Preface

Mathematicians are sometimes categorized as "theory builders" or "problem solvers". The authors of this book belong firmly in the problem solver class and find most pleasure in delving into the details of a particular differential equation, usually one arising from science or engineering, with the aim of understanding how the solutions behave and determining existence and uniqueness of solutions with particular properties. On the other hand, no such classification is hard and fast, and usually our goal is to determine what properties of the equation are important in generating the desired behavior. For example, in what ranges of the parameters do we see this type of solution or that, and sometimes, how broad a class of equations can we discuss without losing the essential behavior. This is a step toward building a theory, but we have not usually been inclined to pursue this goal very far.

We are, of course, delighted if others are able to put our results in a broader context. This has been done for some examples in this text, and we have tried to point the reader towards these new theories. However it is our belief that usually, to derive our particular results, many of the details which we study are still important and need attention specific to the problem at hand. Exceptions, or borderline cases, are discussed, and we have tried to assess fairly the strengths of various approaches.

These problems arise in a variety of areas in science and engineering. Often the mathematical models in these fields consist of nonlinear partial differential equations, and the analysis of these equations leads to a system of nonlinear ordinary differential equations, for example by seeking a steady state, or by a similarity substitution. In other cases the original model is a system of ode's (ordinary differential equations). Knowledge of the behavior of the solutions to these ode systems can be vital to understanding the solutions of related pde's (partial differential equations), if any, and the corresponding physical phenomena. Thus our interests come into play and are, we hope, helpful to the modeler who originally obtained the equations.

The emphasis in this book is on mathematical techniques, rather than results or applications. We choose a variety of applied problems to illustrate these techniques, but we often do not give much discussion of the background of these problems. However we are careful to cite references where this background may be found. We also do not aim for great generality in our results. Instead, for ease of exposition, we usually discuss the simplest examples which illustrate the methods of interest. Again, we give citations where the reader will find more comprehensive discussions.

We wish to emphasize our belief that many of the important problems in differential equations arise from applications. There may be more general theories to be developed; indeed we hope this is the case. But we think that the inspiration for these theories will often come from particular models of new phenomena, discovered either by scientific research or by numerical experiments. We hope that the techniques we discuss in this book will be among those that are useful in analyzing the new phenomena on which the future development of the theory may depend.

The book is written under the assumption that the reader has had a basic course in ordinary differential equations which includes the following topics:

(1) The Picard theorem on existence and uniqueness of solutions to an initial value problem of the form

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),$$
$$\mathbf{x}(t_0) = \mathbf{x}_0$$

when the vector-valued function $\mathbf{f}$ is continuous and satisfies a local Lipshitz condition in $\mathbf{x}$.

(2) The continuous dependence of solutions on initial conditions and parameters.

(3) The general theory of linear systems of ode's with variable coefficients.

(4) Sturm-Liouville problems and Green's functions.

(5) An introduction to qualitative theory and phase plane analysis.

(6) Stability theory of equilibrium points for nonlinear autonomous systems, including the concepts of stable and unstable manifolds.

The material listed above is usually included in a standard graduate course in ode's, and also in some more advanced undergraduate courses.

Much of the material may be difficult for those with only a basic undergraduate course.

Some sections of the book use more advanced material, particularly nonlinear functional analysis and some topics in the calculus of variations. We attempt to outline some of the required background, but frankly, a student with only an ode prerequisite will find this material challenging. Such a student will have to consult the cited basic literature for a better understanding. However, in almost every case there is a classical approach to the same results offered later in the chapter, and these sections can be read independently.

In Chapter 1 we describe what we mean by "classical methods" for ode's and give some simple illustrative examples. Chapter 2 gives an introduction to the so-called "shooting" method for proving the existence of solutions to boundary value problems for ode's. Detailed examples of the shooting method will appear in a number of other chapters.

Chapters 3–18 are the heart of the book. Each chapter introduces one or more techniques, perhaps classical, perhaps modern, in a relatively simple setting that still includes the essential points. These are mostly examples which we have worked on, and often they have also been studied by other authors with alternative approaches. When this is the case, we discuss some of these alternative methods as well. Usually each approach has its own strong points, which we try to bring out in our discussion. For example, one approach may give a simpler proof while another may yield more information. Which type of proof, modern or classical, has which advantage varies from one problem to another. In some cases, the alternative method may not give the simplest proofs or most complete results for the ode problem at hand but has the advantage that it can be extended to cover related problems in partial differential equations. We do not attempt to cover these extensions, however.

