Preface

1. In these notes we develop a theory of strong solutions to linear evolution equations of the type

$$\varepsilon\, u_{tt} + \sigma\, u_t - a_{ij}(t,x)\,\partial_i\partial_j u = f(t,x)\,, \tag{0.0.1}$$

and their quasi-linear counterpart

$$\varepsilon\, u_{tt} + \sigma\, u_t - a_{ij}(t,x,u,u_t,\nabla u)\,\partial_i\partial_j u = f(t,x)\,. \tag{0.0.2}$$

In (0.0.1) and (0.0.2), ε and σ are non-negative parameters; $u = u(t,x)$, $t > 0$, $x \in \mathbb{R}^N$, and summation over repeated indices i, j, $1 \le i,\, j \le N$, is understood. In addition, and in a sense to be made more precise, the quadratic form $\mathbb{R}^N \ni \xi \mapsto a_{ij}(\cdots)\,\xi^i\,\xi^j$ is positive definite.

We distinguish the following three cases.

(1) $\varepsilon > 0$ and $\sigma = 0$. Then, (0.0.1) and (0.0.2) are hyperbolic equations; in particular, when $\varepsilon = 1$, they reduce to

$$u_{tt} - a_{ij}\,\partial_i\partial_j u = f\,, \tag{0.0.3}$$

and when $a_{ij}(\cdots) = \delta_{ij}$ (the so-called Kronecker δ, defined by $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$), (0.0.3) further reduces to the classical wave equation

$$u_{tt} - \Delta\, u = f\,. \tag{0.0.4}$$

(2) $\varepsilon = 0$ and $\sigma > 0$. Then, (0.0.1) and (0.0.2) are parabolic equations; in particular, when $\sigma = 1$, they reduce to

$$u_t - a_{ij}\,\partial_i\partial_j u = f\,, \tag{0.0.5}$$

and when $a_{ij}(\cdots) = \delta_{ij}$, (0.0.5) further reduces to the classical heat equation

$$u_t - \Delta u = f\,. \tag{0.0.6}$$

(3) $\varepsilon > 0$ and $\sigma > 0$. Then, (0.0.1) and (0.0.2) are dissipative hyperbolic equations; in particular, when $\varepsilon = \sigma = 1$, they reduce to

$$u_{tt} + u_t - a_{ij}\,\partial_i\partial_j u = f\,, \tag{0.0.7}$$

and when $a_{ij}(\cdots) = \delta_{ij}$, (0.0.7) further reduces to the so-called telegraph equation

$$u_{tt} + u_t - \Delta u = f\,. \tag{0.0.8}$$

We prescribe that u should satisfy the initial conditions (or Cauchy data)

$$u(0,x) = u_0(x)\,, \qquad \varepsilon\, u_t(0,x) = \varepsilon\, u_1(x)\,, \tag{0.0.9}$$

where u_0 and u_1 are given functions on $\mathbb{R}^N$, and the second condition is vacuous if $\varepsilon = 0$ (that is, in the parabolic case we only prescribe the initial condition $u(0,x) = u_0(x)$).

Our purpose is to show that the Cauchy problems (0.0.1) + (0.0.9) and (0.0.2) + (0.0.9) are solvable in a suitable class of Sobolev spaces; we call the corresponding solutions strong. By this, we mean that the solutions we seek should be functions $t \mapsto u(t)$, which are valued in a Sobolev space $H^r := H^r(\mathbb{R}^N)$, and possess a sufficient number of derivatives, either classical or distributional, so that equations (0.0.1) and (0.0.2) hold for (almost) all t and all x. More precisely, when $\varepsilon > 0$ we seek for solutions of (0.0.1) and (0.0.2) in the space

$$C^0([0,T]; H^{s+1}) \cap C^1([0,T]; H^s) \cap C^2([0,T]; H^{s-1})\,, \tag{0.0.10}$$

for some $T > 0$, where $s \in \mathbb{N}$ is such that $s > \frac{N}{2} + 1$; this condition implies that strong solutions are also classical. When $\varepsilon = 0$, we seek instead for solutions of (0.0.1) and (0.0.2) in the space

