the special case $d = 2$ in Example 1.3.19, and the general case was described in Exercise 1.3.11. With the notation of Proposition 10.1.2, $\zeta \in \mu_d$ acts on $(x,y) \in \mathbb{C}^2$ via $\zeta\cdot(x,y) = (\zeta x, \zeta y)$ and the ring of invariants is

$$\mathbb{C}[x,y]^{\mu_d} = \mathbb{C}[x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d],$$

so

$$U_\sigma \simeq \mathbb{C}^2/\mu_d \simeq \mathrm{Spec}(\mathbb{C}[x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d]).$$

On the other hand, from Example 1.2.22, we also have the description

$$U_\sigma \simeq \mathrm{Spec}(\mathbb{C}[s, st, st^2, \ldots, st^d]).$$

Exercise 10.1.3 studies the relation between these representations of the coordinate ring of U_σ. ◊

Example 10.1.5. Next consider a cone σ with parameters d and $k = d-1$, so $d = k+1$. We will express everything in terms of the parameter k in the following. Unlike the previous example, this is a case we have not encountered previously. Note that $k \equiv -1 \bmod d$. Hence by Proposition 10.1.2, the action of $G = N/N'$ on $\mathbb{C}^2$ is given by

$$\zeta \cdot (x,y) = (\zeta x, \zeta^{-1} y).$$

It is easy to check that the ring of invariants here is

$$\mathbb{C}[x,y]^{\mu_{k+1}} = \mathbb{C}[x^{k+1}, y^{k+1}, xy].$$

Moreover we have an isomorphism of rings

$$\begin{aligned} \varphi : \mathbb{C}[X,Y,Z]/\langle Z^{k+1} - XY\rangle &\simeq \mathbb{C}[x^{k+1}, y^{k+1}, xy] \\ X &\mapsto x^{k+1} \\ Y &\mapsto y^{k+1} \\ Z &\mapsto xy, \end{aligned}$$

so we may identify the toric surface U_σ with the variety $\mathbf{V}(Z^{k+1} - XY) \subseteq \mathbb{C}^3$. ◊

The origin is the unique singular point of the affine variety of Example 10.1.5 and is called a *rational double point* (or *Du Val singularity*) of type A_k. Another standard form of these singularities is given in Exercise 10.1.4. They are called *double* points because the lowest degree nonzero term in the defining equation has degree two (i.e., the *multiplicity* of the singularity is two). The *rational* double points are the simplest singularities from a certain point of view. The exact definition, which we will give in §10.4, depends on the notion of a resolution of singularities, which will be introduced shortly. All rational double points appear as singularities of quotient surfaces $\mathbb{C}^2/G$ where G is a finite subgroup of $\mathrm{SU}(2,\mathbb{C})$. There is a complete classification of such points in terms of the *Dynkin diagrams* of types A_k, D_k, E_6, E_7, and E_8. The groups corresponding to the diagrams D_k, E_6, E_7, E_8 are not abelian, so by the comment after (10.1.2), such points do not appear on toric surfaces. We will see one way that the Dynkin diagram A_k appears from the geometry of the toric surface U_σ in Exercise 10.1.5, and we will return to this example in §10.4. More details on these singularities can be found in [**246**, Ch. VI] and in the article [**85**].

Here is another interesting aspect of Example 10.1.5. Recall that a normal variety X is *Gorenstein* if its canonical divisor is Cartier (Definition 8.2.14). The following result was proved in Exercise 8.2.13 of Chapter 8.

Proposition 10.1.6. *For a cone* $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ *in normal form, the affine toric surface* U_σ *is Gorenstein if and only if* $k = d-1$. □

Toric Resolution of Singularities. Let X be a normal toric surface, and denote by X_{sing} the finite set of singular points of X (possibly empty).

Definition 10.1.7. A proper morphism $\varphi : Y \to X$ is a ***resolution of singularities*** of X if Y is a smooth surface and φ induces an isomorphism of varieties

$$Y \setminus \varphi^{-1}(X_{\text{sing}}) \simeq X \setminus X_{\text{sing}}. \tag{10.1.4}$$

Such a mapping modifies X to produce a smooth variety without changing the smooth locus $X \setminus X_{\text{sing}}$. One of the most appealing aspects of toric varieties is the way that many questions that are difficult for general varieties admit simple and concrete solutions in the toric case. The problem of finding resolutions of singularities is a perfect example. We illustrate this by constructing explicit resolutions of singularities of the toric surfaces from Examples 10.1.4 and 10.1.5.

Example 10.1.8. Consider the rational normal cone of degree d, the affine toric surface U_σ for $\sigma = \text{Cone}(e_2, de_1 - e_2)$ studied in Example 10.1.4. Let Σ be the fan in Figure 1 obtained by inserting a new ray $\tau = \text{Cone}(e_1)$ subdividing σ into two 2-dimensional cones:

$$\begin{aligned}\sigma_1 &= \text{Cone}(e_2, e_1)\\ \sigma_2 &= \text{Cone}(e_1, de_1 - e_2).\end{aligned}$$

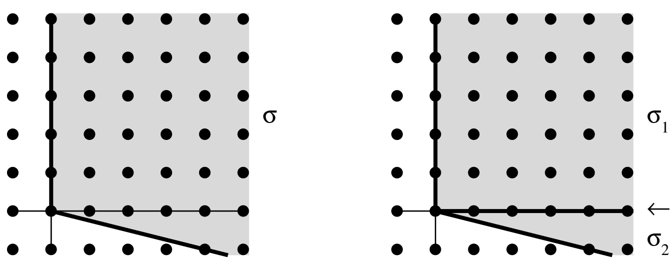

Figure 1. The cone σ and the refinement given by σ_1, σ_2, τ

We now use some results from Chapter 3. The identity mapping on the lattice N is compatible with the fans Σ and σ as in Definition 3.3.1. By Theorem 3.3.4, we have a corresponding toric blowup morphism

$$\phi : X_\Sigma \longrightarrow U_\sigma. \tag{10.1.5}$$

Note that both σ_1 and σ_2 (as well as all of their faces) are smooth cones. Hence Theorem 3.1.19 implies that X_Σ is a smooth surface. In addition, the toric morphism ϕ is proper by Theorem 3.4.11 since Σ is a refinement of σ. Finally, we claim that ϕ satisfies (10.1.4). This follows from the Orbit-Cone Correspondence on the two surfaces: if p_σ is the distinguished point corresponding to the 2-dimensional cone σ (the singular point of U_σ at the origin), then ϕ restricts to an isomorphism

$$X_\Sigma \setminus \phi^{-1}(p_\sigma) \simeq U_\sigma \setminus \{p_\sigma\} = (U_\sigma)_{\text{smooth}}.$$

The inverse image $E = \phi^{-1}(p_\sigma)$ is the curve on X_Σ given by the closure of the T_N-orbit $O(\tau)$ corresponding to the ray τ. That is, the singular point "blows up" to $E \simeq \mathbb{P}^1$ on the smooth surface. It follows that X_Σ and the morphism (10.1.5) give a toric resolution of singularities of the rational normal cone. We call E the *exceptional divisor* on the smooth surface. We will say more about how E sits inside the surface X_Σ in §10.4. ◊

Example 10.1.9. We consider the case $d = 4$ of Example 10.1.5, for which the surface U_σ has a rational double point of type A_3. We will leave the details, as well as the generalization to all $d \geq 2$, to the reader (Exercise 10.1.5). It is easy to find subdivisions of

$$\sigma = \text{Cone}(e_2, 4e_1 - 3e_2)$$

yielding collections of smooth cones. The most economical way to do this is to insert three new rays $\rho_1 = \text{Cone}(e_1)$, $\rho_2 = \text{Cone}(2e_1 - e_2)$, $\rho_3 = \text{Cone}(3e_1 - 2e_2)$ to obtain a fan Σ consisting of four 2-dimensional cones and their faces.

The fan produced by this subdivision is somewhat easier to visualize if we draw the cones relative to a different basis u_1, u_2 for N. For $u_1 = e_2$ and $u_2 = e_1 - e_2$, the cone $\sigma = \text{Cone}(u_1, u_1 + 4u_2)$ and the fan Σ with maximal cones

$$\begin{aligned} \sigma_1 &= \text{Cone}(u_1, u_1 + u_2) \\ \sigma_2 &= \text{Cone}(u_1 + u_2, u_1 + 2u_2) \\ \sigma_3 &= \text{Cone}(u_1 + 2u_2, u_1 + 3u_2) \\ \sigma_4 &= \text{Cone}(u_1 + 3u_2, u_1 + 4u_2) \end{aligned} \tag{10.1.6}$$

appear in Figure 2.

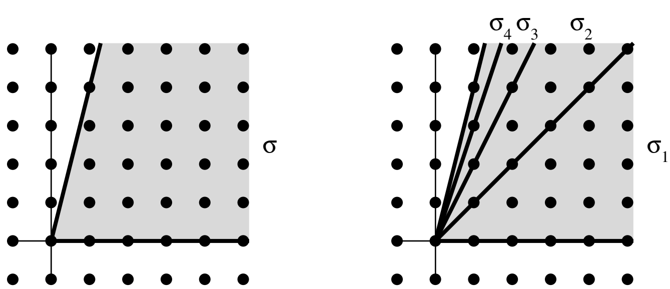

Figure 2. The cone σ and the refinement Σ

You will check that each of these cones is smooth. Hence X_Σ is a smooth surface. Since Σ is a refinement of σ, we have a proper toric morphism

$$\phi : X_\Sigma \to U_\sigma.$$

As in the previous example, ϕ restricts to an isomorphism from $X_\Sigma \setminus \phi^{-1}(p_\sigma)$ to $X_\Sigma \setminus \{p_\sigma\}$. In this case, the exceptional divisor $E = \phi^{-1}(p_\sigma)$ is the union

$$E = V(\tau_1) \cup V(\tau_2) \cup V(\tau_3)$$

on X_Σ. The curves $V(\tau_i)$ are isomorphic to $\mathbb{P}^1$. The first two intersect transversely at the fixed point of the T_N-action on X_Σ corresponding to the cone σ_2, while the second two intersect transversely at the fixed point corresponding to σ_3. $\Diamond$

In these examples, we constructed toric resolutions of affine toric surfaces with just one singular point. The same techniques can be applied to any normal toric surface X_Σ.

Theorem 10.1.10. *Let X_Σ be a normal toric surface. There exists a smooth fan Σ' refining Σ such that the associated toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is a toric resolution of singularities.*

Proof. It suffices to show the existence of the smooth fan Σ' refining Σ. The reasoning given in Example 10.1.8 applies to show that the corresponding toric morphism ϕ is proper and birational, hence a resolution of singularities of X_Σ.

We will prove this by induction on an integer invariant of fans that measures the complexity of the singularities on the corresponding surfaces. Let $\sigma_1, \ldots, \sigma_\ell$ denote the 2-dimensional cones in a fan Σ. For each i, we will write N_i for the sublattice of N generated by the ray generators of σ_i. Then we define

$$s(\Sigma) = \sum_{i=1}^{\ell} (\mathrm{mult}(\sigma_i) - 1),$$

where $\mathrm{mult}(\sigma_i) = [N : N_i]$ as in §6.3. If $s(\Sigma) = 0$, then $\ell = 0$ or $\mathrm{mult}(\sigma_i) = 1$ for all i. It is easy to see that this implies that Σ is a smooth fan. Hence X_Σ is already smooth and we take this as the base case for our induction.

For the induction step, we assume that the existence of smooth refinements has been established for all fans Σ with $s(\Sigma) < s$, and consider a fan Σ with $s(\Sigma) = s$. If $s \geq 1$, then there exists some nonsmooth cone σ_i in Σ. By Proposition 10.1.1, there is a basis e_1, e_2 for N such that $\sigma_i = \mathrm{Cone}(e_2, de_1 - ke_2)$ with parameters $d > 0$, $0 \leq k < d$, and $\gcd(d,k) = 1$. Consider the refinement Σ' of Σ obtained by subdividing the cone σ_i into two new cones

$$\begin{aligned} \sigma_i' &= \mathrm{Cone}(e_2, e_1) \\ \sigma_i'' &= \mathrm{Cone}(e_1, de_1 - ke_2) \end{aligned}$$

with a new 1-dimensional cone $\rho = \mathrm{Cone}(e_1)$. We must show that $s(\Sigma') < s(\Sigma)$ to invoke the induction hypothesis and conclude the proof.

In $s(\Sigma)$, the terms corresponding to the other cones σ_j for $j \neq i$ are unchanged. The cone σ_i' is smooth since e_1, e_2 is the normalized basis of N relative to σ_i. So it contributes a zero term in $s(\Sigma')$. Now consider the cone σ_i''. In order to compute its contribution to $s(\Sigma')$, we must determine the parameters of σ_i''.

In terms of the basis e_1, e_2 for N, the $\mathbb{Z}$-linear mapping defined by the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(a "90-degree rotation") takes σ_i'' to $\mathrm{Cone}(e_2, ke_1 + de_2)$. Since $A \in \mathrm{GL}(2,\mathbb{Z})$, it defines an automorphism of N, and hence σ_i'' will have the same parameters as $\mathrm{Cone}(e_2, ke_1 + de_2)$. But now we apply (10.1.1) to write

$$d = sk - l \tag{10.1.7}$$

where $0 \leq l < k$. Since $\gcd(d,k) = 1$, we have $\gcd(k,l) = 1$ as well. Hence the cone σ_i'' has parameters k and l obtained from (10.1.7). Since $k < d$, if N_i'' is the sublattice generated by the ray generators of σ_i'', then by (10.1.2),

$$[N : N_i''] = k < [N : N_i] = d.$$

It follows that $s(\Sigma') < s(\Sigma)$, and the proof is complete by induction. □

We will see in the next section that in the affine case, the refinement that gives the resolution of singularities of U_σ has a very nice description. As a preview, notice that in Examples 10.1.8 and 10.1.9, the refinement of the given cone σ was produced by subdividing along the rays through the Hilbert basis (the irreducible elements) of the semigroup $\sigma \cap N$.

A resolution of a nonnormal toric surface singularity can be constructed by first saturating the associated semigroup as in Theorem 1.3.5, then applying the results of this section. Toric resolutions of singularities for toric varieties of dimension three and larger also exist. However, we postpone the higher-dimensional case until Chapter 11.

Exercises for §10.1.

10.1.1. Adapt the usual proof of the integer division algorithm to prove (10.1.1).

10.1.2. In this exercise, you will develop further properties of the parameters d,k in the normal form for cones from Proposition 10.1.1 and prove part of Proposition 10.1.3.

(a) Show that if $\widetilde{\sigma}$ is obtained from a cone σ by parameters d,k by a $\mathbb{Z}$-linear mapping of N defined by a matrix in $\mathrm{GL}(2,\mathbb{Z})$, then the parameter $\widetilde{k}$ of $\widetilde{\sigma}$ satisfies either $\widetilde{k} = k$, or $k\widetilde{k} \equiv 1 \bmod d$. Hint: There is a choice of *orientation* to be made in the normalization process. Recall that $\gcd(d,k) = 1$, so there are integers $\widetilde{d}, \widetilde{k}$ such that $d\widetilde{d} + k\widetilde{k} = 1$.

(b) Show that if σ is a cone with parameters d,k, then the dual cone $\sigma^\vee \subseteq M_\mathbb{R}$ has parameters $d, d-k$. Hint: Use the normal form for σ, write down $\sigma^\vee$ in the corresponding dual basis in M, then change bases in M to normalize $\sigma^\vee$.

10.1.3. With the notation in Example 10.1.4, show that

$$\mathbb{C}[s, st, st^2, \ldots, st^d] \simeq \mathbb{C}[x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d]$$

under $s \mapsto x^d$ and $t \mapsto y/x$, and use Proposition 10.1.2 to explain where these identifications come from in terms of the semigroup S_σ. Hint: We have $s = \chi^{m_1}$ and $t = \chi^{m_2}$ where e_1, e_2 is the normalized basis for N and m_1, m_2 is the dual basis for M.

10.1.4. In Example 10.1.5, we gave one form of the rational double point of type A_k, namely the singular point at $(0,0,0)$ on the surface $V = \mathbf{V}(Z^{k+1} - XY) \subseteq \mathbb{C}^3$. Another commonly used normal form for this type of singularity is the singular point at $(0,0,0)$ on the surface $W = \mathbf{V}(X^{k+1} + Y^2 + Z^2)$. Show that V and W are isomorphic as affine varieties, hence the singularities at the origin are analytically equivalent. Hint: There is a linear change of coordinates in $\mathbb{C}^3$ that does this.

10.1.5. In this exercise, you will check the claims made in Example 10.1.9 and show how to extend the results there to the case $\sigma = \mathrm{Cone}(e_2, de_1 - (d-1)e_2)$ for general d.

(a) Check that each of the four cones in (10.1.6) is smooth, so that the toric surface X_Σ is smooth by Theorem 3.1.19.

(b) For general d, show how to insert new rays ρ_i to subdivide σ and obtain a fan Σ whose associated toric surface is smooth. Try to do this with as few new rays as possible. Hence we obtain toric resolutions of singularities $\phi : X_\Sigma \to U_\sigma$ for all d.

(c) Identify the inverse image $C = \phi^{-1}(p_\sigma)$ in general. For instance, how many irreducible components does C have? How are they connected? Hint: One way to represent the structure is to draw a graph with vertices corresponding to the components and connect two vertices by an edge if and only if the components intersect on X_Σ. Do you notice a relation between this graph and the Dynkin diagram $A_k = A_{d-1}$ mentioned before? We will discuss the relation in detail in §10.4.

§10.2. Continued Fractions and Toric Surfaces

To relate continued fractions to toric surfaces, we begin with the affine toric surface U_σ of a cone $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^2$ in normal form with parameters d,k. We will always assume $d > k > 0$, so that U_σ has a unique singular point.

Hirzebruch-Jung Continued Fractions. When we construct a resolution of singularities of U_σ by following the proof of Theorem 10.1.10, the first step is to refine the cone $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ to a fan containing the 2-dimensional cones

$$\sigma' = \mathrm{Cone}(e_2, e_1) \quad \text{and} \quad \sigma'' = \mathrm{Cone}(e_1, de_1 - ke_2).$$

The first is smooth, but the second may not be. However, we saw in the proof of Theorem 10.1.10 that the cone σ'' has parameters k, k_1 satsifying

$$d = b_1 k - k_1,$$

where $b_1 \geq 2$, $0 \leq k_1 < k$ as in (10.1.1). We used slightly different notation before, writing s rather than b_1 and l rather than k_1; the new notation will help us keep track of what happens as we continue the process and refine the cone σ''.

Using the normalized basis for N relative to σ'', we insert a new ray and obtain a new smooth cone and a second, possibly nonsmooth cone with parameters k_1, k_2, where

$$k = b_2 k_1 - k_2$$

using (10.1.1). Doing this repeatedly yields a *modified Euclidean algorithm*

$$\begin{aligned} d &= b_1 k - k_1 \\ k &= b_2 k_1 - k_2 \\ &\;\;\vdots \\ k_{r-3} &= b_{r-1} k_{r-2} - k_{r-1} \\ k_{r-2} &= b_r k_{r-1} \end{aligned} \tag{10.2.1}$$

that computes the parameters of the new cones produced as we successively subdivide to produce the fan giving the resolution of singularities. The process terminates with $k_r = 0$ for some r as shown, since as in the usual Euclidean algorithm, the k_i are a strictly decreasing sequence of nonnegative numbers. Also, by (10.1.1), we have $b_i \geq 2$ for all i.

