# Preface

In the fall of 2010, I taught an introductory one-quarter course on graduate real analysis, focusing in particular on the basics of measure and integration theory, both in Euclidean spaces and in abstract measure spaces. This text is based on my lecture notes of that course, which are also available online on my blog `terrytao.wordpress.com`, together with some supplementary material, such as a section on problem solving strategies in real analysis (Section 2.1) which evolved from discussions with my students.

This text is intended to form a prequel to my graduate text [**Ta2010**] (henceforth referred to as *An epsilon of room, Vol. I*), which is an introduction to the analysis of Hilbert and Banach spaces (such as $L^p$ and Sobolev spaces), point-set topology, and related topics such as Fourier analysis and the theory of distributions; together, they serve as a text for a complete first-year graduate course in real analysis.

The approach to measure theory here is inspired by the text [**StSk2005**], which was used as a secondary text in my course. In particular, the first half of the course is devoted almost exclusively to measure theory on Euclidean spaces $\mathbf{R}^d$ (starting with the more elementary Jordan-Riemann-Darboux theory, and only then moving on to the more sophisticated Lebesgue theory), deferring the abstract aspects of measure theory to the second half of the course. I found that this approach strengthened the student's intuition in the early stages of the course, and helped provide motivation for more abstract constructions, such as Carathéodory's general construction of a measure from an outer measure.

Most of the material here is self-contained, assuming only an undergraduate knowledge in real analysis (and in particular, on the Heine-Borel theorem, which we will use as the foundation for our construction of Lebesgue measure); a secondary real analysis text can be used in conjunction with this one, but it is not strictly necessary. A small number of exercises, however, will require some knowledge of point-set topology or of set-theoretic concepts such as cardinals and ordinals.

A large number of exercises are interspersed throughout the text, and it is intended that the reader perform a significant fraction of these exercises while going through the text. Indeed, many of the key results and examples in the subject will in fact be presented through the exercises. In my own course, I used the exercises as the basis for the examination questions, and indicated this well in advance, to encourage the students to attempt as many of the exercises as they could as preparation for the exams.

The core material is contained in Chapter 1, and already comprises a full quarter's worth of material. Section 2.1 is a much more informal section than the rest of the book, focusing on describing problem solving strategies, either specific to real analysis exercises, or more generally, applicable to a wider set of mathematical problems; this section evolved from various discussions with students throughout the course. The remaining three sections in Chapter 2 are optional topics, which require understanding most of the material in Chapter 1 as a prerequisite (although Section 2.3 can be read after completing Section 1.4).

## Notation

For reasons of space, we will not be able to define every single mathematical term that we use in this book. If a term is italicised for reasons other than emphasis or for definition, then it denotes a standard mathematical object, result, or concept, which can be easily looked up in any number of references. (In the blog version of the book, many of these terms were linked to their Wikipedia pages, or other on-line reference pages.)

Given a subset $E$ of a space $X$, the *indicator function* $1_E : X \to \mathbf{R}$ is defined by setting $1_E(x)$ equal to 1 for $x \in E$ and equal to 0 for $x \notin E$.

For any natural number $d$, we refer to the vector space

$$\mathbf{R}^d := \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \mathbf{R}\}$$

as *(d-dimensional) Euclidean space.* A vector $(x_1, \ldots, x_d)$ in $\mathbf{R}^d$ has length

$$|(x_1, \ldots, x_d)| := (x_1^2 + \ldots + x_d^2)^{1/2}$$

and two vectors $(x_1, \ldots, x_d), (y_1, \ldots, y_d)$ have *dot product*

$$(x_1, \ldots, x_d) \cdot (y_1, \ldots, y_d) := x_1 y_1 + \ldots + x_d y_d.$$

The *extended non-negative real axis* $[0, +\infty]$ is the non-negative real axis $[0, +\infty) := \{x \in \mathbf{R} : x \geq 0\}$ with an additional element adjointed to it, which we label $+\infty$; we will need to work with this system because many sets (e.g. $\mathbf{R}^d$) will have infinite measure. Of course, $+\infty$ is not a real number, but we think of it as an *extended* real number. We extend the addition, multiplication, and order structures on $[0, +\infty)$ to $[0, +\infty]$ by declaring