$$\{u \in C([0,T]; H^{s+1}) \;\mid\; u_t \in L^2(0,T; H^s)\}\,. \tag{0.0.11}$$

In addition, we want to show that the Cauchy problems (0.0.1) + (0.0.9) and (0.0.2) + (0.0.9) are well-posed, in Hadamard's sense, in these spaces; that is, that their solutions should be unique and depend continuously on their data f, u_0 and u_1 (of course, the latter only for $\varepsilon > 0$). Finally, we also consider equations with lower order terms, i.e.,

$$\varepsilon\, u_{tt} + \sigma\, u_t - a_{ij}\,\partial_i\partial_j u = f + b_i\,\partial_i u + c\,u\,, \tag{0.0.12}$$

in particular in the linear case, as well as equations in the divergence form

$$\varepsilon\, u_{tt} + \sigma\, u_t - \partial_j(a_{ij}\,\partial_i u) = f + b_i\,\partial_i u + c\,u\,. \tag{0.0.13}$$

In the quasi-linear case, equations (0.0.2) will in general have only local solutions; that is, even if the source term f is defined on a given interval $[0,T]$, or on all of $[0,+\infty[$, the solution will be defined only on *some* interval $[0,\tau]$, with $\tau < T$, and cannot be extended to all of $[0,T]$.

2. Our main goal is to develop a unified treatment of equations (0.0.1) and (0.0.2), both in the hyperbolic (either dissipative, or not) and the parabolic case, following a common constructive method to solve either problem. In the linear case, of course, a unified theory for both hyperbolic and parabolic equations (0.0.1), in a suitable framework of Hilbert spaces, has been presented by Lions and Magenes in their three-volume treatise [**101, 102, 103**], where they introduced a variety of arguments and techniques to solve fairly general kinds of initial-boundary value problems. The main reason we seek a unified treatment of equations (0.0.1) and (0.0.2) in the quasi-linear case is that this allows us to compare the solutions to the hyperbolic and the parabolic equations, in a number of ways. In particular, when (0.0.2) admits global solutions (that is, defined on all of $[0,+\infty[$), we wish to study their asymptotic behavior as $t \to +\infty$. We assume that the coefficients a_{ij} in (0.0.2) depend only on the first-order derivatives u_t and ∇u, and are interested in the following questions. The first is that of the convergence of the solutions of (0.0.2) to the solution of the stationary equation

$$- a_{ij}(0,\nabla v)\,\partial_i\partial_j v = h\,. \tag{0.0.14}$$

The second, when ε and $\sigma > 0$, is the comparison of the asymptotic profiles of the solutions of the dissipative hyperbolic equation (0.0.2) to those of the solutions of the parabolic equation, corresponding to $\varepsilon = 0$. The third question, related to (0.0.2), is the singular perturbation problem, concerning the convergence, as $\varepsilon \to 0$, of solutions u^ε of the dissipative hyperbolic equation to the solution u^0 of the parabolic equation.

3. Linear hyperbolic equations of the type (0.0.12) and (0.0.13), in particular when $\sigma = 0$, have been studied by many authors, who have considered the corresponding Cauchy problem in different settings. An elementary introduction to both kinds of equations can be found in Evans' textbook [**47**]; for more advanced and specific results, renouncing to any pretense of a comprehensive list, we refer, e.g., to Friedrichs [**51**], Kato [**72**], Mizohata [**122**], and Ikawa [**63**], who resort to a solution method based on a semigroup approach, complicated by the fact that the coefficients a_{ij} depend on t. The semigroup method has later been successfully applied to quasi-linear equations; see, e.g., Okazawa [**130**], Tanaka [**153, 154, 155**], and, for a more abstract approach, Beyer [**15**]. Other methods can be seen, e.g., in Racke [**136**], and Sogge [**151**], based respectively on the Cauchy-Kovaleskaya and the Hahn-Banach theorems. In the solution theory we present in Chapter