The equations (10.2.1) can be rearranged:

$$\begin{aligned} d/k &= b_1 - k_1/k \\ k/k_1 &= b_2 - k_2/k_1 \\ &\;\;\vdots \\ k_{r-3}/k_{r-2} &= b_{r-1} - k_{r-1}/k_{r-2} \\ k_{r-2}/k_{r-1} &= b_r \end{aligned} \tag{10.2.2}$$

and spliced together to give a type of continued fraction expansion for the rational number d/k, with minus signs:

$$d/k = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_r}}}. \tag{10.2.3}$$

This is the *Hirzebruch-Jung continued fraction expansion* of d/k. For obvious typographical reasons, it is desirable to have a more compact way to represent these expressions. We will use the notation

$$d/k = [[b_1, b_2, \ldots, b_r]].$$

The integers b_i are the *partial quotients* of the Hirzebruch-Jung continued fraction, and the truncated Hirzebruch-Jung continued fractions

$$[[b_1, b_2, \dots, b_i]], \quad 1 \le i \le r,$$

are the *convergents*.

Example 10.2.1. Consider the rational number $17/11$. The Hirzebruch-Jung continued fraction expansion is

$$17/11 = [[2,3,2,2,2,2]],$$

as may be verified directly using the modified Euclidean algorithm (10.2.1). ◊

Proposition 10.2.2. *Let $d > k > 0$ be integers with $\gcd(d,k) = 1$ and let $d/k = [[b_1, \dots, b_r]]$. Define sequences P_i and Q_i recursively as follows. Set*

$$P_0 = 1, \quad Q_0 = 0 \qquad \text{(10.2.4)}$$
$$P_1 = b_1, \quad Q_1 = 1,$$

and for all $2 \le i \le r$, let

$$P_i = b_i P_{i-1} - P_{i-2} \qquad \text{(10.2.5)}$$
$$Q_i = b_i Q_{i-1} - Q_{i-2}.$$

Then the P_i, Q_i satisfy:

(a) *The P_i and Q_i are increasing sequences of integers.*

(b) $[[b_1, \dots, b_i]] = P_i/Q_i$ *for all* $1 \le i \le r$.

(c) $P_{i-1}Q_i - P_iQ_{i-1} = 1$ *for all* $1 \le i \le r$.

(d) *The convergents form a strictly decreasing sequence:*

$$\frac{d}{k} = \frac{P_r}{Q_r} < \frac{P_{r-1}}{Q_{r-1}} < \cdots < \frac{P_1}{Q_1}.$$

Proof. The proof of part (a) is left to the reader (Exercise 10.2.1).

To prove part (b), first observe that the expression on the right side of (10.2.3) makes sense when the b_j are any rational numbers (not just integers) such that all denominators in (10.2.3) are nonzero. We will show that the sequences defined by (10.2.5) satisfy

$$[[b_1, \dots, b_s]] = \frac{P_s}{Q_s}$$

for all such lists $b_1, \dots, b_s$. The proof is by induction on the length s of the list. When $s = 1$, we have $[[b_1]] = b_1 = \frac{P_1}{Q_1}$ by (10.2.4). Now assume that the result has been proved for all lists of of length t and consider the expression

$$[[b_1, \dots, b_{t+1}]] = [[b_1, \dots, b_t - \frac{1}{b_{t+1}}]],$$

where the right side comes from a list of length t. By the induction hypothesis, this equals

$$\frac{\left(b_t - \frac{1}{b_{t+1}}\right)P_{t-1} - P_{t-2}}{\left(b_t - \frac{1}{b_{t+1}}\right)Q_{t-1} - Q_{t-2}}.$$

By the recurrences (10.2.5), this equals

$$\frac{P_t - \frac{1}{b_{t+1}}P_{t-1}}{Q_t - \frac{1}{b_{t+1}}Q_{t-1}} = \frac{b_{t+1}P_t - P_{t-1}}{b_{t+1}Q_t - Q_{t-1}} = \frac{P_{t+1}}{Q_{t+1}},$$

which is what we wanted to show.

Part (c) will be proved by induction on i. The base case $i = 1$ follows directly from (10.2.4). Now assume that the result has been proved for $i \leq s$, and consider $i = s+1$. Using the recurrences (10.2.5), we have

$$\begin{aligned} P_sQ_{s+1} - P_{s+1}Q_s &= P_s(b_sQ_s - Q_{s-1}) - (b_sP_s - P_{s-1})Q_s \\ &= P_{s-1}Q_s - P_sQ_{s-1} \\ &= 1 \end{aligned}$$

by the induction hypothesis.

Finally, from part (b), for each $1 \leq i \leq r-1$, we have

$$\frac{P_{i-1}}{Q_{i-1}} = \frac{P_i}{Q_i} + \frac{1}{Q_{i-1}Q_i}.$$

Hence

$$\frac{P_i}{Q_i} < \frac{P_{i-1}}{Q_{i-1}}$$

since $Q_{i-1}Q_i > 0$ by part (a). Hence part (d) follows. □

Hirzebruch-Jung Continued Fractions and Resolutions. When σ is a cone with parameters $d > k > 0$, the process of computing the Hirzebruch-Jung continued fraction of d/k yields a convenient method for finding a refinement Σ of σ such that $\phi : X_\Sigma \to U_\sigma$ is a toric resolution of singularities.

Theorem 10.2.3. *Let* $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ *be in normal form. Let* $u_0 = e_2$ *and use the integers* P_i *and* Q_i *from Proposition 10.2.2 to construct vectors*

$$u_i = P_{i-1}e_1 - Q_{i-1}e_2, \quad 1 \leq i \leq r+1.$$

Then the cones

$$\sigma_i = \mathrm{Cone}(u_{i-1}, u_i), \quad 1 \leq i \leq r+1,$$

have the following properties:

(a) *Each* σ_i *is a smooth cone and* u_{i-1}, u_i *are its ray generators.*

(b) *For each* i, $\sigma_{i+1} \cap \sigma_i = \mathrm{Cone}(u_i)$.

(c) *$\sigma_1 \cup \cdots \cup \sigma_{r+1} = \sigma$, so the fan Σ consisting of the σ_i and their faces gives a smooth refinement of σ.*

(d) *The toric morphism $\phi : X_\Sigma \to U_\sigma$ is a resolution of singularities.*

Proof. Both statements in part (a) follow easily from part (c) of Proposition 10.2.2.

For part (b), we note that the ratio $-Q_{i-1}/P_{i-1}$ represents the *slope* of the line through u_i in the coordinate system relative to the normalized basis e_1, e_2 for σ. By part (d) of Proposition 10.2.2, these slopes form a strictly decreasing sequence for $i \geq 0$, which implies the statement in part (b).

Part (c) follows from part (b) by noting that $u_0 = e_2$ and $P_r/Q_r = d/k$, so $u_{r+1} = de_1 - ke_2$. Hence the cones σ_i fill out σ.

Part (d) now follows by the reasoning used in Examples 10.1.8 and 10.1.5. □

Example 10.2.4. Consider the cone $\sigma = \mathrm{Cone}(e_2, 7e_1 - 5e_2)$ in normal form. To construct the resolution of singularities of the affine toric surface U_σ, we simply compute the Hirzebruch-Jung continued fraction expansion of the rational number $d/k = 7/5$ using the modified Euclidean algorithm:

$$\begin{aligned} 7 &= 2 \cdot 5 - 3 \\ 5 &= 2 \cdot 3 - 1 \\ 3 &= 3 \cdot 1. \end{aligned}$$

Hence $b_0 = b_1 = 2, b_2 = 3$, and

$$7/5 = [[2,2,3]]. \tag{10.2.6}$$

Then from Proposition 10.2.2 we have

$$\begin{aligned} P_0 &= 1, & Q_0 &= 0 \\ P_1 &= 2, & Q_1 &= 1 \\ P_2 &= b_2P_1 - P_0 = 3, & Q_2 &= b_2Q_1 - Q_0 = 2 \\ P_3 &= b_3P_2 - P_1 = 7, & Q_3 &= b_3Q_2 - Q_1 = 5. \end{aligned}$$

Theorem 10.2.3 gives the vectors

$$u_0 = e_2,\ u_1 = e_1,\ u_2 = 2e_1 - e_2,\ u_3 = 3e_1 - 2e_2,\ u_4 = 7e_1 - 5e_2$$

and the cones

$$\begin{aligned} \sigma_1 &= \mathrm{Cone}(e_2, e_1) \\ \sigma_2 &= \mathrm{Cone}(e_1, 2e_1 - e_2) \\ \sigma_3 &= \mathrm{Cone}(2e_1 - e_2, 3e_1 - 2e_2) \\ \sigma_4 &= \mathrm{Cone}(3e_1 - 2e_2, 7e_1 - 5e_2) \end{aligned} \tag{10.2.7}$$

shown in Figure 3 on the next page. The cones σ_i give a smooth refinement of σ. You will see another example of this process in Exercise 10.2.2. ◊

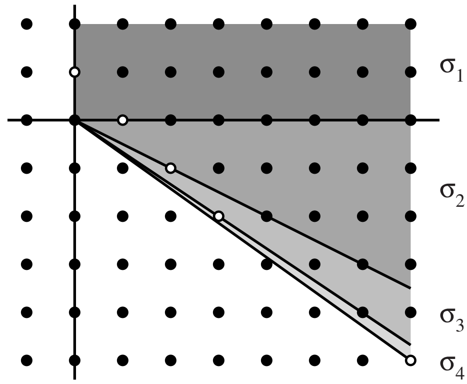

Figure 3. The refinement Σ with open circles at $u_i = P_{i-1}e_1 - Q_{i-1}e_2$ in Example 10.2.4

Next we show that the vectors u_i from Theorem 10.2.3 determine the partial quotients in the Hirzebruch-Jung continued fraction expansion of d/k.

Theorem 10.2.5. *Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ be in normal form, and let*

$$d/k = [[b_1, b_2, \dots, b_r]].$$

Then the vectors $u_0, u_1, \dots, u_{r+1}$ constructed in Theorem 10.2.3 satisfy

$$u_{i-1} + u_{i+1} = b_i u_i, \quad b_i \geq 2, \tag{10.2.8}$$

for $1 \leq i \leq r$.

Proof. By the recurrences (10.2.5),

$$\begin{aligned} u_{i-1} + u_{i+1} &= (P_{i-2}e_1 - Q_{i-2}e_2) + (P_i e_1 - Q_i e_2) \\ &= (P_{i-2} + P_i)e_1 - (Q_{i-2} + Q_i)e_2 \\ &= b_i(P_{i-1}e_1 - Q_{i-1}e_2) = b_i u_i. \end{aligned}$$ □

Later in this chapter we will see several important consequences of (10.2.8) connected with the geometry of smooth toric surfaces.

The nonuniqueness of Proposition 10.1.3 has a nice relation to Theorem 10.2.3. For instance, Example 10.2.4 used the Hirzebruch-Jung expansion $7/5 = [[2,2,3]]$. Since $5 \cdot 3 \equiv 1 \bmod 7$, the cone of Example 10.2.4 also has parameters $d = 7$, $k = 3$. We leave it to the reader to check that

$$7/3 = [[3,2,2]],$$

with the partial quotients the same as those in (10.2.6), but listed in reverse order. This pattern holds for all Hirzebruch-Jung continued fractions. We give a proof that uses the properties of the associated toric surfaces.

Proposition 10.2.6. *Let $0 < k, \widetilde{k} < d$ and assume $k\widetilde{k} \equiv 1 \bmod d$. If the Hirzebruch-Jung continued fraction expansion of d/k is*

$$d/k = [[b_1, b_2, \dots, b_r]],$$

then the Hirzebruch-Jung continued fraction expansion of $d/\widetilde{k}$ is

$$d/\widetilde{k} = [[b_r, b_{r-1}, \dots, b_1]].$$

Proof. Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ and $\widetilde{\sigma} = \mathrm{Cone}(e_2, de_1 - \widetilde{k}e_2)$ be the corresponding cones in normal form. Since $k\widetilde{k} \equiv 1 \bmod m$, there is an integer $\widetilde{d}$ such that $d\widetilde{d} + k\widetilde{k} = 1$. The $\mathbb{Z}$-linear mapping $\varphi : N \to N$ defined with respect to the basis e_1, e_2 by the matrix

$$A = \begin{pmatrix} \widetilde{k} & d \\ \widetilde{d} & -k \end{pmatrix}$$

is bijective, maps $\widetilde{\sigma}$ to σ, and is orientation-reversing. Thus $\varphi(de_1 - \widetilde{k}e_2) = e_2$ and $\varphi(e_2) = de_1 - ke_2$. If we apply Theorem 10.2.3 to σ, then we obtain vectors u_i satisfying the equations

$$u_{i-1} + u_{i+1} = b_i u_i$$

for all $1 \le i \le r$. We claim that when we apply the mapping φ^{-1} defined by the inverse of the matrix A above, then the vectors u_i are taken to corresponding vectors $\widetilde{u}_i$ for the cone $\widetilde{\sigma}$. But since φ and φ^{-1} are orientation-reversing, the partial quotients in the Hirzebruch-Jung continued fraction will be listed in the opposite order. You will complete the proof of this assertion in Exercise 10.2.3. □

Hilbert Bases and Convex Hulls. Our next result gives two alternative ways to understand the vectors u_i in Theorem 10.2.3. The idea is that σ gives two objects:

- The semigroup $\sigma \cap N$. Since σ is strongly convex, its irreducible elements form the unique minimal generating set called the *Hilbert basis* of $\sigma \cap N$. (See Proposition 1.2.23.)
- The convex hull $\Theta_\sigma = \mathrm{Conv}(\sigma \cap (N \setminus \{0\}))$. This is an unbounded polygon in the plane whose bounded edges contain finitely many lattice points.

Example 10.2.7. Consider the cone $\sigma = \mathrm{Cone}(e_2, 7e_1 - 5e_2)$ from Example 10.2.4. Figure 4 on the next page shows the convex hull Θ_σ, where the white circles represent the lattice points on the bounded edges. We will see below that these lattice points give the Hilbert basis of $\sigma \cap N$. ◊

Here is the general result suggested by Example 10.2.7.

Theorem 10.2.8. *Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ be in normal form and let*

$$S = \{u_0, u_1, \dots, u_{r+1}\}$$

be the set of vectors constructed in Theorem 10.2.3. Then:

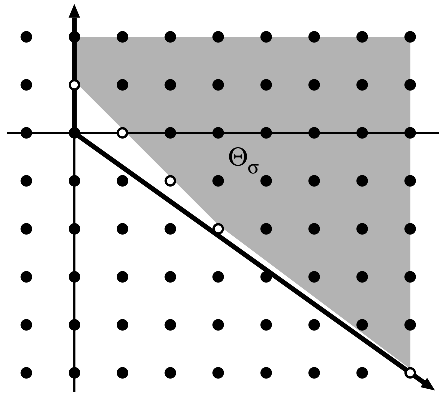

Figure 4. Convex hull Θ_σ and lattice points on bounded edges in Example 10.2.7

(a) *S is the Hilbert basis of the semigroup $\sigma \cap N$.*

(b) *S is the set of lattice points on the bounded edges of Θ_σ.*

Proof. For part (a), we use the notation of Theorem 10.2.3, where the cone σ_i is generated by u_{i-1}, u_i. Then $\sigma_i \cap N$ is generated as a semigroup by u_{i-1}, u_i since σ_i is smooth. Using $\sigma = \sigma_1 \cup \cdots \cup \sigma_{r+1}$, one sees easily that S generates $\sigma \cap N$.

We claim next that all the u_i are irreducible elements of $\sigma \cap N$. This is clear for $u_0 = e_2$ and $u_{r+1} = de_1 - ke_2$ since they are the ray generators for σ. If $1 \le i \le r$ and u_i is not irreducible, then u_i would have to be a linear combination of the vectors in $S \setminus \{u_i\}$ with nonnegative integer coefficients, i.e.,

$$u_i = P_{i-1}e_1 - Q_{i-1}e_2 = \sum_{j \neq i} c_j u_j = \Big(\sum_{j \neq i} c_j P_{j-1}\Big)e_1 - \Big(\sum_{j \neq i} c_j Q_{j-1}\Big)e_2$$

with $c_j \ge 0$ in $\mathbb{Z}$. Hence

$$P_{i-1} = \sum_{j \neq i} c_j P_{j-1}, \qquad Q_{i-1} = \sum_{j \neq i} c_j Q_{j-1}.$$

Since the P_i and Q_i are strictly increasing by part (a) of Proposition 10.2.2, we must have $c_j = 0$ for all $j > i$. But this would imply that u_i is a linear combination with nonnegative integer coefficients of the vectors in $\{u_0, \ldots, u_{i-1}\}$. This contradicts the observation made in the proof of Theorem 10.2.3 that the slopes of the u_i are strictly decreasing. It follows that the u_i are irreducible elements of $\sigma \cap N$.

Finally, we must show that there are no other irreducible elements in $\sigma \cap N$. But this follows from what we have already said. Since $\sigma = \sigma_1 \cup \cdots \cup \sigma_{r+1}$, if u is irreducible, then $u \in \sigma_i \cap N$ for some i. But then $u = c_{i-1}u_{i-1} + c_i u_i$ for some $c_{i-1}, c_i \ge 0$ in $\mathbb{Z}$. Thus u is irreducible only if $u = u_{i-1}$ or u_i.

For part (b), first observe that by Proposition 10.2.2, we have

$$\begin{aligned} P_{i-2}Q_i - P_iQ_{i-2} &= P_{i-2}(b_iQ_{i-1} - Q_{i-2}) - (b_iP_{i-1} - P_{i-2})Q_{i-2} \\ &= b_i(P_{i-2}Q_{i-1} - P_{i-1}Q_{i-2}) = b_i \geq 2. \end{aligned}$$

Combining this with part (c) of Proposition 10.2.2, one obtains the inequality

$$\frac{-(Q_{i-1} - Q_{i-2})}{P_{i-1} - P_{i-2}} \leq \frac{-(Q_i - Q_{i-1})}{P_i - P_{i-1}}.$$

Since $u_i = P_{i-1}e_1 - Q_{i-1}e_2$, this inequality tells us that the slopes of the line segments $\overline{u_{i-1}u_i}$ and $\overline{u_iu_{i+1}}$ are related by

$$\text{slope of } \overline{u_{i-1}u_i} \leq \text{slope of } \overline{u_iu_{i+1}}.$$

This implies that these line segments lie on boundary of Θ_σ. From here, it is easy to see that the u_i are the lattice points of the bounded edges of Θ_σ. □

Ordinary Continued Fractions. The Hirzebruch-Jung continued fractions studied above are less familiar than ordinary continued fraction expansions in which the minus signs are replaced by plus signs. If $d > k > 0$ are integers, then the ordinary continued fraction expansion of d/k may be obtained by performing the same sequence of integer divisions used in the usual Euclidean algorithm for the gcd. Starting with $k_{-1} = d$ and $k_0 = k$, we write a_i for the quotient and k_i for the remainder at each step, so that the ith division is given by

(10.2.9) $$k_{i-2} = a_ik_{i-1} + k_i,$$

where $0 \leq k_i < k_{i-1}$.

Let k_{s-1} be the final nonzero remainder (which equals $\gcd(d,k)$). The resulting equations splice together to form the *ordinary continued fraction*

$$d/k = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_s}}}.$$

To distinguish these from Hirzebruch-Jung continued fractions, we will use the notation

(10.2.10) $$d/k = [a_1, a_2, \ldots, a_s]$$

for the ordinary continued fraction. The a_i are the (ordinary) *partial quotients* of d/k, and the truncated continued fractions

$$[a_1, a_2, \ldots, a_i], \quad 1 \leq i \leq s,$$

are the (ordinary) *convergents* of d/k.

Example 10.2.9. By Example 10.2.1, the Hirzebruch-Jung continued fraction of $17/11$ is

$$17/11 = [[2,3,2,2,2,2]],$$

and the ordinary continued fraction is

$$17/11 = [1,1,1,5].$$

The partial quotients and the lengths are different. However, each expansion determines the other, and there are methods for computing the Hirzebruch-Jung partial quotients b_j in terms of the ordinary partial quotients a_i and vice versa. See [**75**, Prop. 3.6], [**145**, p. 257], [**232**, Prop. 2.3], and Exercise 10.2.4. $\Diamond$

The following result is mostly parallel to Proposition 10.2.2, but shows that ordinary continued fractions are slightly *more* complicated than Hirzebruch-Jung continued fractions. The proof is left to the reader (Exercise 10.2.5).