$$+\infty + x = x + +\infty = +\infty$$

for all $x \in [0, +\infty]$,

$$+\infty \cdot x = x \cdot +\infty = +\infty$$

for all non-zero $x \in (0, +\infty]$,

$$+\infty \cdot 0 = 0 \cdot +\infty = 0,$$

and

$$x < +\infty \text{ for all } x \in [0, +\infty).$$

Most of the laws of algebra for addition, multiplication, and order continue to hold in this extended number system; for instance, addition and multiplication are commutative and associative, with the latter distributing over the former, and an order relation $x \leq y$ is preserved under addition or multiplication of both sides of that relation by the same quantity. However, we caution that the laws of cancellation do *not* apply once some of the variables are allowed to be infinite; for instance, we cannot deduce $x = y$ from $+\infty + x = +\infty + y$ or from $+\infty \cdot x = +\infty \cdot y$. This is related to the fact that the forms $+\infty - +\infty$ and $+\infty / + \infty$ are indeterminate (one cannot assign a value to them without breaking many of the rules of algebra). A general rule of thumb is that if one wishes to use cancellation (or proxies for cancellation, such as subtraction or division), this is only safe if one can guarantee that all quantities involved are finite (and in the case of multiplicative cancellation, the quantity being cancelled also needs to be non-zero, of course). However, as long as one avoids using cancellation and works exclusively with non-negative quantities, there is little danger in working in the extended real number system.

We note also that once one adopts the convention $+\infty \cdot 0 = 0 \cdot +\infty = 0$, then multiplication becomes *upward continuous* (in the sense that whenever $x_n \in [0, +\infty]$ increases to $x \in [0, +\infty]$, and $y_n \in [0, +\infty]$ increases to $y \in [0, +\infty]$, then $x_n y_n$ increases to $xy$) but not *downward continuous* (e.g. $1/n \to 0$ but $1/n \cdot +\infty \not\to 0 \cdot +\infty$). This asymmetry will ultimately cause us to define integration from below rather than from above, which leads to other asymmetries (e.g. the monotone convergence theorem (Theorem 1.4.43) applies for monotone increasing functions, but not necessarily for monotone decreasing ones).

**Remark 0.0.1.** Note that there is a tradeoff here: if one wants to keep as many useful laws of algebra as one can, then one can add in infinity, or have negative numbers, but it is difficult to have both at the same time. Because of this tradeoff, we will see two overlapping types of measure and integration theory: the *non-negative* theory, which involves quantities taking values in $[0, +\infty]$, and the *absolutely integrable* theory, which involves quantities taking values in $(-\infty, +\infty)$ or $\mathbf{C}$. For instance, the fundamental convergence theorem for the former theory is the monotone convergence theorem (Theorem 1.4.43), while the fundamental convergence theorem for the latter is the dominated convergence theorem (Theorem 1.4.48). Both branches of the theory are important, and both will be covered in later notes.

One important feature of the extended non-negative real axis is that all sums are convergent: given any sequence $x_1, x_2, \ldots \in [0, +\infty]$, we can always form the sum

$$\sum_{n=1}^{\infty} x_n \in [0, +\infty]$$

as the limit of the partial sums $\sum_{n=1}^{N} x_n$, which may be either finite or infinite. An equivalent definition of this infinite sum is as the supremum of all finite subsums:

$$\sum_{n=1}^{\infty} x_n = \sup_{F \subset \mathbf{N}, F \text{ finite}} \sum_{n \in F} x_n.$$