2, we prefer to follow the so-called Faedo-Galerkin method, which is a generalization of the method of separation of variables, and which explicitly constructs the solution to (0.0.12) as the limit of a sequence of functions, each of which solves an approximate version of the problem, determined by its projection onto suitable finite-dimensional subspaces. The results we establish for (0.0.1) when $\varepsilon > 0$ are not specifically dependent on the fact that the equation is hyperbolic; in fact, the Faedo-Galerkin method can be readily adapted to obtain strong solutions of the linear parabolic equation. For general references to parabolic equations, both linear and quasi-linear, we refer, e.g., to Ladyzenshkaya, Solonnikov and Ural'tseva [**86**], Amann [**6**], Pao [**131**], Lunardi [**107**], Lieberman [**96**], and Krylov [**83, 84**], where these equations are mostly studied in the Hölder spaces $C^{m+\alpha/2,2m+\alpha}(\overline{Q})$. For the numerical treatment of equation (2.1.1), we refer, e.g., to Meister and Struckmeier [**112**].

4. The Cauchy problem for quasi-linear hyperbolic equations such as (0.0.2), as well as their counterpart in divergence form (0.0.13), has been studied by many authors, who have provided local (and, when possible, global, or, at least, almost global) solutions with a number of methods, including a nonlinear version of the Galerkin scheme and various versions of the Moser-Nash algorithm. Renouncing again to any pretense of a comprehensive list, we refer, e.g., to the classical treatise by Courant and Hilbert [**38**], as well as the more recent works by Kato [**72**], John [**66**], Kichenassamy [**74**], Racke [**136**], Sogge [**151**], Hörmander [**57**], as well as Lax [**88**], Li Ta-Tsien [**91**], and Li Ta-Tsien and Wang Li-Ping [**93**]. In the above context, local solution means a solution defined on *some* interval $[0,\tau]$; almost global solution means a solution defined on a *prescribed* interval $[0,T]$, of finite but arbitrary length, possibly subject to some restrictions on the size of the data, depending on T; global solution means a solution also defined on arbitrary intervals $[0,T]$, but with restrictions on the size of the data, if any, independent of T (thus, these solutions are defined on the entire interval $[0,+\infty[$). Finally, we also consider global bounded solutions; that is, global solutions which remain bounded as $t \to +\infty$. In Chapter 3, we present a solution method based on a linearization and fixed point method, introduced by Kato [**70, 71, 72**], in which we apply the results for the linear theory, developed in Chapter 2. As for the linear case, the results we establish for (0.0.2), when $\varepsilon > 0$, are not specifically dependent on the fact that the equation is hyperbolic; in fact, the linearization and fixed point method can be adapted to obtain local, strong solutions of the quasi-linear parabolic equation

$$u_t - a_{ij}(t,x,u,\nabla u)\,\partial_i\partial_j u = f\ , \tag{0.0.15}$$

as well as of the analogous equation in divergence form. Moreover, these methods can also be applied to other types of evolution equations, such as the so-called dispersive equations considered in Tao [**156**], and Linares and Ponce [**97**]; these include, among others, the Schrödinger and the Korteweg-de-Vries equations.

5. Not surprisingly, many more results are available on semi-linear hyperbolic and parabolic equations; that is, equations of the form

$$u_{tt} - \Delta u = f(t,x,u,Du)\,, \tag{0.0.16}$$

$$u_t - \Delta u = f(t,x,u,\nabla u)\,. \tag{0.0.17}$$

Among the many works on this subject, we limit ourselves to cite Strauss [**150**], Todorova and Yordanov [**159**], Zheng [**168**], Quittner and Souplet [**133**], Cazenave and Haraux [**24**], and the references therein. Most of these results concern the well-posedness of the Cauchy problem for (0.0.16) or (0.0.17) in a suitable weak sense; strong solutions are then obtained by appropriate regularity theorems, and the asymptotic behavior of such weak solutions can be studied in terms of suitable attracting sets in the phase space; see, e.g., Milani and Koksch, [**119**]. In fact, we could try to develop a corresponding weak solution theory for hyperbolic and parabolic quasi-linear equations in the conservation form