Proposition 10.2.10. *Let* $d > k > 0$ *be integers with* $\gcd(d,k) = 1$*, and let* $d/k = [a_1,\dots,a_s]$*. Define sequences* p_i *and* q_i *recursively as follows. First set*

$$\begin{aligned} p_0 &= 1, & q_0 &= 0 \\ p_1 &= a_1, & q_1 &= 1, \end{aligned} \tag{10.2.11}$$

and for all $2 \le i \le s$*, let*

$$\begin{aligned} p_i &= a_i p_{i-1} + p_{i-2} \\ q_i &= a_i q_{i-1} + q_{i-2}. \end{aligned} \tag{10.2.12}$$

Then the p_i, q_i *satisfy:*

(a) $[a_1,\dots,a_i] = p_i/q_i$ *for all* $1 \le i \le s$.

(b) $p_i q_{i-1} - p_{i-1} q_i = (-1)^i$ *for all* $1 \le i \le s$.

(c) *The convergents converge to* d/k*, but in an oscillating fashion:*

$$\frac{p_1}{q_1} < \frac{p_3}{q_3} < \cdots \le \frac{d}{k} \le \cdots < \frac{p_4}{q_4} < \frac{p_2}{q_2}. \qquad \square$$

Ordinary Continued Fractions and Convex Hulls. Felix Klein discovered a lovely geometric interpretation of ordinary continued fractions. Given a basis $u^{\mathrm{o}}_{-1}, u^{\mathrm{o}}_0$ of $N \simeq \mathbb{Z}^2$ and relatively prime integers $d > k > 0$, compute the continued fraction $d/k = [a_1,\dots,a_s]$ and set

$$u^{\mathrm{o}}_i = q_i u^{\mathrm{o}}_{-1} + p_i u^{\mathrm{o}}_0, \quad 1 \le i \le s. \tag{10.2.13}$$

In this notation, the superscript "o" stands for "ordinary". Then $u^{\mathrm{o}}_s = k u^{\mathrm{o}}_{-1} + d u^{\mathrm{o}}_0$, and part (c) of Proposition 10.2.10 implies that the u^{o}_i lie on one side of the ray $\mathrm{Cone}(u^{\mathrm{o}}_s)$ for even indices and on the other side for odd indices. To give a careful description of what is happening, we introduce the cones

$$\sigma_{-1} = \mathrm{Cone}(u^{\mathrm{o}}_{-1}, u^{\mathrm{o}}_s),\ \sigma_0 = \mathrm{Cone}(u^{\mathrm{o}}_0, u^{\mathrm{o}}_s)$$

and associated convex hulls $\Theta_i = \Theta_{\sigma_i} = \mathrm{Conv}(\sigma_i \cap (N \setminus \{0\}))$, $i = -1, 0$.

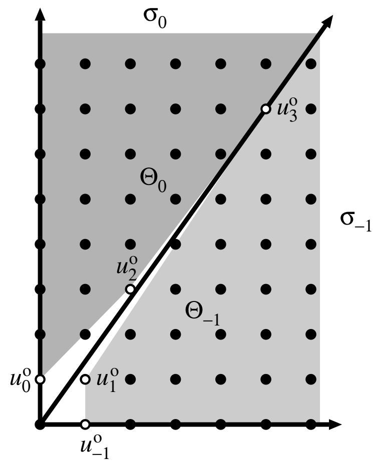

Figure 5. The cones σ_{-1}, σ_0, the convex hulls Θ_{-1}, Θ_0, and their vertices

Example 10.2.11. For $d = 7,\ k = 5$, the expansion $7/5 = [1,2,2]$ gives the vectors $u^{\circ}_{-1}, \dots, u^{\circ}_{3}$ shown in Figure 5. In this figure, it is clear that the u°_{i} are the vertices of the convex hulls Θ_{-1} and Θ_0. $\Diamond$

This example is a special case of the following general result.

Theorem 10.2.12. *For $u^{\circ}_{-1}, \dots, u^{\circ}_{s}$ and Θ_{-1}, Θ_0 as above, we have:*

(a) Θ_{-1} *has vertex set* $\{u^{\circ}_{2j-1} \mid 1 \le j \le \lfloor s/2 \rfloor\} \cup \{u^{\circ}_{s}\}$.

(b) Θ_0 *has vertex set* $\{u^{\circ}_{2j} \mid 0 \le j \le \lfloor s/2 \rfloor\} \cup \{u^{\circ}_{s}\}$.

(c) *For $1 \le i \le s$, $\overline{u^{\circ}_{i-2} u^{\circ}_{i}}$ is an edge of Θ_{-1} (resp. Θ_0) for i odd (resp. even) with $a_i + 1$ lattice points.*

Proof. First note that by Proposition 10.2.10, the vectors u°_{i} satisfy the recursion

$$u^{\circ}_{i} = a_i u^{\circ}_{i-1} + u^{\circ}_{i-2}, \quad 1 \le i \le s. \tag{10.2.14}$$

Since u°_{i-1} is primitive (Proposition 10.2.10), part (c) follows from parts (a) and (b).

We now prove the theorem using induction on the length s of the continued fraction expansion. Consider Figure 6 on the next page, which shows the first quadrant determined by $u^{\circ}_{-1}, u^{\circ}_{0}$, together with the vector u°_{s}. In the picture, σ_0 (darker) lies above the ray determined by u°_{s} and σ_{-1} (lighter) lies below. The figure also shows u°_{1} and the smaller cone $\sigma_1 = \mathrm{Cone}(u^{\circ}_{1}, u^{\circ}_{s}) \subseteq \sigma_{-1} = \mathrm{Cone}(u^{\circ}_{-1}, u^{\circ}_{s})$.

Since $u^{\circ}_{s} = k u^{\circ}_{-1} + d u^{\circ}_{0}$, the ray starting from u°_{-1} through u°_{1} passes through the upper edge of σ_{-1} at a point between $u^{\circ}_{-1} + \lfloor d/k \rfloor u^{\circ}_{1}$ and $u^{\circ}_{-1} + (\lfloor d/k \rfloor + 1) u^{\circ}_{1}$. But by the computation of the ordinary continued fraction, $\lfloor d/k \rfloor = a_1$. Therefore the segment from u°_{-1} to u°_{1} is the first bounded edge of Θ_{-1}. It follows that

$$\Theta_{-1} \cap \mathrm{Cone}(u^{\circ}_{-1}, u^{\circ}_{1})$$

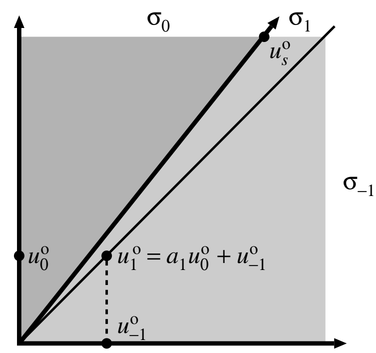

Figure 6. The cones σ_0, $\sigma_1 \subseteq \sigma_{-1}$ and vectors u_{-1}°, u_0°, u_1°, u_s°

has vertices u_{-1}°, u_1°. It remains to understand Θ_0 and $\Theta_{-1} \cap \sigma_1$. This is where induction comes into play.

Apply the above construction to the new basis u_0°, u_1° and the continued fraction

$$\frac{k}{d - a_1 k} = \frac{1}{\frac{d}{k} - a_1} = [a_2, \ldots, a_s]$$

of length $s-1$. If we start numbering at 0 rather than at -1, then the vectors $u_0^\circ, u_1^\circ, \ldots, u_s^\circ$ are the same as before by the recursion (10.2.14). This gives the cones σ_0, σ_1 shown in Figure 6. By induction, the vectors $u_0^\circ, u_1^\circ, \ldots, u_s^\circ$ give the vertices of the corresponding convex hulls $\Theta_0 = \mathrm{Conv}(\sigma_0 \cap (N \setminus \{0\}))$ and $\Theta_1 = \mathrm{Conv}(\sigma_1 \cap (N \setminus \{0\}))$. It follows easily that the theorem holds for continued fraction expansions of length s. □

A discussion of Klein's formulation of Theorem 10.2.12 can be found in [**232**]. In Exercise 10.2.6 you will apply the theorem to the resolution of pairs of singular points on certain toric surfaces. Geometric pictures similar to Figure 5 have also appeared in recent work of McDuff on symplectic embeddings of 4-dimensional ellipsoids (see [**203**]).

Ordinary Continued Fractions and the Supplementary Cone. To relate the above theorem to toric geometry, we follow the approach of [**232**]. Given a cone $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2) \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^2$ in normal form, its *supplement* is the cone $\sigma' = \mathrm{Cone}(-e_2, de_1 - ke_2)$. Thus $\sigma \cup \sigma'$ is the right half-plane, and the cones σ, σ' give the convex hulls

$$\begin{aligned}\Theta_\sigma &= \mathrm{Conv}(\sigma \cap (N \setminus \{0\}))\\ \Theta_{\sigma'} &= \mathrm{Conv}(\sigma' \cap (N \setminus \{0\})).\end{aligned}$$

Example 10.2.13. When $d = 7$, $k = 5$, Figure 7 on the next page shows the cones σ, σ' and the convex hulls Θ_σ, $\Theta_{\sigma'}$. The open circles in the figure are the vertices

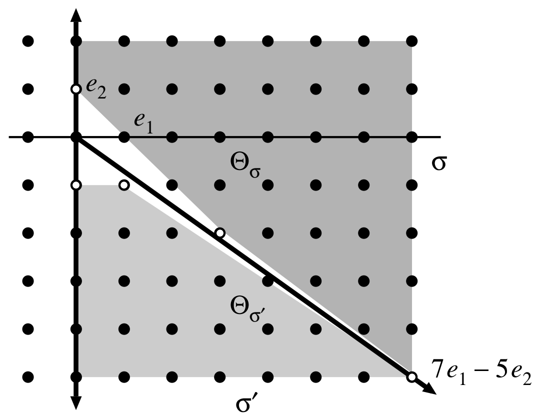

Figure 7. The cones σ, σ', the convex hulls $\Theta_\sigma, \Theta_{\sigma'}$, and their vertices

of Θ_σ and $\Theta_{\sigma'}$. Also observe that the fourth quadrant portion of Figure 7 becomes Figure 5 after a 90° counterclockwise rotation. ◊

For $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$, the ordinary continued fraction expansion $d/k = [a_1, a_2, \ldots, a_s]$ gives the sequences p_i, q_i, $0 \le i \le s$, defined in Proposition 10.2.10. Then define the vectors

$$u^{\circ}_{-1} = -e_2,\ u^{\circ}_i = p_i e_1 - q_i e_2,\ 0 \le i \le s. \tag{10.2.15}$$

These vectors enable us to describe the vertices of $\Theta_\sigma, \Theta_{\sigma'}$ as follows.

Theorem 10.2.14. *Let* $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ *be a cone in normal form with supplement* σ'. *Also let* u°_i, $-1 \le i \le s$, *be as defined in* (10.2.15). *Then:*

(a) *The set of vertices of* $\Theta_{\sigma'}$ *is*

$$\{u^{\circ}_{2j-1} \mid 1 \le j \le \lfloor s/2 \rfloor\} \cup \{u^{\circ}_s\}.$$

(b) *If* $a_1 = 1$, *then the set of vertices of* Θ_σ *is*

$$\{e_2\} \cup \{u^{\circ}_{2j} \mid 1 \le j \le \lfloor s/2 \rfloor\} \cup \{u^{\circ}_s\}.$$

(c) *If* $a_1 > 1$, *then the set of vertices of* Θ_σ *is*

$$\{e_2\} \cup \{u^{\circ}_{2j} \mid 0 \le j \le \lfloor s/2 \rfloor\} \cup \{u^{\circ}_s\}.$$

Proof. First note that since $u^{\circ}_{-1} = -e_2$ and $u^{\circ}_0 = e_1$, we can rewrite (10.2.15) as

$$u^{\circ}_i = p_i e_1 - q_i e_2 = q_i u_{-1} + p_i u_0, \quad 0 \le i \le s.$$

Thus we are in the situation of Theorem 10.2.12, where we have the cones $\sigma_{-1} = \mathrm{Cone}(u^{\circ}_{-1}, u^{\circ}_s)$ and $\sigma_0 = \mathrm{Cone}(u^{\circ}_0, u^{\circ}_s)$ and associated convex hulls $\Theta_{-1} = \Theta_{\sigma_{-1}}$ and $\Theta_0 = \Theta_{\sigma_0}$. Then Theorem 10.2.12 implies that the vectors $u^{\circ}_{-1}, u^{\circ}_0, \ldots, u^{\circ}_s$ give the vertices of Θ_{-1} and Θ_0.

However, $\sigma = \text{Cone}(e_1, e_2) \cup \sigma_0$ and $\sigma' = \sigma_{-1}$. In particular, $\Theta_{\sigma'} = \Theta_{-1}$, so that part (a) of the theorem follows immediately. For parts (b) and (c), note that

$$\Theta_\sigma = (\Theta_\sigma \cap \text{Cone}(e_1, e_2)) \cup \Theta_0.$$

The intersection $\Theta_\sigma \cap \text{Cone}(e_1, e_2)$ has vertices $e_1 = u_0^{\text{o}}, e_2$, while Θ_0 has vertices $u_0^{\text{o}}, u_2^{\text{o}}, \ldots, u_s^{\text{o}}$. If $a_1 = 1$, then $e_2, u_0^{\text{o}}, u_2^{\text{o}}$ are collinear, so that u_0^{o} is not a vertex. This proves part (b) of the theorem. Finally, if $a_1 > 1$, then one can prove without difficulty that u_0^{o} is a vertex (Exercise 10.2.7), and part (c) follows. □

Example 10.2.15. Figure 7 above illustrates Theorem 10.2.12 for the cone $\sigma = \text{Cone}(e_2, 7e_1 - 5e_2)$. Since $7/5 = [1,2,2]$, we use part (b) of the theorem. ◊

Ordinary Continued Fractions and the Dual Cone. For a cone σ in normal form, one surprise is that its supplement σ' is essentially the dual of σ.

Lemma 10.2.16. *Given a cone $\sigma = \text{Cone}(e_2, de_1 - ke_2) \subseteq N_\mathbb{R}$ in normal form, its supplementary cone $\sigma' \subseteq N_\mathbb{R}$ is isomorphic to $\sigma^\vee \subseteq M_\mathbb{R}$.*

Proof. Let e_1^*, e_2^* be the basis of M dual to the basis e_1, e_2 of N. Then the isomorphism defined by $e_1 \mapsto e_2^*$ and $e_2 \mapsto -e_1^*$ takes $\sigma' = \text{Cone}(-e_2, de_1 - ke_2)$ to

$$\text{Cone}(-(-e_1^*), d(e_2^*) - k(-e_1^*)) = \text{Cone}(e_1^*, ke_1^* + de_2^*) = \sigma^\vee. \qquad \square$$

The isomorphism $\sigma' \simeq \sigma^\vee$ from this lemma leads to some nice results about $\sigma^\vee$ and the associated convex hull $\Theta_{\sigma^\vee} = \text{Conv}(\sigma^\vee \cap (M \setminus \{0\}))$. Specifically, this isomorphism takes the vectors

$$u_{-1}^{\text{o}} = -e_2,\ u_i^{\text{o}} = p_i e_1 - q_i e_2, \quad 0 \le i \le s$$

from (10.2.15) to the dual vectors

$$m_{-1} = e_1^*,\ m_i = q_i e_1^* + p_i e_2^*, \quad 0 \le i \le s.$$

In particular, $m_s = ke_1^* + de_2^*$, so that $\sigma^\vee = \text{Cone}(m_{-1}, m_s)$ in this notation. Then Theorem 10.2.14 implies that the vertex set of the convex hull $\Theta_{\sigma^\vee}$ is

$$\{m_{2j-1} \mid 1 \le j \le \lfloor s/2 \rfloor\} \cup \{m_s\}. \tag{10.2.16}$$

Hence, in the language of §7.1, $\Theta_{\sigma^\vee} \subseteq M_\mathbb{R}$ is a full dimensional lattice polyhedron. We can describe its recession cone and normal fan as follows.

Proposition 10.2.17. *Let $\sigma = \text{Cone}(e_2, de_1 - ke_2) \subseteq N_\mathbb{R} \simeq \mathbb{R}^2$ be in normal form and set $P = \Theta_{\sigma^\vee} = \text{Conv}(\sigma^\vee \cap (M \setminus \{0\}))$. Then:*

(a) *$P \subseteq M_\mathbb{R}$ is a full dimensional lattice polyhedron with recession cone $\sigma^\vee$.*

(b) *The normal fan Σ_P of P is the refinement of σ obtained by adding the minimal generators*

$$u_0^{\text{o}},\ u_2^{\text{o}},\ u_4^{\text{o}},\ \ldots,\ u_{s-1}^{\text{o}} \qquad (s \text{ odd})$$
$$u_0^{\text{o}},\ u_2^{\text{o}},\ u_4^{\text{o}},\ \ldots,\ u_{s-2}^{\text{o}}, u_s^{\text{o}} - u_{s-1}^{\text{o}} \quad (s \text{ even}).$$

(c) *The toric morphism $X_P \to U_\sigma$ is projective and X_P is Gorenstein with at worst rational double points.*

Remark 10.2.18. When s is even, we can explain $u_s^\circ - u_{s-1}^\circ$ as follows. By Theorem 10.2.14, u_{s-2}° and u_s° give an edge Θ_σ. The vector determined by this edge is $u_s^\circ - u_{s-2}^\circ = a_s u_{s-1}^\circ$. Hence the edge has $a_s + 1$ lattice points since u_{s-1}° is primitive. Thus, if we start from u_s°, the next lattice point along the edge is

$$u_s^\circ - u_{s-1}^\circ.$$

Since $d > k > 0$ and $d/k = [a_1, \ldots, a_s]$ is computed using the Euclidean algorithm, we have $a_s \geq 2$ (Exercise 10.2.8). It follows that $u_s^\circ - u_{s-1}^\circ$ is not a vertex of Θ_σ.

Proof. Part (a) is straightforward (Exercise 10.2.9). For part (b), we will assume that s is even and leave the case when s is odd to the reader (Exercise 10.2.9).

If we write $s = 2\ell$, then the vertices (10.2.16) of P are $m_{-1}, m_1, \ldots, m_{2\ell-1}, m_{2\ell}$. Thus the bounded edges of $P = \Theta_{\sigma^\vee}$ are

$$\overline{m_{-1}m_1},\ \overline{m_1m_3},\ \ldots,\ \overline{m_{2\ell-3}m_{2\ell-1}},\ \overline{m_{2\ell-1}m_{2\ell}}.$$

The inward-pointing normals of these edges give the rays that refine σ in the normal fan of P.

