Preface

The subject of topology can be described as the study of the category **Top** of all topological spaces and the continuous maps between them. But many topological problems, and their solutions, do not change if the maps involved are replaced with 'continuous deformations' of themselves. The equivalence relation—called *homotopy*—generated by continuous deformations of maps respects composition, so that there is a 'quotient' *homotopy category* H**Top** and a functor **Top** → H**Top**. Homotopy theory is the study of this functor. Thus homotopy theory is not entirely confined to the category H**Top**: it is frequently necessary, or at least useful, to use constructions available only in **Top** in order to prove statements that are entirely internal to H**Top**; and the homotopy category H**Top** can shed light even on questions in **Top** that are not homotopy invariant.

History. The core of the subject I'm calling 'classical homotopy theory' is a body of ideas and theorems that emerged in the 1950s and was later largely codified in the notion of a model category. This includes the notions of fibration and cofibration, CW complexes, long fiber and cofiber sequences, loop space, suspension, and so on. Brown's representability theorems show that homology and cohomology are also contained in classical homotopy theory.

One of the main complications in homotopy theory is that many, if not most, diagrams in the category H**Top** do not have limits or colimits. Thus many theorems were proved using occasionally ingenious and generally *ad hoc* constructions performed in the category **Top**. Eventually many of these constructions were codified in the dual concepts of homotopy colimit and homotopy limit, and a powerful calculus for working with them was developed. The language of homotopy limits and colimits and the techniques for manipulating them made it possible to easily state and conceptually prove many results that had previously seemed quite difficult and inscrutable.

Once the basic theory has been laid down, the most interesting and useful theorems are those that break the categorical barrier between domain and target. The basic example of such a theorem is the Blakers-Massey theorem, which compares homotopy pushout squares to homotopy pullback squares. Other excellent examples of duality-breaking theorems are the Hilton-Milnor theorem on the loop space of a wedge and Ganea's theorem (which is dual to the most important special case of the Blakers-Massey theorem). All of these results were first proved with a great deal of technical finesse but can now be established easily using homotopy pushouts and pullbacks.

The Aim of This Book. The aim of this book is to develop classical homotopy theory and some important developments that flow from it using the more modern techniques of homotopy limits and colimits. Thus homotopy pushouts and homotopy pullbacks play a central role.

The book has been written with the theory of model categories firmly in mind. As is probably already evident, we make consistent and unapologetic use of the language of categories, functors, limits and colimits. But we are genuinely interested in the homotopy theory of *spaces* so, with the exception of a brief account of the abstract theory of model categories, we work with spaces throughout and happily make use of results that are special for spaces. Indeed, the third part of the book is devoted to the development of four basic properties that set the category of spaces apart from generic model categories.

I have generally used topological or homotopy-theoretical arguments rather than algebraic ones. This almost always leads to simpler statements and simpler arguments. Thus my book attempts to upset the balance (observed in many algebraic topology texts) between algebra and topology, in favor of topology. Algebra is just one of many tools by which we understand topology. This is *not* an anti-algebra crusade. Rather, I set out hoping to find homotopy-theoretical arguments wherever possible, with the expectation that at certain points, the simplicity or clarity afforded by the standard algebraic approach would outweigh the philosophical cleanliness of avoiding it. But I ended up being surprised: at no point did I find that 'extra' algebra made any contribution to clarity or simplicity.

Omissions. This is a very long book, and many topics that were in my earliest plans have had to be (regretfully) left out. I had planned three chapters on stable homotopy, extraordinary cohomology and nilpotence and another on Goodwillie calculus. But in the book that emerged it seemed thematically appropriate to draw the line at stable homotopy theory, so space and thematic consistency drove these chapters to the cutting room floor.

Problems and Exercises. Many authors of textbooks assert that the only way to learn the subject is to do the exercises. I have taken this to heart, and so *there are no outright proofs in the book.* Instead, theorems are followed by multi-part problems that guide the readers to find the proofs for themselves. To the expert, these problems will read as terse proofs, perhaps suitable for exposition in a journal article. Reading this text, then, is a preparation for the experience of reading research articles. There are also a great many other problems incorporated into the main flow of the text, problems that develop interesting tangential results, explore applications, or carry out explicit calculations.