$$\varepsilon\, u_{tt} + \sigma\, u_t - \operatorname{div}[a(\nabla u)] = f\,, \tag{0.0.18}$$

where $a : \mathbb{R}^N \to \mathbb{R}^N$ is monotone. However, there appears to be a striking difference between the hyperbolic and the parabolic situation. For the latter, i.e., when $\varepsilon = 0$ in (0.0.18), existence, uniqueness and well-posedness results for weak solutions, at least when a is strongly monotone, are available; see, e.g., Lions [**99**, ch. 2, §1], and Brézis [**19**]. In contrast, when $\varepsilon > 0$ the question of the existence of even a local weak solution to equation (0.0.18) (that is, in the space (0.0.10) with $s = 0$) is, as far as we know, totally open (unless, of course, a is linear).

6. To our knowledge, there are not yet satisfactory answers to the question of finding sharp life-span estimates for problem (0.0.2), at least in the functional framework we consider. On the other hand, rather precise results have long been available, at least for more regular solutions of the homogeneous equation; that is, when $f \equiv 0$ and $u_0 \in H^{s+1} \cap W^{r+1,p}$, $u_1 \in H^s \cap W^{r,p}$, for suitable integers $s \gg \frac{N}{2} + 1$, $r < s$, and $p \in \,]1, 2[$. In this case, the situation also depends on the space dimension N; more precisely, one obtains global existence of strong solutions if $N \geq 4$, and also if $N = 3$ if the nonlinearity satisfies an additional structural restriction, known as the null condition. The proof of these results is based on supplementing the direct energy estimates used to establish local solutions, with rather refined decay estimates of the solution to the linear wave equation. We refer to Racke [**136**], and John [**66**], for a comprehensive survey of the results of, among many others, John [**65**], Klainerman [**75, 76**], Klainerman and Ponce [**81**], as well as Klainerman [**77**] and Christodoulou [**31**], for the null condition.

7. The theory of quasi-linear evolution equations has many important applications. A non exhaustive list would include fluid dynamics (see, e.g., Majda [**108**], and Nishida [**126**]); general relativity, and, specifically, the so-called Einstein vacuum equations (Klainerman and Christodoulou [**78**], and Klainerman and Nicolò [**80**]); wave maps (Shatah and Struwe [**145**], and Tao [**156**]); von Karman type thin plate equations (Cherrier and Milani [**26, 27, 29**], Chuesov and Lasiecka [**33**]); control and observability theory (Li [**94**]). Other applications, specifically of dissipative hyperbolic equations (0.0.2) with $\sigma > 0$ and ε small, include models of heat equations with delay (Li [**92**], Cattaneo [**21**], Jordan, Dai and Mickens [**67**], Liu [**104**]), where ε is a measure of the delay or heat relaxation time; Maxwell's equations in ferromagnetic materials (Milani [**114**]), where ε is a measure of the displacement currents, usually negligible; simple models of laser optic equations (Haus [**54**]), where ε is related to measures of low frequencies of the electromagnetic field; traffic flow models (Schochet [**140**]), where ε is a measure of the drivers' response time to sudden disturbances (which, one hopes, should be small); models of random walk systems (Hadeler [**53**]), where ε is related to the reciprocal of the turning rates of the moving particles; construction materials with strong internal stress-strain relations, measured by parameters related to the reciprocal of ε (see, e.g., Banks et al. [**9, 10**] for the case of a one-dimensional elastomer); and models of time-delayed information propagation in economics (Ahmed and Abdusalam [**2**]).