The m_i's satisfy the same recursion $m_i = a_i m_{i-1} + m_{i-2}$, $1 \leq i \leq s$, as the u_i°'s in (10.2.14). Hence, for the edges $\overline{m_{2j-1}m_{2j+1}}$, $1 \leq j \leq \lfloor s/2 \rfloor$, we have

$$m_{2j+1} - m_{2j-1} = a_{2j+1} m_{2j} = a_{2j+1}(q_{2j} e_1^* + p_{2j} e_2^*).$$

Since $u_{2j}^\circ = p_{2j} e_1 - q_{2j} e_2$, one easily computes that $\langle m_{2j+1} - m_{2j-1}, u_{2j}^\circ \rangle = 0$. It follows that u_{2j}° is the inward-pointing normal of this edge since it lies in σ and is primitive. This takes care of all of the bounded edges except for $\overline{m_{2\ell-1}m_{2\ell}}$. Here, we compute

$$m_{2\ell} - m_{2\ell-1} = (q_{2\ell} - q_{2\ell-1}) e_1^* + (p_{2\ell} - p_{2\ell-1}) e_2^*.$$

This is clearly normal to $u_{2\ell}^\circ - u_{2\ell-1}^\circ = (p_{2\ell} - p_{2\ell-1}) e_1 - (q_{2\ell} - q_{2\ell-1}) e_2$. The latter vector is easily seen to be primitive by part (b) of Proposition 10.2.10. Furthermore, $u_{2\ell}^\circ - u_{2\ell-1}^\circ \in \sigma$ by Remark 10.2.18. Hence this is the inward-pointing normal of the final bounded edge $\overline{m_{2\ell-1}m_{2\ell}}$.

For part (c), note that $X_P \to U_\sigma$ is projective by Theorem 7.1.10. To complete the proof, we need to show that each maximal cone of Σ_P gives a Gorenstein affine toric variety. For simplicity, we assume that s is odd (see Exercise 10.2.9 for the even case). Since $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2) = \mathrm{Cone}(e_2, u_s^\circ)$, the maximal cones of Σ_P consist of two "boundary cones" $\mathrm{Cone}(e_1, u_0^\circ)$ and $\mathrm{Cone}(u_{s-1}^\circ, u_s^\circ)$, plus the "interior cones" $\mathrm{Cone}(u_{2j-2}, u_{2j}) \in \Sigma_P$, $1 \leq j \leq \lfloor s/2 \rfloor$. The boundary cones are easily seen to be smooth and hence Gorenstein (Exercise 10.2.9). For an interior cone $\mathrm{Cone}(u_{2j-2}, u_{2j})$, we use part (b) of Proposition 10.2.10 to compute

$$\begin{aligned}\langle m_{2j-1}, u^{\circ}_{2j-2}\rangle &= \langle q_{2j-1}e_1^* + p_{2j-1}e_2^*, p_{2j-2}e_1 - q_{2j-2}e_2\rangle \\ &= -(p_{2j-1}q_{2j-2} - p_{2j-2}q_{2j-1}) = -(-1)^{2j-1} = 1,\end{aligned}$$

and a similar computation gives $\langle m_{2j-1}, u^{\circ}_{2j}\rangle = p_{2j}q_{2j-1} - p_{2j-1}q_{2j} = (-1)^{2j} = 1$. By Proposition 8.2.12, we conclude that the corresponding affine toric variety is Gorenstein, as desired. Then the singular points of X_P are rational double points by Example 10.1.5 and Proposition 10.1.6. □

In §11.3 we will revisit this result, where we will learn that the morphism $X_P \to U_\sigma$ from Proposition 10.2.17 is the blowup of the singular point of U_σ.

Just as the morphism $X_P \to U_\sigma$ is projective, one can show more generally that the resolution of singularities $X_\Sigma \to U_\sigma$ from Theorem 10.2.3 is a projective morphism. This requires finding a lattice polyhedron with the correct normal fan. You will explore one way of doing this in Exercise 10.2.10.

We next consider the Hilbert basis $\mathscr{H}$ of $\sigma^\vee \cap M$. Recall from Lemma 1.3.10 that $|\mathscr{H}|$ is the dimension of the Zariski tangent space at the singular point of U_σ and is the dimension of the most efficient embedding of U_σ into affine space.

Theorem 10.2.8 tells us that the Hilbert basis of $\sigma \cap N$ is computed using the Hirzebruch-Jung continued fraction expansion of d/k. Since $\sigma^\vee$ has parameters $d, d-k$ (part (b) of Exercise 10.1.2), it follows that we need the Hirzebruch-Jung continued fraction expansion of $d/(d-k)$ to get the Hilbert basis of $\sigma^\vee \cap M$. By Exercise 10.2.4, the ordinary continued fraction

$$d/k = [a_1, \ldots, a_s]$$

gives the Hirzebruch-Jung continued fraction

$$d/(d-k) = \begin{cases} [[(2)^{a_1-1}, a_2+2, (2)^{a_3-1}, a_4+2, \ldots, (2)^{a_{s-1}-1}, a_s+1]] & s \text{ even} \\ [[(2)^{a_1-1}, a_2+2, (2)^{a_3-1}, a_4+2, \ldots, a_{s-1}+2, (2)^{a_s-1}]] & s \text{ odd}. \end{cases}$$

Theorem 10.2.8, applied to this expansion, gives the Hilbert basis of $\sigma^\vee \cap M$. To see the underlying geometry, we need the following observation (Exercise 10.2.11):

(10.2.17) In Theorem 10.2.5, three consecutive lattice points u_{i-1}, u_i, u_{i+1} are collinear if and only if $u_{i-1} + u_{i+1} = 2u_i$, i.e., $b_i = 2$.

This means that a string of consecutive 2's in a Hirzebruch-Jung continued fraction of $d/(d-k)$ gives a string of lattice points in the relative interior of a bounded edge of the convex hull $\Theta_{\sigma^\vee}$. For the vertices $m_{-1}, m_1, \ldots$ of $\Theta_{\sigma^\vee}$, this gives two ways to think about lattice points on an edge connecting two adjacent vertices. For example, the edge $\overline{m_{-1}m_1}$ has $a_1 - 1$ lattice points in its relative interior because:

- The Hirzebruch-Jung expansion of $d/(d-k)$ starts with $a_1 - 1$ consecutive 2's.
- $m_1 - m_{-1} = a_1 u_0$, u_0 primitive, gives $a_1 + 1$ lattice points on the edge.

This pattern continues for the other bounded edges of $\Theta_{\sigma^\vee}$.

Oda gives a different argument for this pattern in [**219**, Sec. 1.6] and uses it to relate the Hirzebruch-Jung continued fractions for d/k and $d/(d-k)$. His relation also follows from our approach (see part (d) of Exercise 10.2.4). The connection between continued fractions and toric surfaces is surprisingly rich and varied and is one of the reasons why toric varieties are so much fun to study. See [**75**] and [**232**] for a further discussion of this wonderful topic.

Exercises for §10.2.

10.2.1. Prove part (a) of Proposition 10.2.2: Show that the sequences P_i and Q_i from (10.2.5) are increasing sequences of nonnegative numbers. Hint: Use $b_i \geq 2$ for all i.

10.2.2. In §10.1, we constructed several resolutions of singularities in a rather ad hoc way. In this exercise, we will see that the resolutions given by Theorem 10.2.3 are the same as what we saw before.

(a) When σ has parameters $d, 1$, show that the Hirzebruch-Jung continued fraction method gives the same resolution of U_σ as the one given in Example 10.1.8.

(b) Do the same for Example 10.1.9. Hint: First show that
$$\frac{d}{d-1} = [[2,2,\ldots,2]],$$
where there are $d-1$ 2's.

10.2.3. Verify the last claim in the proof of Proposition 10.2.6.

10.2.4. This exercise will consider some relations between ordinary continued fraction expansions and Hirzebruch-Jung continued fraction expansions.

(a) Given integers $a_1, a_2 > 0$ and a variable x, prove that
$$[a_1, a_2, x] = [[a_1+1, (2)^{a_2-1}, x+1]],$$
where for any $l \geq 0$, $(2)^l$ denotes a string of l 2's. Hint: Argue by induction on a_2.

(b) Use part (a) to prove the equality $[1,1,1,5] = [[2,3,2,2,2,2]]$ from Example 10.2.9.

(c) Given $d/k = [a_1, \ldots, a_s]$, prove that $d/(d-k)$ has the Hirzebruch-Jung expansion given in the discussion leading up to (10.2.17). Hint: You will want to consider the cases $a_1 = 1$ and $a_1 > 1$ separately, but the formula can be written as in (10.2.17) in either case. If you get stuck, see [**232**].

(d) Starting from the ordinary continued fraction for d/k, use parts (a) and (c) to show that if $d/k = [[b_1, \ldots, b_r]]$ and $d/(d-k) = [[c_1, \ldots, c_s]]$, then $(\sum_{i=1}^r b_i) - r = (\sum_{j=1}^s c_j) - s$.

10.2.5. In this exercise we will consider Proposition 10.2.10.

(a) Prove the proposition. Hint: For part (a), argue by induction on the length $s \geq 1$ of the expansion. The expression $[a_1, a_2, \ldots, a_i]$ is well defined when the a_i are positive rational numbers, so we can write
$$[a_1, a_2, \ldots, a_{i-1}, a_i] = [a_1, a_2, \ldots, a_{i-1} + 1/a_i].$$
Then use (10.2.12) and follow the reasoning from the proof of Proposition 10.2.2.

(b) Suppose we modify the initialization and the recurrences (10.2.12) as follows. Let
$$(r_{-1}, s_{-1}) = (1,0) \quad \text{and} \quad (r_0, s_0) = (0,1).$$

Then, for all $1 \le i \le s$, compute

$$r_i = r_{i-2} - a_i r_{i-1}$$
$$s_i = s_{i-2} - a_i s_{i-1}$$

(note the change in sign!). What is true about $r_i d + s_i k$ for all i? What do we get with $i = s$? Hint: This fact is the basis for the extended Euclidean algorithm.

10.2.6. In this exercise, you will show that the ordinary continued fraction expansion of a rational number d/k can be used to construct *simultaneous resolutions* of pairs of singularities of certain toric surfaces. Let $1 < k < d$ be relatively prime integers, and let P be the triangle $\mathrm{Conv}(0, de_1, ke_2)$ in $\mathbb{R}^2$.

(a) Draw the normal fan Σ_P and show that X_P has exactly two singular points. Note that X_P is isomorphic to the weighted projective plane $\mathbb{P}(1,k,d)$.

(b) Adapt Theorem 10.2.12 to produce a resolution of singularities of X_P from the ordinary continued fraction expansion of d/k. Hint: First refine Σ_P by introducing 1-dimensional cones $\mathrm{Cone}(-e_1)$ and $\mathrm{Cone}(-e_2)$. Then apply Theorem 10.2.12 to the third quadrant of your drawing.

10.2.7. Complete the proof of Theorem 10.2.14 by showing that u_0° is a vertex of Θ_σ if and only if $a_1 > 1$.

10.2.8. For relative prime integers $d > k > 0$, we used the Euclidean algorithm to construct $d/k = [a_1, \dots, a_s]$. Prove that $a_s \ge 2$.

10.2.9. Prove part (a) of Proposition 10.2.17. Also prove part (b) for s odd and part (c) for s even.

10.2.10. In this exercise, given a cone σ in normal form with parameters d,k, you will see how to construct an unbounded polyhedron P in $M_{\mathbb{R}} \simeq \mathbb{R}^2$ whose recession cone is $\sigma^\vee = \mathrm{Cone}(e_1, ke_1 + de_2)$, and whose normal fan defines the resolution of U_σ from Theorem 10.2.3. Let P_i, Q_i be the sequences constructed in Proposition 10.2.2. Let m be the smallest positive integer such that $md > P_1 + \cdots + P_{r-1}$ and $mk > Q_1 + \cdots + Q_{r-1}$. Starting from the point $A_r = (mk, md)$ on the upper boundary ray of $\sigma^\vee$, construct the points

$$\begin{aligned} A_{r-1} &= (mk - Q_{r-1}, md - P_{r-1}) \\ A_{r-2} &= (mk - Q_{r-1} - Q_{r-2}, md - P_{r-1} - P_{r-2}) \\ &\ \vdots \\ A_1 &= (mk - \textstyle\sum_{i=1}^{r-1} Q_i, md - \sum_{i=1}^{r-1} P_i) \\ A_0 &= (mk - \textstyle\sum_{i=1}^{r-1} Q_i, 0). \end{aligned}$$

Then let P be the polyhedron with one edge along the positive x-axis starting at A_0, edges $\overline{A_i A_{i+1}}$ for $i = 0, \dots, r-1$, and one edge along the upper edge of $\sigma^\vee$ starting from A_r.

(a) Draw the polyhedron P for the cone with parameters $(d,k) = (7,5)$. What is the integer m in this case?

(b) Show that $\sigma^\vee$ is the recession cone of P.

(c) Show that the normal fan of P is the fan giving the resolution of U_σ constructed in Theorem 10.2.3.

10.2.11. Prove (10.2.11).

10.2.12. Let d/k be a rational number in lowest terms with $0 < k < d$, and let

$$d/k = [[b_1,\dots,b_r]] = [a_1,\dots,a_s]$$

be its continued fraction expansions. You will show that the sequences P_i, Q_i and p_i, q_i considered in this section can be expressed in matrix form.

(a) Let $M^-(b) = \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix}$. Show that for all $1 \le i \le r$,

$$\begin{pmatrix} P_i & -P_{i-1} \\ Q_i & -Q_{i-1} \end{pmatrix} = M^-(b_1)M^-(b_2)\cdots M^-(b_i).$$

(b) Let $M^+(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$. Show that for all $1 \le i \le s$,

$$\begin{pmatrix} p_i & p_{i-1} \\ q_i & q_{i-1} \end{pmatrix} = M^+(a_1)M^+(a_2)\cdots M^+(a_i).$$

10.2.13. In this exercise, you will apply the results of this section to the weighted projective plane $\mathbb{P}(q_0,q_1,q_2)$ from §2.0 and Example 3.1.17.

(a) Construct a resolution of singularities for any $\mathbb{P}(1,1,q_2)$, where $q_2 \ge 2$. What smooth toric surface is obtained in this way? A complete classification of the smooth complete toric surfaces will be developed in §10.4.

(b) Do the same for $\mathbb{P}(1,q_1,q_2)$ in general. Hint: Exercise 10.2.6.

§10.3. Gröbner Fans and McKay Correspondences

The fans obtained by resolving the singularities of the affine toric toric surfaces U_σ have unexpected descriptions that involve Gröbner bases and representation theory. In this section we will present these ideas, following [**156**] and [**157**].

Gröbner Bases and Gröbner Fans. We assume the reader knows about Gröbner bases (see [**69**]) and Gröbner fans (see [**70**, Ch. 8, §4] or [**265**]). A nonzero ideal $I \subset \mathbb{C}[x_1,\dots,x_n]$ has a unique reduced Gröbner basis with respect to each monomial order $>$ on the polynomial ring. However, the set of distinct *reduced marked Gröbner bases* for I (i.e., reduced Gröbner bases with marked leading terms in each polynomial) is finite. Hence the ideal I has a finite *universal Gröbner basis*, i.e., a finite subset $\mathscr{U} \subset I$ that is a Gröbner basis for all monomial orders simultaneously.

Let $\mathbf{w} \in \mathbb{R}^n_{\ge 0}$ be a weight vector in the positive orthant (so $\mathbf{w}$ could be taken as the first row of a weight matrix defining a monomial order). Let

$$\mathscr{G} = \{g_1,\dots,g_t\}$$

be one of the reduced marked Gröbner bases for I, where

$$g_i = \underline{x^{\alpha(i)}} + \textstyle\sum_\beta c_{i\beta}x^\beta,$$

and $x^{\alpha(i)}$ is marked as the leading term of g_i. If $\mathbf{w}\cdot\alpha(i) > \mathbf{w}\cdot\beta$ whenever $c_{i\beta} \neq 0$, then I will have Gröbner basis $\mathscr{G}$ with respect to any monomial order defined by a weight matrix with first row $\mathbf{w}$. The set

$$C_{\mathscr{G}} = \{\mathbf{w} \in \mathbb{R}^n_{\geq 0} \mid \mathbf{w}\cdot\alpha(i) \geq \mathbf{w}\cdot\beta \text{ whenever } c_{i\beta} \neq 0\} \tag{10.3.1}$$

is the intersection of a finite collection of half-spaces, hence has the structure of a closed convex polyhedral cone in $\mathbb{R}^n_{\geq 0}$. The cones $C_{\mathscr{G}}$ as $\mathscr{G}$ runs over all distinct marked Gröbner bases of I, together with all of their faces, have the structure of a fan in $\mathbb{R}^n_{\geq 0}$ called the *Gröbner fan* of I. In particular, for each pair $\mathscr{G}, \mathscr{G}'$ of marked Gröbner bases, the cones $C_{\mathscr{G}}$ and $C_{\mathscr{G}'}$ intersect along a common face where the $\mathbf{w}$-weights of terms in some polynomials in $\mathscr{G}$ (and in $\mathscr{G}'$) coincide.

A First Example. Let σ be a cone in normal form with parameters d,k, and recall $\gcd(d,k) = 1$ by hypothesis. By Proposition 10.1.2, the group

$$G_{d,k} = \{(\zeta,\zeta^k) \in (\mathbb{C}^*)^2 \mid \zeta^d = 1,\ 0 \leq k \leq d-1\} \simeq \mu_d$$

acts on $\mathbb{C}^2$ by componentwise multiplication

$$(\zeta,\zeta^k)\cdot(x,y) = (\zeta x, \zeta^k y), \tag{10.3.2}$$

with quotient $\mathbb{C}^2/G_{d,k} \simeq U_\sigma$.

Let $\mathbf{I}(G_{d,k})$ be the ideal defining $G_{d,k}$ as a variety in $\mathbb{C}^2$. In the next extended example, we will introduce the first main result of this section.

Example 10.3.1. Let $d = 7$, $k = 5$ and $I = \mathbf{I}(G_{7,5})$. It is easy to check that

$$I = \langle x^7 - 1,\ y - x^5\rangle.$$

Moreover, for lexicographic order with $y > x$, the set

$$\mathscr{G}^{(1)} = \{\underline{x^7} - 1,\ \underline{y} - x^5\}$$

is the reduced marked Gröbner basis for I, where the underlines indicate the leading terms. The corresponding cone in the Gröbner fan of I is

$$C_{\mathscr{G}^{(1)}} = \{\mathbf{w} = (a,b) \in \mathbb{R}^2_{\geq 0} \mid b \geq 5a\} = \mathrm{Cone}(e_2, e_1 + 5e_2).$$

There are three other marked reduced Gröbner bases of I:

$$\begin{aligned}
\mathscr{G}^{(2)} &= \{\underline{x^5} - y,\ \underline{x^2y} - 1,\ \underline{y^2} - x^3\},\\
\mathscr{G}^{(3)} &= \{\underline{x^3} - y^2,\ \underline{x^2y} - 1,\ \underline{y^3} - x\},\\
\mathscr{G}^{(4)} &= \{\underline{y^7} - 1,\ \underline{x} - y^3\}.
\end{aligned}$$

It is easy to check that each of these sets is a Gröbner basis for I using Buchberger's criterion. The corresponding cones are

$$\begin{aligned}
C_{\mathscr{G}^{(2)}} &= \{(a,b) \in \mathbb{R}^2_{\geq 0} \mid b \leq 5a,\ 2b \geq 3a\} = \mathrm{Cone}(e_1 + 5e_2, 2e_1 + 3e_2),\\
C_{\mathscr{G}^{(3)}} &= \{(a,b) \in \mathbb{R}^2_{\geq 0} \mid 2b \leq 3a,\ 3b \geq a\} = \mathrm{Cone}(2e_1 + 3e_2, 3e_1 + e_2),\\
C_{\mathscr{G}^{(4)}} &= \{(a,b) \in \mathbb{R}^2_{\geq 0} \mid 3b \leq a\} = \mathrm{Cone}(3e_1 + e_2, e_1).
\end{aligned}$$

Since these three cones fill out rest of the first quadrant in $\mathbb{R}^2$, the Gröbner fan of I consists of the four cones $C_{\mathscr{G}(i)}$ and their faces, as shown in Figure 8. We will denote this fan by Γ in the following.