Motivated by this, given any collection $(x_\alpha)_{\alpha \in A}$ of numbers $x_\alpha \in [0, +\infty]$ indexed by an arbitrary set $A$ (finite or infinite, countable or uncountable), we can define the sum $\sum_{\alpha \in A} x_\alpha$ by the formula

$$\sum_{\alpha \in A} x_\alpha = \sup_{F \subset A, F \text{ finite}} \sum_{\alpha \in F} x_\alpha. \tag{0.1}$$

Note from this definition that one can relabel the collection in an arbitrary fashion without affecting the sum; more precisely, given any bijection $\phi : B \to A$, one has the change of variables formula

$$\sum_{\alpha \in A} x_\alpha = \sum_{\beta \in B} x_{\phi(\beta)}. \tag{0.2}$$

Note that when dealing with signed sums, the above rearrangement identity can fail when the series is not absolutely convergent (cf. the *Riemann rearrangement theorem*).

**Exercise 0.0.1.** If $(x_\alpha)_{\alpha \in A}$ is a collection of numbers $x_\alpha \in [0, +\infty]$ such that $\sum_{\alpha \in A} x_\alpha < \infty$, show that $x_\alpha = 0$ for all but at most countably many $\alpha \in A$, even if $A$ itself is uncountable.

We will rely frequently on the following basic fact (a special case of the *Fubini-Tonelli theorem*, Corollary 1.7.23):

**Theorem 0.0.2** (Tonelli's theorem for series)**.** *Let $(x_{n,m})_{n,m\in\mathbf{N}}$ be a doubly infinite sequence of extended non-negative reals $x_{n,m} \in [0,+\infty]$. Then*

$$\sum_{(n,m)\in\mathbf{N}^2} x_{n,m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{n,m}.$$

Informally, Tonelli's theorem asserts that we may rearrange infinite series with impunity as long as all summands are non-negative.

**Proof.** We shall just show the equality of the first and second expressions; the equality of the first and third is proven similarly.

We first show that

$$\sum_{(n,m)\in\mathbf{N}^2} x_{n,m} \leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m}.$$

Let $F$ be any finite subset of $\mathbf{N}^2$. Then $F \subset \{1,\ldots,N\} \times \{1,\ldots,N\}$ for some finite $N$, and thus (by the non-negativity of the $x_{n,m}$)

$$\sum_{(n,m)\in F} x_{n,m} \leq \sum_{(n,m)\in\{1,\ldots,N\}\times\{1,\ldots,N\}} x_{n,m}.$$

The right-hand side can be rearranged as

$$\sum_{n=1}^{N}\sum_{m=1}^{N} x_{n,m},$$

which is clearly at most $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m}$ (again by non-negativity of $x_{n,m}$). This gives

$$\sum_{(n,m)\in F} x_{n,m} \leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m},$$

for any finite subset $F$ of $\mathbf{N}^2$, and the claim then follows from (0.1).

It remains to show the reverse inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m} \leq \sum_{(n,m)\in\mathbf{N}^2} x_{n,m}.$$

It suffices to show that

$$\sum_{n=1}^{N}\sum_{m=1}^{\infty} x_{n,m} \leq \sum_{(n,m)\in\mathbf{N}^2} x_{n,m}$$

for each finite $N$.

Fix $N$. As each $\sum_{m=1}^\infty x_{n,m}$ is the limit of $\sum_{m=1}^M x_{n,m}$, the left-hand side is the limit of $\sum_{n=1}^N \sum_{m=1}^M x_{n,m}$ as $M \to \infty$. Thus it suffices to show that

$$\sum_{n=1}^N \sum_{m=1}^M x_{n,m} \leq \sum_{(n,m)\in \mathbf{N}^2} x_{n,m}$$

for each finite $M$. But the left-hand side is $\sum_{(n,m)\in\{1,\ldots,N\}\times\{1,\ldots,M\}} x_{n,m}$, and the claim follows. □

**Remark 0.0.3.** Note how important it was that the $x_{n,m}$ were non-negative in the above argument. In the signed case, one needs an additional assumption of absolute summability of $x_{n,m}$ on $\mathbf{N}^2$ before one is permitted to interchange sums; this is *Fubini's theorem for series*, which we will encounter later in this text. Without absolute summability or non-negativity hypotheses, the theorem can fail (consider, for instance, the case when $x_{n,m}$ equals $+1$ when $n = m$, $-1$ when $n = m + 1$, and $0$ otherwise).