8. These notes have their origin in a series of graduate courses and seminars we gave at Fudan University, Shanghai, at the Université Pierre et Marie Curie (Paris VI), the Technische Universität Dresden, and the Pontificia Universidad Católica of Santiago, Chile. Some of the material we cover is relatively well known, although many results, in particular on hyperbolic equations, seem to be somewhat scattered in the literature, and often subordinate to other topics or applications. Other results, in particular on the diffusion phenomenon for quasi-linear hyperbolic waves, appear to be new. Our intention is, in part, to provide an introduction to the theory of quasi-linear evolution equations in Sobolev spaces, organizing the material in a progression that is as gradual and natural as possible. To this end, we have tried to put particular care in giving detailed proofs of the results we present; thus, if successful, our effort should give readers the necessary basis to proceed to the more specialized texts we have indicated above. In this sense, these notes are not meant to serve as an advanced PDEs textbook;

rather, their didactical scope and subject range is restricted to the effort of explaining, as clearly as we are able to, one possible way to study two simple and fundamental examples of evolution equations (hyperbolic, both dissipative or not, and parabolic) on the whole space $\mathbb{R}^N$. In addition, we also hope that these notes may serve as a fairly comprehensive and self-contained reference for researchers in other areas of applied mathematics and sciences, in which, as we have mentioned, the theory of quasi-linear evolution equations has many important applications. We should perhaps mention explicitly the fact that, given the introductory level of these notes, we have limited ourselves to present only those results that can be obtained by resorting to one of the most standard methods for the study of equations (0.0.1) and (0.0.2); namely, that of the *a priori*, or *energy*, estimates. Of course, this choice forces us to neglect other methods that are more specific to the type of equation under consideration and which are extensively studied in more specialized texts. For example, we do not cover, but only mention, the theory of Hölder solutions of parabolic quasi-linear equations (0.0.15) (see, e.g., Krylov [**83**]), or the theory of weak solutions to quasi-linear first-order hyperbolic systems of conservation laws, as in (3.1.8) of Chapter 3 (see, e.g., Alinhac [**5**], or Serre [**143, 144**]), and we do not even mention other very specific and highly refined techniques which have been developed and are being developed for the study of these equations, such as, to cite a few, the theory of nonlinear semigroup (see, e.g., Beyer [**15**]), the methods of pseudo-differential operators (see, e.g., Taylor [**157**]) and of microlocal analysis (see, e.g., Bony [**17**]). On the other hand, one can perhaps be surprised by the extent of the results one can obtain, by means of the one and same technique; that is, the *energy method*. As we have stated, this method has, among others, the advantage of allowing us to present our results in a highly unified way, and to show that, even today, classical analysis allows us to deal in a simple way, by means of standard and well-tested techniques, with relevant questions in the theory of PDEs of evolution, which are still the subject of considerable study.

9. The material of these notes is organized as follows. In Chapter 1 we provide a summary of the main functional analysis results we need for the development of the theory we wish to present. In Chapter 2 we develop a strong solution theory for the Cauchy problem for the linear equation (0.0.12), with existence, uniqueness, regularity, and well-posedness results for both the hyperbolic equation ($\varepsilon = 1$, $\sigma = 0$) and the parabolic one ($\varepsilon = 0$, $\sigma = 1$). In Chapter 3 we construct local in time solutions to the quasi-linear equations (0.0.2) and (0.0.15), by means of a linearization and fixed-point technique, in which we apply the results on the linear equations we established in the previous Chapter. Again, we give existence, uniqueness, regularity, and well-posedness results for both types of equation. In