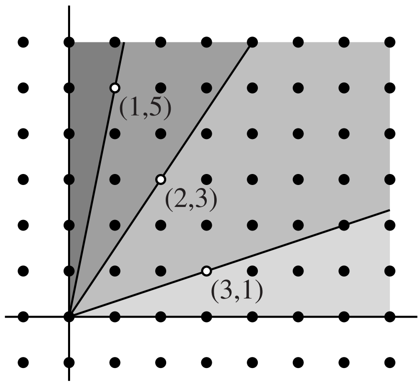

Figure 8. The Gröbner fan Γ

Next, let us consider the resolution of singularities

$$X_\Sigma \longrightarrow U_\sigma$$

for the cone σ with parameters $d = 7, k = 5$ computed in Example 10.2.4 in the last section. The reader can check that the linear transformation $T : N_\mathbb{R} \to N_\mathbb{R}$ with matrix relative to the basis e_1, e_2 given by

$$A = \begin{pmatrix} 7 & 0 \\ -5 & 1 \end{pmatrix} \tag{10.3.3}$$

maps the cones $C_{\mathscr{G}(i)}$ in the Gröbner fan Γ to the corresponding cones σ_i in the fan Σ. The matrix in (10.3.3) is invertible, but its inverse is not an integer matrix. The image of the lattice $N = \mathbb{Z}^2$ under T is the proper sublattice $7\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and T^{-1} maps N to the lattice

$$N' = \{(a/7, b/7) \mid a,\ b \in \mathbb{Z}, b \equiv 5a \bmod 7\} = N + \mathbb{Z}\left(\frac{1}{7}e_1 + \frac{5}{7}e_2\right).$$

There is an exact sequence

$$0 \longrightarrow N \longrightarrow N' \longrightarrow G \longrightarrow 0$$

induced by the map $N' \to (\mathbb{C}^*)^2$ defined by $(a/7, b/7) \mapsto (e^{2\pi i a/7}, e^{2\pi i b/7})$. Letting $\tau = \mathrm{Cone}(e_1, e_2)$, the corresponding toric morphism $U_{\tau,N} \longrightarrow U_{\tau,N'}$ is the quotient mapping $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/G$.

It is easy to check that $\mathbf{w}_0 = (0,1)$ and $\mathbf{w}_1 = (1/7, 5/7)$ form a basis of the lattice N'. In Figure 9 on the next page, the fan defined by the cones with ray

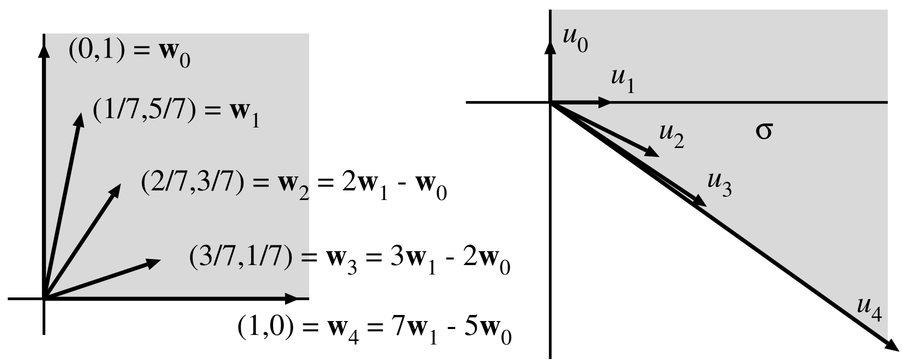

Figure 9. The toric variety $X_{\Gamma,N'}$ is the resolution of U_σ in Example 10.3.1

generators on the left is the same as in Figure 8 above, and the fan on the right is the same as in Figure 3 above. Note that $T(\mathbf{w}_i) = u_i$ for $0 \le i \le 4$ in Figure 9.

With respect to N', the cones in the Gröbner fan Γ are smooth cones, and it follows from the discussion of toric morphisms in §3.3 that the toric surfaces $X_{\Gamma,N'}$ and X_Σ are isomorphic. In other words, the Gröbner fan Γ of the ideal I encodes the structure of the resolution of singularities of U_σ.

Example B.2.4 shows how to compute this example using `GFan` [**161**]. $\Diamond$

A Tale of Two Fans. We next show that the last observation in Example 10.3.1 holds in general. As in the example, consider the ideal $I = \mathbf{I}(G_{d,k})$ and the action of $G_{d,k}$ given in (10.3.2). Each monomial in $\mathbb{C}[x,y]$ is equivalent modulo I to one of the monomials x^j, $j = 0,\dots,d-1$. This may be seen, for instance, from the remainders on division by the lexicographic Gröbner basis $\{\underline{x^d} - 1,\ \underline{y} - x^k\}$. As a result, we have a direct sum decomposition of the coordinate ring of the variety $G_{d,k}$ as a $\mathbb{C}$-vector space:

$$\mathbb{C}[x,y]/I \simeq \bigoplus_{j=0}^{d-1} V_j, \tag{10.3.4}$$

where V_j is the 1-dimensional subspace spanned by x^j mod I.

The following result establishes a first connection between the ideal I and the resolution of singularities of U_σ described in Theorem 10.2.3.

Proposition 10.3.2. *Let $I = \mathbf{I}(G_{d,k})$ where $0 < k < d$ and $\gcd(d,k) = 1$. Consider*

$$\begin{aligned} u_0 &= e_2 \\ u_i &= P_{i-1}e_1 - Q_{i-1}e_2, \quad i = 1,\dots,r+1, \end{aligned}$$

from Theorem 10.2.3. Let $T : N_\mathbb{R} \to N_\mathbb{R}$ be the linear transformation with matrix

$$A = \begin{pmatrix} d & 0 \\ -k & 1 \end{pmatrix}$$

and let $\mathbf{w}_i = T^{-1}(u_i)$ *for* $i = 0,\dots,r+1$. *Write* $\mathbf{w}_i = \frac{1}{d}(a_i e_1 + b_i e_2)$ *and define*

$$g_i = x^{b_i} - y^{a_i}, \quad 0 \le i \le r+1.$$

(a) *The polynomials* $g_0,\dots,g_{r+1}$ *are contained in the ideal* I.

(b) $\mathsf{S} = \{ae_1 + be_2 \in N \mid a,b \ge 0,\ x^b - y^a \in I\} \subseteq N$ *is an additive semigroup.*

(c) $\{d\mathbf{w}_i \mid i = 0,\dots,r+1\}$ *is the Hilbert basis of the semigroup* S *of part* (b).

Proof. It is an easy calculation to show T^{-1} maps $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ to the first quadrant $\mathbb{R}^2_{\ge 0}$. Moreover, $\mathbf{w}_0 = e_2$, and for $i = 1,\dots,r+1$,

$$\mathbf{w}_i = \frac{1}{d}\big(P_{i-1}e_1 + (kP_{i-1} - dQ_{i-1})e_2\big).$$

Therefore, $g_0 = x^d - 1$ and

$$g_i = x^{kP_{i-1} - dQ_{i-1}} - y^{P_{i-1}}$$

for $i = 1,\dots,r+1$. Since $\zeta^d = 1$, these polynomials clearly vanish at $(\zeta,\zeta^k) \in G_{d,k}$. Therefore $g_i \in \mathbf{I}(G_{d,k}) = I$ for all i.

The proof of part (b) is left to the reader as Exercise 10.3.3. For part (c), it follows from parts (a) and (b) that $d\mathbf{w}_i$ is contained in S. On the other hand, let $ae_1 + be_2 \in \mathsf{S}$. Then $x^b - y^a \in I$, which implies that $b \equiv ak \bmod d$. Hence

$$T\Big(\frac{1}{d}(ae_1 + be_2)\Big) = ae_1 + \frac{b - ak}{d}e_2$$

must be an element of $\sigma \cap N$. Since the u_i are the Hilbert basis for the semigroup $\sigma \cap N$ by Theorem 10.2.8, this vector is a nonnegative integer combination of the u_i. Hence $ae_1 + be_2$ is a nonnegative integer combination of the $d\mathbf{w}_i$. It follows that S is generated by the $d\mathbf{w}_i$. The $d\mathbf{w}_i$ are irreducible in S because the corresponding $u_i = T(\mathbf{w}_i)$ are irreducible in the semigroup $\sigma \cap N$. □

We also have a first result about reduced Gröbner bases of the ideal I.

Lemma 10.3.3. *Every element of a reduced Gröbner basis of* $I = \mathbf{I}(G_{d,k})$ *is either of the form* $x^b - y^a$ *or of the form* $x^s y^t - 1$ *for* $s,t > 0$.

Proof. Since I is generated by $x^d - 1, y - x^k$, the Buchberger algorithm implies that a reduced Gröbner basis $\mathscr{G}$ of I consists of binomials. By taking out common factors, every $g \in \mathscr{G}$ can be written $g = x^i y^j h$, where $h = x^b - y^a$ or $x^s y^t - 1$. Then $h \in I$ since it vanishes on $G_{d,k}$. Its leading term is divisible by the leading term of an element of $\mathscr{G}$, which is impossible in a reduced Gröbner basis unless $g = h$. □

The Gröbner bases in Example 10.3.1 give a nice illustration of Lemma 10.3.3. Our next lemma relates the polynomials $g_i = x^{b_i} - y^{a_i}$ from Proposition 10.3.2 to the reduced Gröbner bases of I.

Lemma 10.3.4. *Let* $g_i = x^{b_i} - y^{a_i}$ *be as in Proposition 10.3.2 and fix a monomial order* $>$ *on* $\mathbb{C}[x,y]$.

(a) *The a_i are increasing and the b_i are decreasing with i.*

(b) *There is some index $i = i_0$ (depending on $>$) such that*

$$\text{LT}_>(g_i) = x^{b_i} \text{ for all } i \leq i_0, \text{ and } \text{LT}_>(g_i) = y^{a_i} \text{ for all } i > i_0.$$

(c) *If $i = i_0$ is the index from part* (b)*, then g_{i_0} and g_{i_0+1} are elements of the reduced Gröbner basis of $I = \mathbf{I}(G_{d,k})$ with respect to $>$.*

Proof. You will prove parts (a) and (b) in Exercise 10.3.4. For part (c), let $\mathscr{G}$ be the reduced Gröbner basis of I with respect to $>$. Since $g_{i_0} \in I$ by part (a) of Proposition 10.3.2, there is $g \in \mathscr{G}$ whose leading term divides $\text{LT}_>(g_{i_0}) = x^{b_{i_0}}$. By Lemma 10.3.3, it follows that $g = x^b - y^a$ with $\text{LT}_>(g) = x^b$. In particular, $b \leq b_{i_0}$ and $ae_1 + be_2 \in \mathsf{S}$, where S is the semigroup from part (b) of Proposition 10.3.2. Then part (c) of the same proposition implies that $ae_1 + be_2$ must be a nonnegative integer combination

$$ae_1 + be_2 = \textstyle\sum_{i=0}^{r+1} \ell_i d\mathbf{w}_i = \sum_{i=0}^{r+1} \ell_i(a_i e_1 + b_i e_2), \quad \ell_i \in \mathbb{N}. \tag{10.3.5}$$

Since $b \leq b_{i_0}$ and the b_i decrease with i, (10.3.5) can include only the $d\mathbf{w}_i$ with $i \geq i_0$. Suppose that $d\mathbf{w}_i$ appears in (10.3.5) with $i > i_0$. Then $a \geq a_i$ and $y^{a_i} > x^{b_i}$, so that $\text{LT}_>(g_i) = y^{a_i}$ divides y^a. Since $g_i \in I$, $\text{LT}_>(g_i)$ is divisible by the leading term of some $h \in \mathscr{G}$. Hence $\text{LT}_>(h)$ divides y^a, which is a term of $g = x^b - y^a \in \mathscr{G}$. This is impossible in a reduced Gröbner basis, hence $i > i_0$ cannot occur in (10.3.5). From here, it follows easily that $g = g_{i_0}$, giving $g_{i_0} \in \mathscr{G}$ as desired.

The statement for g_{i_0+1} follows by an argument parallel to the one above. The details are left to the reader (Exercise 10.3.4). □

We are now ready for the first major result of this section.

Theorem 10.3.5. *Let Γ be the fan in $\mathbb{R}^2$ with maximal cones*

$$\gamma_i = \text{Cone}(a_{i-1}e_1 + b_{i-1}e_2, a_i e_1 + b_i e_2), \quad 1 \leq i \leq r+1,$$

for a_i, b_i as in Proposition 10.3.2. Then Γ is the Gröbner fan of the ideal $\mathbf{I}(G_{d,k})$.

Proof. The cones γ_i fill out the first quadrant $\mathbb{R}^2_{\geq 0}$. Take any $\mathbf{w} = ae_1 + be_2$ lying in the interior of some γ_i and let $>$ be a monomial order defined by a weight matrix with $\mathbf{w}$ as first row. This gives a reduced Gröbner basis $\mathscr{G}$. The theorem will follow once we prove that γ_i is the Gröbner cone $C_{\mathscr{G}}$ of $\mathscr{G}$.

First observe that for $>$, we must have must have $i_0 = i - 1$ in part (b) of Lemma 10.3.4. It follows from part (c) of the lemma that g_{i-1} and g_i are elements of $\mathscr{G}$. Since a reduced Gröbner basis has only one element with leading term a power of x and only one with leading term a power of y, all other elements of $\mathscr{G}$ will have the form $x^s y^t - 1$, $s, t > 0$, by Lemma 10.3.3. Therefore, the Gröbner cone $C_{\mathscr{G}}$ is exactly γ_i and we are done. □

As in Example 10.3.1, the following statement is an immediate consequence.

Corollary 10.3.6. *Let N' be the lattice*

$$N' = \{(a/d, b/d) \mid a,b \in \mathbb{Z}, b \equiv k \bmod d\} = \mathbb{Z}\big(\tfrac{1}{d}e_1 + \tfrac{k}{d}e_2\big) \oplus \mathbb{Z}e_2.$$

The toric surfaces X_Σ and $X_{\Gamma,N'}$ are isomorphic.

In other words, the Gröbner fan of $\mathbf{I}(G_{d,k})$ can be used to construct a resolution of singularities of the affine toric surface U_σ when σ has parameters d,k.

Connections with Representation Theory. We now consider the above results from a different point of view. We assume the reader is familiar with the beginnings of representation theory for finite abelian groups.

The group $G = G_{d,k}$ from (10.3.2) acts on $V = \mathbb{C}^2$ by the 2-dimensional linear representation of the group μ_d of dth roots of unity defined by

$$(10.3.6)\qquad \begin{aligned} \rho : \mu_d &\longrightarrow \mathrm{GL}(V) = \mathrm{GL}(2,\mathbb{C}) \\ \zeta &\longmapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^k \end{pmatrix}. \end{aligned}$$

Since μ_d is abelian, its irreducible representations are 1-dimensional over $\mathbb{C}$, and hence each is defined by a character

$$\begin{aligned} \chi_j : \mu_d &\longrightarrow \mathbb{C}^* \\ \zeta &\longmapsto \zeta^{-j} \end{aligned}$$

for $j = 0,\ldots,d-1$. The reason for the minus sign will soon become clear.

Via (10.3.2), we get the induced action of μ_d on the polynomial ring $\mathbb{C}[x,y]$ by

$$\zeta \cdot x = \zeta^{-1}x, \quad \zeta \cdot y = \zeta^{-j}y,$$

as explained in §5.0. Each monomial $x^a y^b$ spans an invariant subspace where the action of μ_d is given by the irreducible representation with character χ_{-j} for $j \equiv a + kb \bmod d$. We call $a + kb \bmod d$ the *weight* of the monomial $x^a y^b$ with respect to this action of μ_d. Since the ideal of the group $G \subset (\mathbb{C}^*)^2$ is invariant, the action descends to the quotient $\mathbb{C}[x,y]/\mathbf{I}(G)$, and we have a representation of μ_d on $\mathbb{C}[x,y]/\mathbf{I}(G)$. The direct sum decomposition (10.3.4) shows that the irreducible representation with character χ_{-j} appears exactly once in this representation, as the subspace V_j in (10.3.4). This means that the representation on $\mathbb{C}[x,y]/\mathbf{I}(G)$ is isomorphic to the *regular representation* of μ_d (Exercise 10.3.5).

A 2-dimensional M^cKay Correspondence. In 1979, M^cKay pointed out that there is a one-to-one correspondence between the irreducible representations of μ_d and the components of the exceptional divisor in the resolution $\phi : X_\Sigma \longrightarrow U_\sigma$ when σ was a cone in normal form with parameters $k = d-1$, as in Example 10.1.5. In this case, the singular point of U_σ is a rational double point, and the image of the representation ρ from (10.3.6) lies in $\mathrm{SL}(2,\mathbb{C})$. A great deal of research was devoted to explaining the original M^cKay correspondence in representation-theoretic and geometric terms (work of Gonzalez-Sprinberg, Artin, and Verdier). However, for $1 < k < d-1$, $\rho(\mu_d)$ is a subgroup of $\mathrm{GL}(2,\mathbb{C})$, not $\mathrm{SL}(2,\mathbb{C})$, and there are more irreducible representations of μ_d than components of the exceptional divisor. The McKay correspondence can be extended to these cases by identifying certain *special representations* that correspond to the components of the exceptional divisor (work of Wunram, Esnault, Ito and Nakamura, Kidoh, and others).

We will describe a generalized McKay correspondence that applies for all d,k. Writing $G = G_{d,k}$ as before, consider the ring of invariants $\mathbb{C}[x,y]^G$. You will prove the following in Exercise 10.3.6.

Lemma 10.3.7. *Let V_j be an irreducible representation of μ_d with character χ_{-j}, and consider the action of $G = G_{d,k} \simeq \mu_d$ on $\mathbb{C}[x,y] \otimes_{\mathbb{C}} V_j$. Then the subspace of invariants $(\mathbb{C}[x,y] \otimes_{\mathbb{C}} V_j)^G$ has the structure of a module over the ring $\mathbb{C}[x,y]^G$.* □

Example 10.3.8. Let $G = G_{7,5}$ as in Example 10.3.1. The ring of invariants is $\mathbb{C}[x,y]^G = \mathbb{C}[x^7, x^2y, xy^4, y^7]$ in this case. If v_j is the basis of the representation V_j, then it is easy to check that $x^a y^b \otimes v_j$ is invariant under G if and only if $a + kb - j \equiv 0 \bmod 7$, or in other words if and only if $x^a y^b$ has weight j under this action of μ_7.

First consider the case $j = 1$ in Lemma 10.3.7. The monomials in the complement of the monomial ideal

$$\mathsf{M} = \langle x^7, x^2y, xy^4, y^7 \rangle$$

that have weight 1 are x and y^3. Then $x \otimes v_1$ and $y^3 \otimes v_1$ generate the module $(\mathbb{C}[x,y] \otimes V_1)^G$. Since x and y^3 have the same weight with respect to this action of μ_7, the difference $x - y^3$ is an element of the ideal $\mathbf{I}(G)$, and this is one of the polynomials g_i as in the proof of Proposition 10.3.2.

On the other hand, if $j = 2$, then there are three monomials with weight 2 in the complement of M, and these give three generators of $(\mathbb{C}[x,y] \otimes V_2)^G$, namely $x^2 \otimes v_2, xy^3 \otimes v_2$, and $y^6 \otimes v_2$. It is still true that $x^2 - xy^3$, $x^2 - y^6$, and $xy^3 - y^6$ are elements of $\mathbf{I}(G)$, but these polynomials cannot appear in a reduced Gröbner basis for $\mathbf{I}(G)$. Moreover, no proper subset of the three generators generates the whole module $(\mathbb{C}[x,y] \otimes V_2)^G$. ◊

Definition 10.3.9. Let $G = G_{d,k} \simeq \mu_d$ as above. We say that the representation V_j is ***special with respect to*** k if $(\mathbb{C}[x,y] \otimes V_j)^G$ is minimally generated as a module over the invariant ring $\mathbb{C}[x,y]^G$ by two elements.