In addition, there are numerous exercises. These are intended to help the student develop some habits of mind that are extremely useful when reading mathematics. After definitions, the reader is asked to find examples and nonexamples, to explore how the new concept fits in with previous ideas, etc. Other exercises ask the reader to compare theorems with previous results, to test whether hypotheses are needed, or can be weakened, and so on.

Audience. This book was written with the idea that it would be used by students in their first year or two of graduate school. It is assumed that the reader is familiar with basic algebraic concepts such as groups and rings. It is also assumed that the student has had an introductory course in topology. It would be nice if that course included some mention of the fundamental group, but that is not necessary.

Teaching from This Book. This book covers more topics, in greater depth, than can be covered in detail in a typical two-semester homotopy theory or algebraic topology sequence. That being said, a good goal for a two-semester course would be to cover the high points of Parts 1 – 4 in the first semester and Parts 5 – 6 in the second semester, followed by some or all of Part 7 if time permits.

Here's some more detail.

The first semester would start with a brief (one day) introduction to the language of category theory before heading on to Part 2 to develop the basic theory of cofibrations, fibrations, and homotopy limits and colimits. Part 1 is an overview of the basics of category theory and shouldn't be covered in its own right at all; refer back to it as needed to bring in more advanced category-theoretical topics. Chapters 3 and 4, in which the category of spaces is established and the concept of homotopy is developed should be covered fairly thoroughly. Chapter 5 is on cofibrations and fibrations. The basic properties should be explored, and the mapping cylinder and its dual should be studied carefully; it's probably best to gloss over the distinction between the pointed and unpointed cases. State the Fundamental Lifting Property and the basic factorization theorems without belaboring their proofs. The fact that fiber and cofiber sequences lead to exact sequences of homotopy sets should be explored in detail. Chapter 6 is on homotopy colimits and limits. Cover homotopy pushouts in detail, appealing to duality for homotopy pullbacks, and give a brief discussion of the issues for more general diagrams. Chapter 7 is on homotopy pullback and pushout squares and should be covered in some detail. Chapters 8 and 9 offer a huge collection of topics. For the moment, only Section 8.1 (Long cofiber and fiber sequences) and perhaps Section 9.2 (on H-Spaces and co-H-spaces) are really mandatory. Other sections can be covered as needed or assigned to students as homework. Chapter 10 is a brief account of abstract model categories. It is included for 'cultural completeness' and, since it does not enter into the main flow of the text, it can be skipped in its entirety. Part 3 covers the four major special features of the homotopy theory of spaces. Chapters 11 through 14 should be covered in detail. Chapter 15 is a combination of topics and cultural knowledge. Sections 15.1 and 15.2 are crucial, but the rest can be glossed over if need be. Part 4 is where the four basic topological inputs are developed into effective tools for studying homotopy-theoretical problems. Chapters 16 through 19 should all be covered in detail. Chapter 20 contains topics which can be assigned to students as homework.

The second semester should pick up with Part 5 where we develop cohomology (and homology). Chapters 21 through 24 should be covered pretty thoroughly. Chapter 25 is a vast collection of topics, which can be covered at the instructor's discretion or assigned as homework. Part 6 is about the cohomology of fiber sequences, leading ultimately to the Leray-Serre spectral sequence, which is notoriously forbidding when first encountered. The exposition here is broken into small pieces with a consistent emphasis on the topological content. Many of the basic ideas and a nice application are covered in Chapters 26 through 29; this would be a fine place to stop if time runs out. Otherwise, Chapters 30 and 31 get to the full power of the Leray-Serre spectral sequence. This power is used in Chapter 32 to prove the Bott Periodicity Theorem. Chapter 33 is another topics chapter, which includes the cohomology of Eilenberg-Mac Lane spaces and some computations involving the homotopy groups of spheres. Finally, Part 7 covers some very fun and interesting topics: localization and completion, a discussion of the exponents of homotopy groups of spheres including a proof of Selick's theorem on the exponent of $\pi_*(S^3)$; the theory of closed classes and a dual concept known as strong resolving classes; and a proof of Miller's theorem on the space of maps from $B\mathbb{Z}/p$ to a simply-connected finite complex.

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