Preface

Number theory is one of the most fascinating topics in mathematics, and there are various reasons for this. Here are a few:

- Several number theory problems can be formulated in simple terms with very little or no background required to understand their statements.
- It has a rich history that goes back thousands of years when mankind was learning to count (even before learning to write!).
- Some of the most famous minds of mathematics (including Pascal, Euler, Gauss, and Riemann, to name only a few) have brought their contributions to the development of number theory.
- Like many other areas of science, but perhaps more so with this one, its development suffers from an apparent paradox: giant leaps have been made over time, while some problems remain as of today completely impenetrable, with little or no progress being made.

All this explains in part why so many scientists and so many amateurs have worked on famous problems and conjectures in number theory. The long quest for a proof of Fermat's Last Theorem is only one example.

And what about "analytic number theory"? The use of analysis (real or complex) to study number theory problems has brought light and elegance to this field, in particular to the problem of the distribution of prime numbers. Through the centuries, a large variety of tools has been developed to grasp a better understanding of this particular problem. But the year 1896 saw a turning point in the history of number theory. Indeed, that was the year when two mathematicians, Jacques Hadamard and Charles Jean de la Vallée Poussin, one French, the other Belgian, independently used complex analysis to prove what we now call the Prime Number Theorem, namely the fact that "$\pi(x)$ is asymptotic to $x/\log x$" as x tends to infinity, where $\pi(x)$ stands for the number of prime numbers not exceeding x. This event marked the birth of analytic number theory.

At first one might wonder how analysis can be of any help in solving problems from number theory, which are after all related to the study of positive integers. Indeed, while integers "live" in a discrete world, analysis "lives" in a continuous one. This duality goes back to Euler, who had observed that there was a connection between an infinite product running over the set of all prime numbers and an infinite series which converges or diverges depending on the value of real variable s, a connection described by the relation

$$\left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{3^s}\right)^{-1}\left(1-\frac{1}{5^s}\right)^{-1}\cdots=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\cdots,$$

which holds in particular for all real $s > 1$. Approximately one century later, Riemann studied this identity for complex values of s by carefully exhibiting the analytic properties of the now famous *Riemann Zeta Function*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\qquad(\mathrm{Re}(s)>1).$$

By extending this function to the entire complex plane, he used it to establish in 1859 a somewhat exceptional but nevertheless incomplete proof of the Prime Number Theorem. Thirty-seven years later, Hadamard and de la Vallée Poussin managed to complete the proof initiated by Riemann.

The methods put forward by Riemann and many other 20th-century mathematicians have helped us gain a better understanding of the distribution of prime numbers and a clearer picture of the complexity of the multiplicative structure of the integers or, using a stylistic device, a better comprehension of the anatomy of integers.

In this book, we provide an introduction to analytic number theory. The choice of the subtitle "Exploring the Anatomy of Integers" was coined at a CRM workshop held at Université de Montréal in March 2006 which the two of us, along with Andrew Granville, organized. For the workshop as well as for this book, the terminology "anatomy of integers" is appropriate as it describes the area of multiplicative number theory that relates to the size and distribution of the prime factors of positive integers and of various families of integers of particular interest.

Besides the proof of the Prime Number Theorem, our choice of subjects for this book is very subjective but nevertheless legitimate. Hence, several chapters are devoted to the study of arithmetic functions, in particular those which provide a better understanding of the multiplicative structure of the integers. For instance, we study the average value of the number of prime factors of an integer, the average value of the number of its divisors, the behavior of its smallest prime factor and of its largest prime factor, and so on. A whole chapter is devoted to sieve methods, and many of their applications are presented in the problem section at the end of that chapter. Moreover, we chose to include some results which are hard to find elsewhere. For instance, we state and prove the very useful Birkhoff-Vandiver primitive divisor theorem and the important Turán-Kubilius inequality for additive functions. We also discuss less serious but nevertheless interesting topics such as the Erdős multiplication table problem.

We also chose to discuss the famous *abc* conjecture, because it is fairly recent (it was first stated in 1985) and also because it is central in the study of various conjectures in number theory. Finally, we devote a chapter to the study of the index of composition of an integer, its study allowing us to better understand the anatomy of an integer.

To help the reader better comprehend the various themes presented in this book, we listed 263 problems along with the solutions to the even-numbered ones.

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