Hence, in Example 10.3.8, V_1 is special while V_2 is not. According to our definition, the trivial representation V_0 is never special, since $(\mathbb{C}[x,y] \otimes V_0)^G$ is generated by the single monomial 1 over the invariant ring. Our next theorem gives a rudimentary form of a McKay correspondence for the group $G_{d,k}$.

Theorem 10.3.10 (McKay Correspondence). *Let σ be a cone with parameters d,k, where $0 < k < d$ and $\gcd(d,k) = 1$. Then there is a one-to-one correspondence between the representations of μ_d that are special with respect to k and the components of the exceptional divisor for the minimal resolution $\phi : X_\Sigma \to U_\sigma$.*

Proof. Write $G = G_{d,k}$ as above and consider the set B of monomials in the complement of the ideal M generated by the G-invariant monomials. This set contains

$$L = \{1, x, x^2, \ldots, x^{d-1}, y, y^2, \ldots, y^{d-1}\}.$$

Since $\gcd(d,k) = 1$, for each $1 \le j \le d-1$, there is an integer $1 \le a_j \le d-1$ such that x^j and y^{a_j} have equal weight (equal to j) for the action of G. The representation V_j is special with respect to k if and only if these are the *only* two monomials of weight j in the set B, and nonspecial if and only if there is some monomial $x^a y^b$ with $a, b > 0$ in B which also has weight j. Since $x^j - y^{a_j} \in \mathbf{I}(G)$, saying V_j is special with respect to k is in turn equivalent to saying that the corresponding vector $a_j e_1 + j e_2$ is an irreducible element in the semigroup from part (b) of Proposition 10.3.2 (Exercise 10.3.8). By Theorem 10.3.5, $\mathrm{Cone}(a_j e_1 + j e_2)$ is one of the 1-dimensional cones of the Gröbner fan of $\mathbf{I}(G)$. Then V_j corresponds to one of the 1-dimensional cones in the fan Σ and hence to one of the irreducible components of the exceptional divisor. □

The original McKay correspondence is the following special case.

Corollary 10.3.11. *When $k = d-1$, there is a one-to-one correspondence between the set of all irreducible representations of μ_d and the components of the exceptional divisor of the minimal resolution $\phi : X_\Sigma \longrightarrow U_\sigma$.*

Proof. In this case, the invariant ring is $\mathbb{C}[x^d, xy, y^d]$, so the sets L and B in the proof of the theorem coincide. □

There has also been much work devoted to extend the McKay correspondence to finite abelian subgroups $G \subset \mathrm{GL}(n,\mathbb{C})$ for $n \ge 3$, and several other ways to understand these constructions have also been developed, including the theory of G-Hilbert schemes. See Exercise 10.3.10 for the beginnings of this.

Exercises for §10.3.

10.3.1. In this exercise, you will verify the claims made in Example 10.3.1, and extend some of the observations there.

(a) Show that each of the $\mathscr{G}^{(i)}$ is a Gröbner basis of $\mathbf{I}(G_{7,5})$.

(b) Show that $\mathscr{G} = \{x^5 - 1, y - x^3, x^2 y - 1, x - y^2, y^5 - 1\}$ is a universal Gröbner basis for $\mathbf{I}(G_{7,5})$.

(c) Determine the cones $C_{\mathscr{G}^{(i)}}$ using (10.3.1).

(d) Verify the final claim that linear transformation defined by the matrix A from (10.3.3) maps the Gröbner cones $C_{\mathscr{G}^{(i)}}$ to the σ_i for $i = 1, 2, 3$.

10.3.2. Verify the conclusions of Proposition 10.3.2 and Theorem 10.3.5 for the case $d = 17, k = 11$.

10.3.3. Show that the set S defined in part (b) of Proposition 10.3.2 is an additive semigroup. Hint: A direct proof starts from two general elements $ae_1 + be_2$ and $a'e_1 + b'e_2$ in S. Consider $(x^b - y^a)(x^{b'} + y^{a'})$ and $(x^b + y^a)(x^{b'} - y^{a'})$.

10.3.4. In this exercise you will complete the proof of Lemma 10.3.4.

(a) Show that for each i, $kP_i - dQ_i = k_i$, where the k_i are produced by the modified Euclidean algorithm from (10.2.1).

(b) Prove part (a) of Lemma 10.3.4.

(c) Verify that there is an index i_0 as in part (b) of Lemma 10.3.4.

(d) Verify that if i_0 is as in part (c), then g_{i_0+1} is contained in the reduced Gröbner basis.

10.3.5. If G is any finite group, the *(left) regular representation* of G is defined as follows. Let W be a vector space over $\mathbb{C}$ of dimension $|G|$ with a basis $\{e_h \mid h \in G\}$ indexed by the elements of G. For each $g \in G$ let $\rho(g) : W \longrightarrow W$ be defined by $\rho(g)(e_h) = e_{gh}$.

(a) Show that $g \mapsto \rho(g)$ is a group homomorphism from G to $\mathrm{GL}(W)$.

(b) Now let G be the cyclic group μ_d of order d. Show that W is the direct sum of 1-dimensional invariant subspaces W_j, $j = 0, \ldots, d-1$ on which G acts by the character χ_j defined in the text, so that W decomposes as $W \simeq \bigoplus_{j=0}^{d-1} V_j$.

10.3.6. In this exercise you will consider the module structures from Lemma 10.3.7.

(a) Prove Lemma 10.3.7.

(b) Verify the claims made in Example 10.3.8.

10.3.7. In this exercise you will prove an alternate characterization of the special representations with respect to k from Definition 10.3.9. We write $G = G_{d,k} \simeq \mu_d$ as usual.

(a) Let $\Omega^2_{\mathbb{C}^2} = \{f\, dx \wedge dy \mid f \in \mathbb{C}[x,y]\}$. Show that $\zeta \cdot x^a y^b dx \wedge dy = \zeta^{a+b+1+k} x^a y^b dx \wedge dy$ defines an action of G on $\Omega^2_{\mathbb{C}^2}$.

(b) Show that the spaces of G-invariants $(\Omega^2_{\mathbb{C}^2})^G$ and $(\Omega^2_{\mathbb{C}^2} \otimes V_j)^G$ have the structure of modules over the invariant ring $\mathbb{C}[x,y]^G$.

(c) Show that V_j is special with respect to k if and only if the "multiplication map"

$$(\Omega^2_{\mathbb{C}^2})^G \otimes (\mathbb{C}[x,y] \otimes V_j)^G \longrightarrow (\Omega^2_{\mathbb{C}^2} \otimes V_j)^G$$

is surjective.

10.3.8. In the proof of the M^cKay correspondence, show that $a_j e_1 + j e_2$ is irreducible in the semigroup S from Proposition 10.3.2 if and only if the representation V_j is special with respect to k.

10.3.9. Let g_i, $0 \le i \le r+1$, be the binomials constructed in Proposition 10.3.2. Show that

$$\mathscr{U} = \{g_1, \ldots, g_r\} \cup \{x^a y^b - 1 \mid x^a y^b \text{ is } G_{d,k}\text{-invariant}\}$$

is a universal Gröbner basis for $\mathbf{I}(G_{d,k})$ (not always minimal, however).

10.3.10. Let $G = G_{d,k}$ act on $\mathbb{C}^2$ as in (10.3.2). As a point set, G can be viewed as the orbit of the point $(1,1)$ under this action. The ideal $I = \mathbf{I}(G)$ is invariant under the action of G on $\mathbb{C}[x,y]$ and as we have seen, the corresponding representation on $\mathbb{C}[x,y]/I$ is isomorphic to the regular representation of G.

(a) Show that if $p = (\xi, \eta)$ is any point in $\mathbb{C}^2$ other than the origin, the ideal I of the orbit of p is another G-invariant ideal and the corresponding representation of G on $\mathbb{C}[x,y]/I$ is also isomorphic to the regular representation of G.

(b) The *G-Hilbert scheme* can be defined as the set of all G-invariant ideals in $\mathbb{C}[x,y]$ such that the representation of G on $\mathbb{C}[x,y]/I$ is isomorphic to the regular representation of G. Show that every such ideal has a set of generators of the form

$$\{x^a - \alpha y^c, y^b - \beta x^d, x^{a-d}y^{b-c} - \alpha\beta\}$$

for some $\alpha, \beta \in \mathbb{C}$ and where x^a and y^c (resp. y^b and x^d) have equal weights for the action of G. It can be seen from this result that the G-Hilbert scheme is also isomorphic to the minimal resolution of singularities of U_σ.

§10.4. Smooth Toric Surfaces

This section will use §10.1 and §10.2 to classify smooth complete toric surfaces and study the relation between continued fractions and intersection products of divisors on the resulting resolutions of singularities.

Classification of Smooth Toric Surfaces. We will show that smooth complete toric surfaces are all obtained by toric blowups from either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or one of the Hirzebruch surfaces $\mathscr{H}_r$ with $r \geq 2$ from Example 3.1.16. The proof will be based on the following facts.

First, Proposition 3.3.15 implies that if $\sigma = \mathrm{Cone}(u_1, u_2)$ is a smooth cone and we refine σ by inserting the new 1-dimensional cone $\tau = \mathrm{Cone}(u_1 + u_2)$, then on the resulting toric surface, the smooth point p_σ is blown up to a copy of $\mathbb{P}^1$.

For the second ingredient of the proof, we introduce the following notation. If Σ is a smooth complete fan, then list the ray generators of the 2-dimensional cones in Σ as $u_0, u_1, \ldots, u_{r-1}$ in clockwise order around the origin in $N_\mathbb{R}$, and we will consider the indices as integers modulo r, so $u_r = u_0$. Then we have the following statement parallel to (10.2.8).

Lemma 10.4.1. *Let $u_0, \ldots, u_r$ be the ray generators for a smooth complete fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^2$. There exist integers b_i, $i = 0, \ldots, r-1$, such that*

$$u_{i-1} + u_{i+1} = b_i u_i. \tag{10.4.1}$$

Proof. This is a special case of the wall relation (6.4.4). □

We also have the following result.

Lemma 10.4.2. *Let Σ be a smooth fan that refines a smooth cone σ. Then Σ is obtained from σ by a sequence of star subdivisions as in Definition 3.3.13.*

Proof. Suppose Σ has r cones of dimension 2, with ray generators $u_0, \ldots, u_r$, listed clockwise starting from u_0. We argue by induction on $r = |\Sigma(2)|$. If $r = 1$, there is only one cone in Σ and there is nothing to prove. Assume the result has been proved for all Σ with $|\Sigma(2)| = r$, and consider a fan Σ with $|\Sigma(2)| = r+1$. There are r "interior" rays that by (6.4.4) give wall relations $u_{i-1}+u_{i+1} = b_i u_i$, $b_i \in \mathbb{Z}$, for $1 \le i \le r$. Note that σ is strongly convex so $b_i > 0$ for all i. We claim that there exists some i such that $b_i = 1$. If not, i.e., if $b_i \ge 2$ for all i, then as in §10.2, the Hirzebruch-Jung continued fraction

$$[[b_1, b_2, \ldots, b_r]]$$

represents a rational number d/k and the cone σ has parameters d,k with $d > k > 0$. But then $d \ge 2$, which would contradict the assumption that σ is a smooth cone.

Hence there exists an i, $1 \le i \le r$, such that

$$u_{i-1}+u_{i+1} = u_i.$$

In this situation, $\mathrm{Cone}(u_{i-1}, u_{i+1})$ is also smooth (Exercise 10.4.1). Moreover, $\mathrm{Cone}(u_{i-1}, u_i)$ and $\mathrm{Cone}(u_i, u_{i+1})$ are precisely the cones in the star subdivision of $\mathrm{Cone}(u_{i-1}, u_{i+1})$. Then we are done by induction. □

We are now ready to state our classification theorem.

Theorem 10.4.3. *Every smooth complete toric surface X_Σ is obtained from either*

$$\mathbb{P}^2,\ \mathbb{P}^1 \times \mathbb{P}^1,\ \textit{or}\ \mathscr{H}_r,\ r \ge 2$$

by a finite sequence of blowups at fixed points of the torus action.

Proof. We follow the notation of Lemma 10.4.1. As in the proof of Lemma 10.4.2, if $b_i = 1$ in (10.4.1) for some i, then our surface is a blowup of the smooth surface corresponding to the fan where u_i is removed. Hence we only need to consider the case in (10.4.1) where $b_i \ne 1$ for all i.

Suppose first that $u_j = -u_i$ for some $i < j$. We relabel the vertices to make $u_j = -u_1$ for some j. Note that $j > 2$ since the cones must be strongly convex. Then from (10.4.1),

$$u_0 = -u_2 + b_1 u_1.$$

Using the basis u_1, u_2 of N, we get the picture shown in Figure 10 on the next page. Comparing this with the fans for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathscr{H}_a$ (Figures 2, 3 and 4 from §3.1), we see that Σ is a refinement of the fan of $\mathscr{H}_r$ if $r = b_1 > 2$. The same follows if $b_1 < -2$ and $r = |b_1|$ (see Exercise 10.4.2). Since $b_1 \ne 1$, the remaining possibilities are $b_1 = 0$ or -1, where we get a refinement of the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ respectively. Then the theorem follows from Lemma 10.4.2 in this case.

We will complete the proof using primitive collections (Definition 5.1.5). The first step is to note that X_Σ is projective by Proposition 6.3.25. This will allow us to use Proposition 7.3.6, which asserts that Σ has a primitive collection whose minimal generators sum to 0.

If $\Sigma(1)$ has only three elements, it is easy to see that $X_\Sigma = \mathbb{P}^2$ since Σ is smooth and complete (Exercise 10.4.3). If $|\Sigma(1)| > 3$, every primitive collection

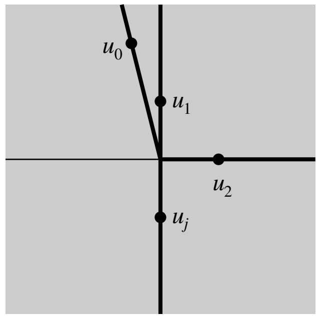

Figure 10. The ray generators u_0, u_1, u_2, u_j when $u_j = -u_1$

of Σ has exactly two elements (Exercise 10.4.3). By Proposition 7.3.6, one of these primitive collections must have minimal generators u_i, u_j that satisfy $u_i + u_j = 0$. Hence $u_j = -u_i$, and we are done by the earlier part of the proof. □

Since there is also a Hirzebruch surface $\mathscr{H}_1$, the statement of this theorem might seem puzzling. The reason that $\mathscr{H}_1$ is not included is that this surface is actually a blowup of $\mathbb{P}^2$ (Exercise 10.4.4).

The problem of classifying smooth compete toric varieties of higher dimension is much more difficult. We did this when $\operatorname{rank}\operatorname{Pic}(X_\Sigma) = 2$ in Theorem 7.3.7. See [**14**] for the case when $\operatorname{rank}\operatorname{Pic}(X_\Sigma) = 3$.

Intersection Products on Smooth Surfaces. A fundamental feature of the theory of smooth surfaces is the intersection product on divisors. In §6.3, we defined $D \cdot C$ when D is a Cartier divisor and C is a complete irreducible curve. On a smooth complete surface, this means that the intersection product $D \cdot C$ is defined for all divisors D and C. In particular, taking $D = C$ gives the *self-intersection* $D \cdot D = D^2$.

Here is a useful result about intersection numbers on a smooth toric surface.

Theorem 10.4.4. *Let D_ρ be the divisor on a smooth toric surface X_Σ corresponding to $\rho = \operatorname{Cone}(u)$ which is the intersection of 2-dimensional cones $\operatorname{Cone}(u, u_1)$ and $\operatorname{Cone}(u, u_2)$ in Σ. Then:*

(a) *$D_\rho \cdot D_\rho = -b$, where $u_1 + u_2 = bu$ as in (10.4.1).*

(b) *For a divisor $D_{\rho'} \neq D_\rho$, we have*

$$D_{\rho'} \cdot D_\rho = \begin{cases} 1 & \rho' = \operatorname{Cone}(u_i),\ i = 1,2 \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Since X_Σ is smooth, part (a) follows easily from Proposition 6.4.4 once you compare (10.4.1) to (6.4.4). Then part (b) follows from Corollary 6.4.3 and Proposition 6.4.4. □

Example 10.4.5. Let $\sigma = \text{Cone}(u_1, u_2)$ be a cone in a smooth fan Σ, and consider the star subdivision, in which $\rho = \text{Cone}(u_1 + u_2)$ is inserted to subdivide σ into two cones. Call the refined fan Σ'. Then the exceptional divisor $E = D_\rho$ of the blowup

$$\phi : X_{\Sigma'} \to X_\Sigma$$

satisfies $E \cdot E = -1$ on $X_{\Sigma'}$. ♢

Complete curves with self-intersection number -1 on a smooth surface are called *exceptional curves of the first kind*. They can always be *contracted* to a smooth point on a birationally equivalent surface, as in the above example.

One of the foundational results in the theory of general algebraic surfaces is that every smooth complete surface S has at least one *relatively minimal model*. This means that there is a birational morphism $S \to \overline{S}$, where $\overline{S}$ is a smooth surface with the property that if $\phi : \overline{S} \to S'$ is a birational morphism to another smooth surface S', then ϕ is necessarily an isomorphism. This is proved in [**131**, V.5.8]. Interestingly, the possible relatively minimal models for rational surfaces are precisely the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathscr{H}_r$, $r \geq 2$, from Theorem 10.4.3.

On a smooth complete toric surface X_Σ, the intersection product can be regarded as a $\mathbb{Z}$-valued symmetric bilinear form on $\text{Pic}(X_\Sigma)$. Here is an example.

Example 10.4.6. Consider the Hirzebruch surface $\mathscr{H}_r$. Using the fan shown in Figure 3 of Example 4.1.8, we get divisors $D_1, \ldots, D_4$ corresponding to minimal generators $u_1 = -e_1 + re_2,\ u_2 = e_2,\ u_3 = e_1,\ u_4 = -e_2$. By Theorem 10.4.4, we have the self-intersections

$$D_1 \cdot D_1 = D_3 \cdot D_3 = 0,\ D_2 \cdot D_2 = -r,\ D_4 \cdot D_4 = r.$$

The Picard group $\text{Pic}(\mathscr{H}_r)$ is generated by the classes of D_3 and D_4. Note also that

$$D_3 \cdot D_4 = D_4 \cdot D_3 = 1$$

by Theorem 10.4.4. The intersection product is described by the matrix

$$\begin{pmatrix} D_3 \cdot D_3 & D_3 \cdot D_4 \\ D_4 \cdot D_3 & D_4 \cdot D_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & r \end{pmatrix}.$$

If $D \sim aD_3 + bD_4$ and $E \sim cD_3 + dD_4$ are any two divisors on the surface, then

$$D \cdot E = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & r \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = bc + ad + rbd. \tag{10.4.2}$$

For instance, with $D = E = D_2 \sim -rD_3 + D_4$, we obtain

$$D_2 \cdot D_2 = 1 \cdot (-r) + (-r) \cdot 1 + r \cdot 1 \cdot 1 = -r.$$

The self-intersection numbers $D_1 \cdot D_1 = D_3 \cdot D_3 = 0$ reflect the fibration structure on $\mathscr{H}_r$ studied in Example 3.3.20. The divisors D_1 and D_3 are fibers of the mapping $\mathscr{H}_r \to \mathbb{P}^1$. Such curves always have self-intersection equal to zero. You will compute several other intersection products on $\mathscr{H}_r$ in Exercise 10.4.5. ♢

Resolution of Singularities Reconsidered. Another interesting class of smooth toric surfaces consists of those that arise from a resolution of singularities of the affine toric surface U_σ of a 2-dimensional cone σ. Here is a simple example.