Chapter 3 begins with an example where the shooting method appears not to work but where a proof using real analysis in infinite dimensions (Helly's theorem) can be replaced by a simple compactness argument in two dimensions. In the second part of this chapter we contrast two different shooting techniques for proving existence of certain important solutions to the second Painlevé transcendent, a second order nonlinear equation which arises in studying the Korteweg-de Vries equation for water waves.

In Chapter 4 we show how the Brouwer fixed point theorem can be used to prove the existence of periodic solutions to some autonomous systems. In Chapter 5 we describe three different approaches to a boundary value problem for a linear system.

In Chapter 6 we consider the existence of traveling wave solutions of the FitzHugh-Nagumo equations from neurobiology. Comparison is made with the technique of geometric perturbation theory.

In Chapter 7 we give elementary and rigorous proofs of the validity of matched asymptotic expansions for two example problems, in one case comparing our methods with those from geometric perturbation theory. Chapter 8 is something of a change of pace and is independent of the other sections. It explores the use of complex function theory techniques by extending a nonlinear ode into the complex plane. One point of interest is that the result established is the nonexistence of solutions to a simple looking third order boundary value problem.

In Chapter 9 we return to the question of periodic solutions. The well-known Falkner-Skan equation from fluid mechanics is rescaled, turning the question of existence of periodic solutions into a singularly perturbed problem. In Chapter 10 we study a problem in Poiseuille flow, comparing an elementary method with use of degree theory in a Sobolev space. Chapter 11 deals with buckling of a tapered rod. Classical methods are contrasted with the use of calculus of variations and bifurcation theory in a Hilbert space.

In Chapter 12 we give an extended discussion of uniqueness and multiplicity problems. We illustrate some techniques for proving that a boundary value problem has only one solution, and in addition we discuss some examples where the solution is not unique and the goal is to determine just how many solutions there are.

Chapter 13 gives an application of two-dimensional shooting to a problem from boundary layer theory in fluid mechanics. In Chapter 14 we give classical ode approaches to some important results of A. Lazer and coauthors, as well as short proofs of related pde theorems. In Chapter 15 we show how shooting techniques can lead to results about “chaos”. Comparison is made with the technique of Melnikov in the same setting of a forced pendulum equation.

This idea is carried forward in Chapter 16, where we discuss solutions with “spike” behavior and also a type of “chaos”. In Chapter 17 we outline a very recent approach of X. Chen and Sadhu to obtaining asymptotic expansions of solutions with boundary layers and spikes for a class of equations with quadratic nonlinear terms. The last of the core chapters is Chapter 18, in which families of spikes and transition layer solutions are found for another class of inhomogeneous reaction-diffusion equations. Three different proofs of a central result are discussed.

Finally, in Chapter 19, we describe three important unsolved problems in our area, problems which have challenged us and other researchers for a number of years and which we hope the reader will find attractive. It would be gratifying to see these problems solved by someone who learned of them from this book.

An experienced reader will now detect that important techniques have been neglected. Undoubtedly many will feel that their favorite method is omitted, or at least under-appreciated. Our main defense is that we have written most extensively about what we know best. Also, many of the omitted topics have been the subject of their own specialized monographs, which we have tried to cite appropriately. It is undoubtedly true that many other techniques have importance in a wide variety of problems, which we have neither the space nor the background to discuss in detail. Topics which are under-represented include Lin's method and others from the important and influential school of Hale, Chow, and Mallet-Paret, applications of the Moser twist theorem, use of bifurcation and degree theory (including center manifolds), comparison methods and ideas from the theory of competitive and cooperative systems (developed particularly by M. Hirsch and H. Smith), many topics generally related to chaos and to be found in the landmark monograph of Guckenheimer and Holmes, and others perhaps even farther from the realm of the classical techniques which are our focus. Our prejudice is that for the particular kinds of problems we study here, problems which appear frequently in applications, the methods illustrated are often effective and efficient. This is not meant to suggest that they would be best in all of the vast array of problems in ode's which are found in modern applied analysis.

Finally, we are delighted to thank the people who have assisted us with various parts of the book. We are indebted to Ina Mette, AMS acquisitions editor, whose emailed question "Have you thought of writing a book?" started the project off and whose steady encouragement helped keep it going. Other AMS staff, including Marcia Almeida, Barbara Beeton, our production editor, Arlene O'Sean, and others have been especially helpful as well.

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