Chapter 4 we study the question of the extendibility of these local solutions to either a prefixed finite but arbitrary time interval $[0, T]$ (almost global and global existence), or to the whole interval $[0, +\infty[$ (global and global bounded existence). We present an explicit example of blow-up in finite time for solutions of the quasi-linear equation (0.0.3) in one dimension of space, as well as some global and almost global existence results for either equation, when the data u_0, u_1 and f are sufficiently small. We also present a global existence result for the parabolic equation (0.0.15), for data of arbitrary size. In Chapter 5 we consider the asymptotic behavior, as $t \to +\infty$, of global, bounded, small solutions of (0.0.2), both dissipative hyperbolic ($\varepsilon = 1$) and parabolic ($\varepsilon = 0$), and we prove some results on their convergence to the solution of the stationary equation (0.0.14). In the homogeneous case $f \equiv 0$, we also establish some stability estimates, on the rate of decay to 0 of the corresponding solutions. We also give a result on the diffusion phenomenon, which consists in showing that, when $f \equiv 0$, solutions of the hyperbolic equation (0.0.7) (both linear and quasi-linear) asymptotically behave as those of the parabolic equation (0.0.5) of corresponding type. In Chapter 6 we consider a second way in which we can compare the hyperbolic and the parabolic problems; namely, we consider (0.0.2) as a perturbation, for small values of $\varepsilon > 0$, of the parabolic equation (0.0.2), with $\varepsilon = 0$. Denoting by u^ε and u^0 the corresponding solutions, we study the problem of the convergence $u^\varepsilon \to u^0$ as $\varepsilon \to 0$, on compact time intervals. We consider either intervals $[0, T]$ or $[\tau, T]$, $\tau \in \,]0, T[$; that is, including $t = 0$ or not. In the former case, the convergence is singular, due to the loss of the initial condition on u_t, and we give rather precise estimates, as t and $\varepsilon \to 0$, on the corresponding *initial layer*. We mention in passing that the estimates we establish on the difference $u^\varepsilon - u^0$ allow us also to deduce a global existence equivalency result between the two types of equations, in the sense that a global solution to the parabolic equation, corresponding to data of *arbitrary* size, exists, if and only if global solutions to the dissipative hyperbolic equation also exist, corresponding to data of arbitrary size, and ε is sufficiently small. We conclude the chapter with a global result for equation (0.0.2), with data of arbitrary size, when ε is sufficiently large. Lastly, in Chapter 7, we present two applications of the theory developed in the previous chapters. In the first example, we consider a model for the complete system of Maxwell's equations, in which the use of suitable electromagnetic potentials allows us to translate the first-order Maxwell's system into a second-order evolution equation of the type (0.0.2). In this model, the parameters ε and σ can be interpreted as a measure, respectively, of the displacement and the eddy currents; in some situations, such as when the equations are considered in a ferro-magnetic medium, displacement currents are negligible with respect to the eddy ones, and this observation leads to the question of the control of the error introduced in the model when the term εu_{tt} is neglected. In related situations, one is interested in periodic phenomena, with relatively low frequencies, thus leading to the question of the existence of solutions on the whole period of time. It is our hope that these questions may be addressed, at least to some extent, by the results of the previous chapters. In the second example, we consider two systems of evolution equations, of hyperbolic and parabolic type, relative to a highly nonlinear elliptic system of von Karman type equations on $\mathbb{R}^{2m}$, $m \geq 2$. These equations generalize the well-known equations of the same name in the theory of elasticity, which correspond to the case $m = 1$, and model the deformation of a thin plate due to both internal and external stresses. For both types of systems we show the existence and uniqueness of local in time strong solutions, which again can be extended to almost global ones if the initial data are small enough. Even though these systems do not fit exactly in the framework of second-order evolution equations for which our theory is developed, their study allows us to show that the unified methods we present can be applied to a much wider class of equations than those of the form (0.0.2).

10. Finally, we mention that an analogous unified theory could be constructed for initial-boundary value problems for equation (0.0.2), in a subdomain $\Omega \subset \mathbb{R}^N$, with u subject to appropriate conditions at the boundary $\partial\Omega$ of Ω, assumed to be adequately smooth. The type of results one obtains is qualitatively analogous, but in a different functional setting for both the data and the solutions. Indeed, the data have to satisfy a number of so-called *compatibility conditions* at $\{t = 0\} \times \partial\Omega$, which are different in the hyperbolic and parabolic cases, and the integrations by parts that are usually carried out in order to establish the necessary energy estimates (see Chapter 2) would involve boundary terms that do not appear when $\Omega = \mathbb{R}^N$. For example, in our papers [**116, 117**] we considered the simple case where equation (0.0.2) is studied in a bounded domain, with homogeneous Dirichlet boundary conditions; other results can be found, e.g., in Dafermos and Hrusa [**40**]. To discuss this topic in a meaningful degree of detail would require a whole new book; here, we limit ourselves to a reference to the above-mentioned papers, and to the literature quoted therein, for a brief overview of the technical issues typically encountered in this situation.

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Pascal Cherrier
Université Pierre et Marie Curie, Paris

Albert Milani
University of Wisconsin - Milwaukee