Example 10.4.7. Let $\sigma = \mathrm{Cone}(e_2, de_1 - e_2)$ have parameters $d, 1$ where $d > 1$. The resolution of singularities $X_\Sigma \to U_\sigma$ constructed in Example 10.1.8 uses the smooth refinement of σ obtained by adding $\mathrm{Cone}(e_1)$. This gives the exceptional divisor E on X_Σ. Since

$$e_2 + (de_1 - e_2) = de_1,$$

we see that

$$E \cdot E = -d$$

is the self-intersection number of E. ◊

More generally, suppose that the smooth toric surface X_Σ is obtained via a resolution of singularities of U_σ, where the 2-dimensional cone σ has parameters d, k with $d > 1$. Let the Hirzebruch-Jung continued fraction expansion of d/k be

$$d/k = [[b_1, b_2, \ldots, b_r]].$$

Recall from Theorem 10.2.3 that Σ is obtained from $\sigma = \mathrm{Cone}(u_0, u_{r+1})$ by adding rays generated by $u_1, \ldots, u_r$, and by Theorem 10.2.5, we have

$$u_{i-1} + u_{i+1} = b_i u_i, \quad 1 \le i \le r.$$

It follows that $D_1, \ldots, D_r$ are complete curves in X_Σ. They are the irreducible components of the exceptional fiber, with self-intersections

$$D_i \cdot D_i = -b_i, \quad 1 \le i \le r,$$

by Theorem 10.4.4. Then the intersection matrix $(D_i \cdot D_j)_{1 \le i,j \le r}$ is given by

$$D_i \cdot D_j = \begin{cases} -b_i & \text{if} \quad j = i \\ 1 & \text{if} \quad |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{10.4.3}$$

In Exercise 10.4.6 you will show that the associated quadratic form is negative definite. This condition is necessary for the *contractibility* of a complete curve C on a smooth surface S, i.e., the existence of a proper birational morphism $\pi : S \to \overline{S}$, where $\pi(C)$ is a (possibly singular) point on $\overline{S}$.

The resolutions described here have another important property.

Definition 10.4.8. A resolution of singularities $\phi : Y \to X$ is *minimal* if for every resolution of singularities $\psi : Z \to X$, there is a morphism $\rho : Z \to Y$ such that

$$\begin{array}{ccc} & & Y \\ & \overset{\rho}{\nearrow} & \big\downarrow{\scriptstyle \phi} \\ Z & \xrightarrow[\psi]{} & X \end{array}$$

is a commutative diagram, i.e., $\psi = \phi \circ \rho$.

It is easy to see that if a minimal resolution of X exists, then it is unique up to isomorphism. If X has a unique singular point p, then using the theory of birational morphisms of surfaces it is not difficult to show that a resolution of singularities $\phi : Y \to X$ is minimal if the exceptional fiber contains no irreducible components E with $E \cdot E = -1$ (see Exercise 10.4.7). By Theorem 10.4.4 and the fact that $b_i \geq 2$ in Hirzebruch-Jung continued fractions, this holds for the resolutions constructed in Theorem 10.2.3. Hence we have the following.

Corollary 10.4.9. *The resolution of singularities of the affine toric surface U_σ constructed in Theorem 10.2.3 is minimal.* □

Rational Double Points Reconsidered. If σ has parameters $d, d-1$, then from Exercise 10.2.2, the Hirzebruch-Jung continued fraction expansion of $d/(d-1)$ is given by

$$d/(d-1) = [[2,2,\ldots,2]],$$

with $d-1$ terms. Hence $b_i = 2$ for all i, and (10.4.3) gives the $(d-1)\times(d-1)$ matrix

$$\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

representing the intersection product on the subgroup of $\mathrm{Pic}(X_\Sigma)$ generated by the components of the exceptional divisor for the resolution of a rational double point of type A_{d-1}. We can now fully explain the terminology for these singularities.

The problem of classifying lattices

$$\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_s$$

with negative definite bilinear forms B satisfying $B(e_i, e_i) = -2$ for all i arises in many areas within mathematics, most notably in the classification of complex simple Lie algebras via root systems. The matrix above is the (negative of) the Cartan matrix for the root system of type A_{d-1}, which is often represented by the Dynkin diagram:

with $d-1$ vertices. The vertices represent the lattice basis vectors. The edges connect the pairs with $B(e_i, e_j) \neq 0$ and $B(e_i, e_i) = -2$ for all i as above. In our case, the vertices represent the components D_i of the exceptional divisor, and the bilinear form is the intersection product.

A precise definition of a surface *rational double point* follows.

Definition 10.4.10. A singular point p of a normal surface X is a ***rational double point*** or ***Du Val singularity*** if X has a minimal resolution of singularities $\phi : Y \to X$ such that if K_Y is a canonical divisor on Y, then every irreducible component E_i of the exceptional divisor E over p satisfies

$$K_Y \cdot E_i = 0.$$

We can relate these concepts to the toric case as follows.

Proposition 10.4.11. *Assume σ has parameters $d > k > 0$ and let $\phi : X_\Sigma \to U_\sigma$ be the resolution of singularities constructed in Theorem 10.2.3. Then the singular point of U_σ is a rational double point if and only if $k = d-1$.*

Proof. The canonical divisor of X_Σ is $K_{X_\Sigma} = -\sum_{i=0}^{r+1} D_i$, and one computes that

$$K_{X_\Sigma} \cdot D_i = b_i - 2, \quad 1 \le i \le r.$$

Thus the singular point is a rational double point if and only if $b_i = 2$ for all i. This easily implies $k = d-1$. You will verify these claims in Exercise 10.4.8. □

There is much more to say about rational double points. For example, one can show that $E = E_1 + \cdots + E_r$ satisfies $E \cdot E = -2$ (you will prove this in the toric case in Exercise 10.4.8). From a more sophisticated point of view, $E \cdot E = -2$ implies that the canonical sheaf on Y is the pullback of the canonical sheaf on X under ϕ. We will explore this in Proposition 11.2.8. See [**85**] for more on rational double points.

Exercises for §10.4.

10.4.1. Here you will verify several statements made in the proof of Lemma 10.4.2.

(a) Show that if the cone σ is strictly convex, then the integers b_i in (10.4.1) must be strictly positive.

(b) Show that if $u_{i-1} + u_{i+1} = u_i$, then $\mathrm{Cone}(u_{i-1}, u_{i+1})$ must also be smooth.

10.4.2. In the proof of Theorem 10.4.3, verify that if $u_j = -u_1$ and $u_0 = -u_2 + b_1 u_1$ with $b_1 < -2$, then Σ is a refinement of a fan Σ' with $X_{\Sigma'} \simeq \mathscr{H}_r$, where $r = |b_1|$.

10.4.3. In this exercise, you will prove some facts used in the proof of Theorem 10.4.3. Let Σ be a smooth complete fan in $N_\mathbb{R} \simeq \mathbb{R}^2$.

(a) If $|\Sigma(1)| = 3$, prove that $X_\Sigma \simeq \mathbb{P}^2$.

(b) If $|\Sigma(1)| > 3$, prove that every primitive collection of $\Sigma(1)$ has two elements.

10.4.4. In the statement of Theorem 10.4.3, you might have noticed the absence of the Hirzebruch surface $\mathscr{H}_1$. Show that this surface is isomorphic to the blowup of $\mathbb{P}^2$ at one of its torus-fixed points. See Exercise 3.3.8 for more details.

10.4.5. This exercise studies several further examples of the intersection product on $\mathscr{H}_r$.

(a) Compute $D_1 \cdot D_1$ using (10.4.2) and also directly from Theorem 10.4.4.

(b) Compute $K^2 = K \cdot K$ on $\mathscr{H}_r$, where $K = K_{\mathscr{H}_r}$ is the canonical divisor.

10.4.6. Show that the matrix defined by (10.4.3) has a negative-definite associated quadratic form. Hint: Recall that if $B(x,y)$ is a bilinear form, the associated quadratic form is $Q(x) = B(x,x)$.

10.4.7. Let X have a unique singular point p and let $\phi : Y \to X$ be a resolution of singularities such that no component E of the exceptional fiber $\phi^{-1}(p)$ has $E \cdot E = -1$. In this exercise, you will show that Y is a minimal resolution of X according to Definition 10.4.8. Let $\psi : Z \to X$ be another resolution of singularities and consider the possibly singular surface $S = Z \times_X Y$. Let R be a resolution of S. Then we have a commutative diagram of morphisms

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & Y \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ Z & \xrightarrow[\psi]{} & X. \end{array}$$

(a) Explain why it suffices to show that β must be an isomorphism.

(b) If not, apply [**131**, V.5.3] to show that β factors as a sequence of blowups of points. Hence R must contain curves L with $L \cdot L = -1$ in the exceptional fiber over p.

(c) Let L be an irreducible curve on R with $L \cdot L = -1$ and show that $E = \alpha(L)$ satisfies $E \cdot E = -1$.

(d) Deduce that β is an isomorphism, hence $\phi : Y \to X$ is a minimal resolution.

10.4.8. This exercise deals with the proof of Proposition 10.4.11.

(a) Show that $K_{X_\Sigma} \cdot D_i = b_i - 2$ for $1 \leq i \leq r$.

(b) Show that $d/k = [[2,\dots,2]]$ if and only if $k = d-1$. Hint: Exercise 10.2.2.

(c) Show that $E = D_1 + \cdots + D_r$ satisfies $E \cdot E = -2$.

10.4.9. Let σ have parmeters $d, d-1$, so that the singular point of U_σ is a rational double point. By Proposition 10.1.6, U_σ is Gorenstein, so that its canonical sheaf ω_{U_σ} is a line bundle. Let $\phi : X_\Sigma \to U_\sigma$ be the resolution constructed in Theorem 10.2.3. Prove that $\phi^*\omega_{U_\sigma}$ is the canonical sheaf of X_Σ.

10.4.10. Another interesting numerical fact about the integers b_i from (10.4.1) is the following. Suppose a smooth fan Σ has 1-dimensional cones labeled as in Lemma 10.4.1. Then

$$b_0 + b_1 + \cdots + b_{r-1} = 3r - 12. \tag{10.4.4}$$

This exercise will sketch a proof of (10.4.4).

(a) Show that (10.4.4) holds for the standard fans of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathscr{H}_r$, $r \geq 2$.

(b) Show that if (10.4.4) holds for a smooth fan Σ, then it holds for the fan obtained by performing a star subdivision on one of the 2-dimensional cones of Σ.

(c) Deduce that (10.4.4) holds for all smooth fans using Theorem 10.4.3.

§10.5. Riemann-Roch and Lattice Polygons

Riemann-Roch theorems are a class of results about the dimensions of sheaf cohomology groups. The original statement along these lines was the theorem of

Riemann and Roch concerning sections of line bundles on algebraic curves. This result and its generalizations to higher-dimensional varieties can be formulated most conveniently in terms of the Euler characteristic of a sheaf, defined in §9.4.

Riemann-Roch for Curves. A modern form of the Riemann-Roch theorem for curves states that if D is a divisor on a smooth projective curve C, then

$$\chi(\mathscr{O}_C(D)) = \deg(D) + \chi(\mathscr{O}_C), \tag{10.5.1}$$

where the degree $\deg(D)$ is defined in Definition 6.3.2. This equality can be rewritten using Serre duality as follows. Namely, if K_C is a canonical divisor on C, then we have

$$H^1(C,\mathscr{O}_C(D)) \simeq H^0(C,\mathscr{O}_C(K_C - D))^\vee$$
$$H^1(C,\mathscr{O}_C) \simeq H^0(C,\mathscr{O}_C(K_C))^\vee.$$

The integer $g = \dim H^0(C,\mathscr{O}_C(K_C))$ is the *genus* of the curve C. Then (10.5.1) can be rewritten in the form commonly used in the theory of curves:

$$\dim H^0(C,\mathscr{O}_C(D)) - \dim H^0(C,\mathscr{O}_C(K_C - D)) = \deg(D) + 1 - g. \tag{10.5.2}$$

A proof of this theorem and a number of its applications are given in [**131**, Ch. IV]. Also see Exercise 10.5.1 below. As a first consequence, note that if $D = K_C$ is a canonical divisor, then

$$\deg(K_C) = 2g - 2. \tag{10.5.3}$$

We will need to use (10.5.2) most often in the simple case $X \simeq \mathbb{P}^1$. Then $g = 0$ and the Riemann-Roch theorem for $\mathbb{P}^1$ is the statement for all divisors D on $\mathbb{P}^1$,

$$\chi(\mathscr{O}_{\mathbb{P}^1}(D)) = \deg(D) + 1. \tag{10.5.4}$$

The Adjunction Formula. For a smooth curve C contained in a smooth surface X, the canonical sheaves ω_C of the curve and ω_X of the surface are related by

$$\omega_C \simeq \omega_X(C) \otimes_{\mathscr{O}_X} \mathscr{O}_C. \tag{10.5.5}$$

This follows without difficulty from Example 8.2.2 (Exercise 10.5.2) and has the following consequence for the intersection product on X.

Theorem 10.5.1 (Adjunction Formula)**.** *Let C be a smooth curve contained in a smooth complete surface X. Then*

$$K_X \cdot C + C \cdot C = 2g - 2,$$

where g is the genus of the curve C.

Proof. Let $i : C \hookrightarrow X$ be the inclusion map. Then

$$\omega_C \simeq \omega_X(C) \otimes_{\mathscr{O}_X} \mathscr{O}_C = i^*\omega_X(C) = i^*\mathscr{O}_X(K_X + C),$$

so that

$$2g - 2 = \deg(\omega_C) = \deg(i^*\mathscr{O}_X(K_X + C)) = (K_X + C) \cdot C,$$

where the first equality is (10.5.3) and the last is the definition of $(K_X+C)\cdot C$ given in §6.3. □

Riemann-Roch for Surfaces. The statement for surfaces corresponding to (10.5.1) is given next.

Theorem 10.5.2 (Riemann-Roch for Surfaces)**.** *Let D be a divisor on a smooth projective surface X with canonical divisor K_X. Then*

$$\chi(\mathscr{O}_X(D)) = \frac{D\cdot D - D\cdot K_X}{2} + \chi(\mathscr{O}_X).$$

We will only prove this for X a smooth complete toric surface; there is a simple and concrete proof in this case.

Proof. The theorem certainly holds for $D=0$ since $\mathscr{O}_X(D)=\mathscr{O}_X$ in this case. Our proof will use the special properties of smooth complete toric surfaces. Recall that if $X=X_\Sigma$, then $\mathrm{Pic}(X)$ is generated by the classes of the divisors D_i, $i=1,\dots,r$, corresponding to the 1-dimensional cones in Σ. Hence, to prove the theorem, it suffices to show that if the theorem holds for a divisor D, then it also holds for $D+D_i$ and $D-D_i$ for all i.

Assume the theorem holds for D. By Proposition 4.0.28, the sequence

$$0 \longrightarrow \mathscr{O}_X(-D_i) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{D_i} \longrightarrow 0$$

is exact. Tensoring this with $\mathscr{O}_X(D+D_i)$ gives the exact sequence

$$0 \longrightarrow \mathscr{O}_X(D) \longrightarrow \mathscr{O}_X(D+D_i) \longrightarrow \mathscr{O}_{D_i}(D+D_i) \longrightarrow 0.$$

By (9.4.1), it follows that

$$\chi(\mathscr{O}_X(D+D_i)) = \chi(\mathscr{O}_X(D)) + \chi(\mathscr{O}_{D_i}(D+D_i)).$$

By the induction hypothesis,

$$\chi(\mathscr{O}_X(D)) = \frac{D\cdot D - D\cdot K_X}{2} + \chi(\mathscr{O}_X). \tag{10.5.6}$$

For $\chi(\mathscr{O}_{D_i}(D+D_i))$, recall that $D_i \simeq \mathbb{P}^1$. Hence, by the Riemann-Roch theorem for $\mathbb{P}^1$ given in (10.5.4), we have

$$\chi(\mathscr{O}_{D_i}(D+D_i)) = D\cdot D_i + D_i\cdot D_i + 1. \tag{10.5.7}$$

Combining (10.5.6) and (10.5.7), we obtain

$$\begin{aligned}\chi(\mathscr{O}_X(D+D_i)) &= \frac{D\cdot D - D\cdot K_X}{2} + D\cdot D_i + D_i\cdot D_i + 1 + \chi(\mathscr{O}_X)\\ &= \frac{(D+D_i)\cdot(D+D_i) + D_i\cdot D_i - D\cdot K_X + 2}{2} + \chi(\mathscr{O}_X).\end{aligned}$$

However, using $K_X = -(D_1+\cdots+D_r)$ and Theorem 10.4.4, one computes that

$$D_i\cdot K_X = -D_i\cdot D_i - 2. \tag{10.5.8}$$

Substituting this into the above expression for $\chi(\mathscr{O}_X(D+D_i))$ and simplifying, we obtain

$$\chi(\mathscr{O}_X(D+D_i)) = \frac{(D+D_i)\cdot(D+D_i)-(D+D_i)\cdot K_X}{2} + \chi(\mathscr{O}_X),$$

which shows that the theorem holds for $D+D_i$.

The proof for $D-D_i$ is similar and is left to the reader (Exercise 10.5.3). □

The following statement is sometimes considered as the topological part of the Riemann-Roch theorem for surfaces.

Theorem 10.5.3 (Noether's Theorem)**.** *Let X be a smooth projective surface with canonical divisor K_X. Then*

$$\chi(\mathscr{O}_X) = \frac{K_X\cdot K_X + e(X)}{12},$$

where $e(X)$ is the topological Euler characteristic of X defined by

$$e(X) = \sum_{k=0}^{4}(-1)^k \dim H^k(X,\mathbb{C}).$$

As before, we will give a proof only for a smooth complete toric surface. We will also use the Hodge decomposition

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(X,\Omega_X^p)$$

from (9.4.11).

Proof. Demazure vanishing (Theorem 9.2.3) implies that for a smooth complete toric surface $X = X_\Sigma$,

$$\begin{aligned}\chi(\mathscr{O}_X) &= \dim H^0(X,\mathscr{O}_X) - \dim H^1(X,\mathscr{O}_X) + \dim H^2(X,\mathscr{O}_X)\\ &= 1-0+0 = 1.\end{aligned} \tag{10.5.9}$$

Thus Noether's theorem for a smooth complete toric surface is equivalent to

$$K_X\cdot K_X + e(X) = 12. \tag{10.5.10}$$

We prove this as follows. Set $r = |\Sigma(1)|$ and let the minimal generators of the rays be $u_0,\dots,u_{r-1}$ as in Lemma 10.4.1. Since $K_X = -\sum_{i=0}^{r-1} D_i$, (10.5.8) implies

$$K_X\cdot K_X = -\sum_{i=0}^{r-1} D_i\cdot K_X = -\sum_{i=0}^{r-1}(-D_i\cdot D_i - 2) = 2r + \sum_{i=0}^{r-1} D_i\cdot D_i.$$

If $u_{i-1}+u_{u+i} = b_i u_i$ as in (10.4.1), then $D_i\cdot D_i = -b_i$ by Theorem 10.4.4. Hence

$$K_X\cdot K_X = 2r - \sum_{i=0}^{r-1} b_i = 2r-(3r-12) = 12-r,$$

where the equality $\sum_{i=0}^{r-1} b_i = 3r-12$ is from Exercise 10.4.10.