**Exercise 0.0.2** (Tonelli's theorem for series over arbitrary sets)**.** Let $A, B$ be sets (possibly infinite or uncountable), and $(x_{n,m})_{n\in A, m\in B}$ be a doubly infinite sequence of extended non-negative reals $x_{n,m} \in [0, +\infty]$ indexed by $A$ and $B$. Show that

$$\sum_{(n,m)\in A\times B} x_{n,m} = \sum_{n\in A}\sum_{m\in B} x_{n,m} = \sum_{m\in B}\sum_{n\in A} x_{n,m}.$$

(*Hint:* Although not strictly necessary, you may find it convenient to first establish the fact that if $\sum_{n\in A} x_n$ is finite, then $x_n$ is non-zero for at most countably many $n$.)

Next, we recall the *axiom of choice*, which we shall be assuming throughout the text:

**Axiom 0.0.4** (Axiom of choice)**.** *Let $(E_\alpha)_{\alpha\in A}$ be a family of non-empty sets $E_\alpha$, indexed by an index set $A$. Then we can find a family $(x_\alpha)_{\alpha\in A}$ of elements $x_\alpha$ of $E_\alpha$, indexed by the same set $A$.*

This axiom is trivial when $A$ is a singleton set, and from mathematical induction one can also prove it without difficulty when $A$ is finite. However, when $A$ is infinite, one cannot deduce this axiom from the other axioms of set theory, but must explicitly add it to the list of axioms. We isolate the countable case as a particularly useful corollary (though one which is strictly weaker than the full axiom of choice):

**Corollary 0.0.5** (Axiom of countable choice)**.** *Let $E_1, E_2, E_3, \ldots$ be a sequence of non-empty sets. Then one can find a sequence $x_1, x_2, \ldots$ such that $x_n \in E_n$ for all $n = 1, 2, 3, \ldots$.*

**Remark 0.0.6.** The question of how much of real analysis still survives when one is not permitted to use the axiom of choice is a delicate one, involving a fair amount of logic and descriptive set theory to answer. We will not discuss these matters in this text. We will, however, note a theorem of Gödel [**Go1938**] that states that any statement that can be phrased in the first-order language of *Peano arithmetic*, and which is proven with the axiom of choice, can also be proven without the axiom of choice. So, roughly speaking, Gödel's theorem tells us that for any "finitary" application of real analysis (which includes most of the "practical" applications of the subject), it is safe to use the axiom of choice; it is only when asking questions about "infinitary" objects that are beyond the scope of Peano arithmetic that one can encounter statements that are provable using the axiom of choice, but are not provable without it.

## Acknowledgments

This text was strongly influenced by the real analysis text of Stein and Shakarchi [**StSk2005**], which was used as a secondary text when teaching the course on which these notes were based. In particular, the strategy of focusing first on Lebesgue measure and Lebesgue integration, before moving onwards to abstract measure and integration theory, was directly inspired by the treatment in [**StSk2005**], and the material on differentiation theorems also closely follows that in [**StSk2005**]. On the other hand, our discussion here differs from that in [**StSk2005**] in other respects; for instance, a far greater emphasis is placed on Jordan measure and the Riemann integral as being an elementary precursor to Lebesgue measure and the Lebesgue integral.

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`terrytao.wordpress.com/category/teaching/245a-real-analysis`

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