We next compute $e(X)$. Proposition 6.3.25 implies that Σ is the normal fan of a polygon. Such a polygon clearly has $r = |\Sigma(1)|$ sides. Then the formula for $\dim H^q(X,\Omega_X^p)$ given in Theorem 9.4.11 implies

$$\begin{aligned}
\dim H^0(X,\mathbb{C}) &= \dim H^0(X,\mathscr{O}_X) = 1\\
\dim H^1(X,\mathbb{C}) &= \dim H^0(X,\Omega_X^1) + \dim H^1(X,\mathscr{O}_X) = 0+0 = 0\\
\dim H^2(X,\mathbb{C}) &= \dim H^0(X,\Omega_X^2) + \dim H^1(X,\Omega_X^1) + \dim H^2(X,\mathscr{O}_X)\\
&= 0 + f_1 - 2f_2 = r-2\\
\dim H^3(X,\mathbb{C}) &= \dim H^1(X,\Omega_X^2) + \dim H^2(X,\Omega_X^1) = 0+0 = 0\\
\dim H^4(X,\mathbb{C}) &= \dim H^2(X,\Omega_X^2) = 1.
\end{aligned}$$

where $f_1 = r$ and $f_2 = 1$ are the face numbers of a polygon with r sides. It follows that $e(X) = 1 + (r-2) + 1 = r$. Then (10.5.10) follows easily from the above computation of $K_X \cdot K_X$. □

We will give a topological proof of $e(X) = r$ in Chapter 12, and in Chapter 13, we will interpret $e(X)$ in terms of the Chern classes of the tangent bundle.

The Riemann-Roch theorems for curves and surfaces have been vastly generalized by results of Hirzebruch and Grothendieck, and the precise relation of Noether's theorem to the Riemann-Roch theorem for surfaces is a special case of their approach. We will discuss Riemann-Roch theorems for higher-dimensional toric varieties in Chapter 13.

Lattice Polygons. For the remainder of this section, we will explore the relation between toric surfaces and the geometry and combinatorics of lattice polygons. We will see that the results from §9.4 for lattice polytopes have an especially nice form for lattice polygons.

Let $X = X_\Sigma$ be a smooth complete toric surface. Since $\chi(\mathscr{O}_X) = 1$ by (10.5.9), Riemann-Roch for a divisor D on X becomes

$$\chi(\mathscr{O}_X(D)) = \frac{D\cdot D - D\cdot K_X}{2} + 1. \tag{10.5.11}$$

Thus, for any $\ell \in \mathbb{Z}$,

$$\chi(\mathscr{O}_X(\ell D)) = \frac{\ell D\cdot \ell D - \ell D\cdot K_X}{2} + 1 = \tfrac{1}{2}(D\cdot D)\,\ell^2 - \tfrac{1}{2}(D\cdot K_X)\,\ell + 1. \tag{10.5.12}$$

The theory developed in §9.4 guarantees that $\chi(\mathscr{O}_X(\ell D))$ is a polynomial in ℓ; the above computation gives explicit formulas for the coefficients in terms of intersection products.

Here is an example of how this formula works.

Example 10.5.4. Let X be the Hirzebruch surface $\mathscr{H}_2$. We will use the notation of Example 10.4.6. The divisor $D = D_1 + D_2$ is clearly effective, but the inequalities (9.3.6) defining the nef cone show that D is not nef. Using $K_X = -D_1 - \cdots - D_4$ and Example 10.4.6, one computes that

$$D \cdot D = 0, \quad D \cdot K_X = -2.$$

Then (10.5.12) implies that

$$\chi(\mathscr{O}_X(\ell D)) = \ell + 1.$$

An easy application of Serre duality gives $H^2(X, \mathscr{O}_X(\ell D)) = 0$ (Exercise 10.5.4). Thus

$$\dim H^0(X, \mathscr{O}_X(\ell D)) - \dim H^1(X, \mathscr{O}_X(\ell D)) = \ell + 1.$$

Things get more surprising when we compute $\dim H^0(X, \mathscr{O}_X(\ell D))$. Using the ray generators $u_1, \ldots, u_4$ from Example 10.4.6, the polygon P_D corresponding to $D = D_1 + D_2$ is defined by the inequalities

$$\langle m, u_1 \rangle \geq -1, \ \langle m, u_2 \rangle \geq -1, \ \langle m, u_3 \rangle \geq 0, \ \langle m, u_4 \rangle \geq 0.$$

The polygon P_D is shown in Figure 11.

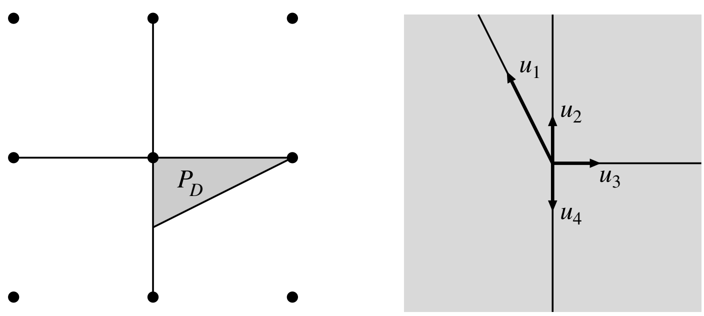

Figure 11. The polygon of the divisor D and the fan of $\mathscr{H}_2$

Even though P_D is *not* a lattice polytope, Proposition 4.3.3 still applies. Thus

$$\dim H^0(X, \mathscr{O}_X(\ell D)) = |P_{\ell D} \cap M| = |\ell P_D \cap M| = \begin{cases} \frac{1}{4}\ell^2 + \ell + 1 & \ell \text{ even} \\ \frac{1}{4}\ell^2 + \ell + \frac{3}{4} & \ell \text{ odd}, \end{cases}$$

where the final equality follows from Exercise 9.4.13. Combining this with the above computation of $\chi(\mathscr{O}_X(\ell D))$, we obtain

$$\dim H^1(X, \mathscr{O}_X(\ell D)) = \begin{cases} \frac{1}{4}\ell^2 & \ell \text{ even} \\ \frac{1}{4}\ell^2 - \frac{1}{4} & \ell \text{ odd}. \end{cases}$$

This is a vivid example how the Euler characteristic smooths out the complicated behavior of the individual cohomology groups. ◊

On the other hand, if D is nef, the higher cohomology is trivial by Demazure vanishing, so that the Euler characteristic reduces to $\dim H^0$. We exploit this as follows. Suppose that $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^2$ is a lattice polygon. By Theorem 9.4.2 and Example 9.4.4, the Ehrhart polynomial $\mathrm{Ehr}_P(x) \in \mathbb{Q}[x]$ of P satisfies

$$\mathrm{Ehr}_P(\ell) = |(\ell P)\cap M| = \mathrm{Area}(P)\,\ell^2 + \tfrac{1}{2}|\partial P \cap M|\,\ell + 1 \tag{10.5.13}$$

for $\ell \in \mathbb{N}$. We next describe this polynomial in terms of intersection products.

By the results of §2.3, we get the projective toric surface X_P coming from the normal fan Σ_P of P. In general X_P will not be smooth, so we compute a minimal resolution of singularities

$$\phi : X_\Sigma \longrightarrow X_P$$

using the methods of this chapter. Recall that X_P has the ample divisor D_P whose associated polygon is P.

Proposition 10.5.5. *There is a unique torus-invariant nef divisor D on X_Σ such that:*

(a) *The support function of D equals the support function of D_P.*

(b) *$\chi(\mathscr{O}_{X_\Sigma}(\ell D))$ is the Ehrhart polynomial of P.*

Proof. Proposition 6.2.7 implies that X_Σ has a divisor D that satisfies part (a). As in §6.1, we call D the *pullback* of D_P. Since D_P has a convex support function, the same is true for D, so that D is nef. Furthermore, P is the polytope associated to D_P and hence is the polytope associated to D since the polytope of a nef divisor is determined by its support function (Theorem 6.1.7).

It follows that $\dim H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D)) = |P_{\ell D} \cap M| = |(\ell P)\cap M|$ when $\ell \geq 0$, so that $H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D))$ equals the Ehrhart polynomial of P when $\ell \geq 0$. However, ℓD is nef when $\ell \geq 0$ and hence has trivial higher cohomology by Demazure vanishing (Theorem 9.2.3). Thus $\ell \geq 0$ implies

$$\chi(\mathscr{O}_{X_\Sigma}(\ell D)) = \dim H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D)) = |(\ell P)\cap M|.$$

Since $\chi(\mathscr{O}_{X_\Sigma}(\ell D))$ is a polynomial in ℓ, it must be the Ehrhart polynomial of P. □

Proposition 10.5.5 and (10.5.12) imply that the Ehrhart polynomial of P is

$$\mathrm{Ehr}_P(\ell) = \tfrac{1}{2}(D\cdot D)\,\ell^2 - \tfrac{1}{2}(D\cdot K_{X_\Sigma})\,\ell + 1.$$

Comparing this to the formula (10.5.13) for $\mathrm{Ehr}_P(\ell)$, we get the following result.

Proposition 10.5.6. *Let P be a lattice polygon and let D be the pullback of D_P constructed in Proposition 10.5.5. Then*

$$D\cdot D = 2\,\mathrm{Area}(P)$$
$$-D\cdot K_{X_\Sigma} = |\partial P \cap M|.$$

□

Example 10.5.7. Take the fan of $\mathscr{H}_2$ shown in Figure 11 from Example 10.5.4 and combine the two 2-dimensional cones containing u_2 into a single cone. The resulting fan has minimal generators u_1, u_3, u_4 that satisfy $u_1 + u_3 + 2u_4 = 0$, so the resulting toric variety is $\mathbb{P}(1,1,2)$.

Let $P = \text{Conv}(0, 2e_1, -e_2) \subseteq M_{\mathbb{R}}$, which is the double of the polytope shown in Figure 11. The normal fan of P is the fan of $\mathbb{P}(1,1,2)$. The minimal generators u_1, u_3, u_4 of this fan give divisors D'_1, D'_3, D'_4 on $\mathbb{P}(1,1,2)$, and the divisor D_P is easily seen to be the ample divisor $2D'_1$.

Since the fan of $\mathscr{H}_2$ refines the fan of $\mathbb{P}(1,1,2)$, the resulting toric morphism $\mathscr{H}_2 \to \mathbb{P}(1,1,2)$ is a resolution of singularities. By considering the support function of D_P, we find that the pullback of $D_P = 2D'_1$ is $D = 2D_1 + 2D_2$. We leave it as Exercise 10.5.5 to compute $D \cdot D$ and $D \cdot K_{\mathscr{H}_2}$ and verify that they give the numbers predicted by Proposition 10.5.6. ◊

Sectional Genus. The divisor D_P on X_P is very ample since $\dim P = 2$. Hence it gives a projective embedding $X_P \hookrightarrow \mathbb{P}^s$ such that $\mathscr{O}_{\mathbb{P}^s}(1)$ restricts to $\mathscr{O}_{X_\Sigma}(D_P)$. In geometric terms, this means that hyperplanes $H \subseteq \mathbb{P}^s$ give curves $X_P \cap H \subseteq X_P$ that are linearly equivalent to D_P. For some hyperplanes, the intersection $X_P \cap H$ can be complicated. Since X_P has only finitely many singular points, the *Bertini theorem* (see [**131**, II 8.18 and III 7.9.1]) guarantees that when H is generic, $C = X_P \cap H$ is a smooth connected curve contained in the smooth locus of X_P. The genus g of C is called the *sectional genus* of the surface X_P.

We will compute g in terms of the geometry of P using the adjunction formula. Since we need a smooth surface for this, we use a resolution $\phi : X_\Sigma \to X_P$ and note that C can be regarded as a curve in X_Σ since ϕ is an isomorphism away from the singular points of X_P. Since $C \sim D_P$ on X_P, we have $C \sim D$ on X_Σ, where D is the pullback of D_P. Then the adjunction formula (Theorem 10.5.1) implies

$$2g - 2 = K_{X_\Sigma} \cdot C + C \cdot C = K_{X_\Sigma} \cdot D + D \cdot D,$$

so that

$$g = \tfrac{1}{2} D \cdot (K_{X_\Sigma} + D) + 1. \tag{10.5.14}$$

Then we have the following result.

Proposition 10.5.8. *The sectional genus of X_P is $g = |\text{Int}(P) \cap M|$.*

Proof. Pick's formula from Example 9.4.4 can be written as

$$|\text{Int}(P) \cap M| = \text{Area}(P) - \tfrac{1}{2}|\partial P \cap M| + 1,$$

which by Proposition 10.5.6 becomes

$$|\text{Int}(P) \cap M| = \tfrac{1}{2} D \cdot D + \tfrac{1}{2} D \cdot K_{X_\Sigma} + 1.$$

The right-hand side is g by (10.5.14), completing the proof. □

Example 10.5.9. Let $P = d\Delta_2 = \mathrm{Conv}(0, de_1, de_2)$. Then X_P is the projective plane $\mathbb{P}^2$ in its dth Veronese embedding, and $D_P \sim dL$, where $L \subseteq \mathbb{P}^2$ is a line. The hyperplane sections are the curves of degree d in $\mathbb{P}^2$, and the smooth ones have genus

$$g = |\mathrm{Int}(d\Delta_2) \cap M| = \frac{(d-1)(d-2)}{2}.$$

You will check this assertion and another example in Exercise 10.5.6. $\Diamond$

The curves $C \subseteq X_\Sigma$ studied here can be generalized to the study of hypersurfaces in projective toric varieties coming from sections of a nef line bundle. The geometry and topology of these hypersurfaces have been studied in many papers, including [**15**], [**19**], [**77**] and [**198**].

Reflexive Polygons and The Number 12. Our final topic gives a way to understand a somewhat mysterious formula we noted in the last section of Chapter 8. Recall from Theorem 8.3.7 that there are exactly 16 equivalence classes of reflexive lattice polytopes in $\mathbb{R}^2$, shown in Figure 3 of §8.3. The article [**231**] gives four different proofs of the following result.

Theorem 10.5.10. *Let P be a reflexive lattice polygon in $M_\mathbb{R} \simeq \mathbb{R}^2$. Then*

$$|\partial P \cap M| + |\partial P^\circ \cap N| = 12.$$

One proof consists of a case-by-case verification of the statement for each of 16 equivalence classes. You proved the theorem this way in Exercise 8.3.5. The argument was straightforward but not very enlightening! Here we will give another proof using Noether's theorem.

Proof. Since Noether's theorem requires a smooth surface, we need to refine the normal fan Σ_P of $P \subseteq M_\mathbb{R}$. Since P is reflexive, we can do this using the dual polygon $P^\circ \subseteq N_\mathbb{R}$. We know from (8.3.2) that the vertices of P° are the minimal generators of Σ_P. Let Σ be the refinement of Σ_P whose 1-dimensional cones are generated by the rays through the lattice points on the boundary of P°. This is illustrated in Figure 12.

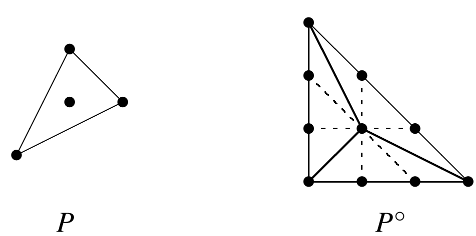

Figure 12. A reflexive polygon P and its dual P°

The fan Σ has the following properties:

- For each cone of Σ, its minimal generators and the origin form a triangle whose only lattice points are the vertices. Thus Σ is smooth by Exercise 8.3.4.
- The minimal generators of Σ are the lattice points of P° lying on the boundary. Thus $|\Sigma(1)| = |\partial P^\circ \cap N|$.

From the first bullet, we get a resolution $\phi : X_\Sigma \to X_P$. Recall that $D_P = -K_{X_P}$ since P is reflexive. The wonderful fact is that its pullback via ϕ is again anticanonical, i.e., $D = -K_{X_\Sigma}$. To prove this, recall that D and D_P have the same support function φ, which takes the value 1 at the vertices of P° since $D_P = -K_{X_P}$. It follows that $\varphi = 1$ on the boundary of P°. Then $D = -K_{X_\Sigma}$ because the minimal generators of Σ all lie on the boundary.

Now apply Noether's theorem to the toric surface X_Σ. By (10.5.10), we have

$$K_{X_\Sigma} \cdot K_{X_\Sigma} + e(X_\Sigma) = 12.$$

We analyze each term on the left as follows. First, $D = -K_{X_\Sigma}$ implies

$$K_{X_\Sigma} \cdot K_{X_\Sigma} = -D \cdot K_{X_\Sigma} = |\partial P \cap M|,$$

where the last equality follows from Proposition 10.5.6. Second, $e(X_\Sigma)$ is the number of minimal generators of Σ by the proof of Theorem 10.5.3. In other words,

$$e(X_\Sigma) = |\Sigma(1)| = |\partial P^\circ \cap N|,$$

where the second equality follows from the above analysis of Σ. Hence the theorem is an immediate consequence of Noether's theorem. □

A key step in the above proof was showing that the pullback of the canonical divisor on X_P was the canonical divisor on X_P. This may fail for a general resolution of singularities. We will say more about this when we study *crepant resolutions* in Chapter 11.

Exercises for §10.5.

10.5.1. The Riemann-Roch theorem for curves, in the form (10.5.1), can be proved by much the same method as used in the proof of Theorem 10.5.2. Namely, show that if (10.5.1) holds for a divisor D then it also holds for the divisors $D+P$ and $D-P$, where P is an arbitrary point on the curve.

10.5.2. Prove the adjunction formula (Theorem 10.5.1) using (10.5.3) and (10.5.5).

10.5.3. Complete the proof of Theorem 10.5.2 by showing that if the theorem holds for D, then it also holds for $D - D_i$ where D_i is any one of the divisors corresponding to the 1-dimensional cones in Σ.

10.5.4. Let $D = \sum_\rho a_\rho D_\rho$ be an effective $\mathbb{Q}$-Cartier Weil divisor on a complete toric variety X_Σ of dimension n. Use Serre duality (Theorem 9.2.10) to prove that $H^n(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = 0$.

10.5.5. In Example 10.5.7, compute $D \cdot D$ and $D \cdot K_{\mathscr{H}_2}$ and check that they agree with the numbers given by Proposition 10.5.6.

10.5.6. This exercise studies the sectional genus of toric surfaces.

(a) Verify the formula given in Example 10.5.9 for the sectional genus of $\mathbb{P}^2$ in its dth Veronese embedding.

(b) Let $P = \mathrm{Conv}(0, ae_1, be_2, ae_1 + be_2)$. What is the smooth toric surface X_P in this case? Show that its sectional genus is $(a-1)(b-1)$.

10.5.7. Let P be a reflexive polygon.

(a) Prove that the singularities (if any) of the toric surface X_P are rational double points. Hint: Proposition 10.1.6.

(b) Prove that X_P has sectional genus $g = 1$. This means that smooth anticanonical curves in X_P are all elliptic curves.

(c) Explain how part (b) relates to Exercise 10.5.6.

10.5.8. According to Theorem 10.4.3, every smooth toric surface is a blowup of either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathscr{H}_r$ for $r \geq 2$. For each of the 16 reflexive polygons in Figure 3 of §8.3, the process described in the proof of Theorem 10.5.10 produces a smooth toric surface X_Σ. Where does X_Σ fit in this classification in each case? (This gives a classification of smooth toric Del Pezzo surfaces.)

10.5.9. As in §9.3, the *p-Ehrhart polynomials* of a lattice polygon $P \subset M_\mathbb{R}$ are defined by

$$\mathrm{Ehr}_P^p(\ell) = \chi(\widehat{\Omega}^p_{X_P}(\ell D_P)), \quad p = 0, 1, 2.$$

We know that Ehr_P^0 is the usual Ehrhart polynomial Ehr_P, and then $\mathrm{Ehr}_P^2(x) = \mathrm{Ehr}_P(-x)$ by Theorem 9.4.7. The remaining case is Ehr_P^1. Prove that

$$\mathrm{Ehr}_P^1(x) = 2\mathrm{Area}(P)\, x^2 + f_1 - 2,$$

where f_1 is the number of edges of P. Hint: Use Theorem 9.4.11 for the constant term and part (c) of Theorem 9.4.7 for the coefficient of x. For the leading coefficient, tensor the exact sequence of Theorem 8.1.6 with $\mathscr{O}_{X_P}(\ell D_P)$, take the Euler characteristic, and then let $\ell \to \infty$.