Chapter 2

Linear Equations

2.1. Introduction

1. In this chapter we consider the Cauchy problem for the linear hyperbolic evolution equation

$$u_{tt} - a_{ij}(t,x)\,\partial_i\partial_j u = f(t,x) + b_i(t,x)\,\partial_i u + c(t,x)\,u\,, \tag{2.1.1}$$

where summation for i, j from 1 to N is understood. This means that we take (t,x) in the cylinder $Q = \,]0,T[\,\times\mathbb{R}^N$, where $T > 0$ is fixed but arbitrary, and seek solutions of (2.1.1) which satisfy the initial conditions (or Cauchy data)

$$u(0,x) = u_0(x)\,, \qquad u_t(0,x) = u_1(x)\,, \tag{2.1.2}$$

where u_0 and u_1 are given functions on $\mathbb{R}^N$. Our goal is to show that the Cauchy problem (2.1.1)+(2.1.2) is well-posed in a suitable class of Sobolev spaces; we call the corresponding solutions *strong*. In section 2.4, we will also briefly consider linear equations in the divergence form

$$u_{tt} - \,\partial_j(a_{ij}\,\partial_i u) = f + b_i\,\partial_i u + c\,u\,, \tag{2.1.3}$$

and show that the Cauchy problem (2.1.3)+(2.1.2) is well-posed in a suitable class of weak solutions. Note that (2.1.1) can formally be rewritten in the divergence form

$$u_{tt} - \,\partial_j(a_{ij}\,\partial_i u) = f + (b_i - \partial_j\,a_{ij})\partial_i u + c\,u\,. \tag{2.1.4}$$

As we mentioned in the Preface, the results we establish for (2.1.1) are not specifically dependent on the fact that the equation is hyperbolic; in fact, our solution theory, based on the Faedo-Galerkin method, can be readily adapted to obtain strong solutions of linear parabolic equations in non divergence form

$$u_t - a_{ij}(t,x)\,\partial_i\partial_j u = f(t,x) + b_i(t,x)\,\partial_i u + c(t,x)u\,; \tag{2.1.5}$$

indeed, in section 2.5 we will present corresponding well-posedness results for the Cauchy problem relative to (2.1.5).

2. Throughout this chapter, we will adopt the following notations and conventions, some of which we have already introduced in Chapter 1. We denote first-order derivatives by $D := \{\partial_t, \nabla\}$. The abbreviations "a.e." and "a.a." stand, respectively, for "almost everywhere" and "almost all", either in Q or in $\mathbb{R}^N$, with reference to the Lebesgue measure in these sets. For $1 \le p \le \infty$, we set $L^p := L^p(\mathbb{R}^N)$, and denote by $|\cdot|_p$ its norm. For $m \in \mathbb{N}$, we set $H^m := H^m(\mathbb{R}^N)$, and denote by $\|\cdot\|_m$ and $\langle\cdot,\cdot\rangle_m$ its norm and scalar product. We identify $L^2 = H^0$, and abbreviate $\|\cdot\|_0 = |\cdot|_2 = \|\cdot\|$, $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle$. When there is no risk of confusion, we often write $\|u(t)\|_m$ instead of $\|u(t,\cdot)\|_m$, or even $\|u\|_m$, especially under integration over time intervals. Finally, when we say that a constant "depends on the data" (respectively, "on the coefficients"), we assume that we are in a context where the data u_0, u_1, and f (respectively, the coefficients a_{ij}, b_i, c) have been specified in some function spaces, and we mean that the constant can be estimated by a continuous function of the norm of the data (respectively, the coefficients) in these spaces. Typically, we denote such functions by γ, κ, or ψ, with γ, κ, $\psi \in \mathcal{K}$. These functions can be explicitly determined and, without loss of generality, we may assume them to have range in $\mathbb{R}_{\ge 1}$.

Finally, we reserve the letter s to *always* denote an integer strictly larger than $\frac{N}{2} + 1$. This condition plays a crucial role in the sequel, because it implies that H^{s-1} is an algebra (see Corollary 1.5.2), and also, by the imbeddings (1.5.61) and (1.5.56), that

$$H^{s-1} \hookrightarrow C^{0,\alpha}(\overline{\mathbb{R}^N}) \hookrightarrow L^\infty\,. \tag{2.1.6}$$

2.2. The Hyperbolic Cauchy Problem

1. We consider the Cauchy problem for the hyperbolic equation (2.1.1). We assume that the coefficients a_{ij} in (2.1.1) are bounded, symmetric, and uniformly strongly elliptic in some cylinder Q; that is, $a_{ij} \in L^\infty(Q)$, $a_{ij} = a_{ji}$ a.e. in Q, and there are numbers $\alpha_1 > \alpha_0 > 0$ such that, for a.a $(t,x) \in Q$ and all $q \in \mathbb{R}^N$,

$$\alpha_0\,|q|^2 \le a_{ij}(t,x)\,q^i\,q^j \le \alpha_1\,|q|^2\,. \tag{2.2.1}$$

In addition, we assume that, for some integer $s > \frac{N}{2} + 1$,

(2.2.2)

$$D\,a_{ij} \;\in\; L^1(0,T;H^{s-1})\,, \quad b_i\,,\; c \;\in\; L^1(0,T;H^s) \cap L^2(0,T;H^{s-1})\,.$$

Correspondingly, we set

$$\mu_1(t) \;:=\; \|Da_{ij}(t)\|_{s-1} + \|b_i(t)\|_s + \|c(t)\|_s\,, \tag{2.2.3}$$

$$M_1 \;:=\; \int_0^T \mu_1(t)\,\mathrm{d}t\,; \tag{2.2.4}$$

in addition, noting that, by the second claim of Proposition 1.7.6 (with $m = s-1$), $\nabla\, a_{ij} \in W^{1,1}(0,T;H^{s-1},H^{s-2}) \hookrightarrow AC([0,T];H^{s-2})$, we set

$$A \;:=\; \max_{i,j}\left(|a_{ij}|_{L^\infty(Q)} + \|\nabla a_{ij}\|_{C([0,T];H^{s-2})}\right)\,, \tag{2.2.5}$$

$$M_2^2 \;:=\; A^2\,T + \max_{1\le i\le N}\int_0^T \|b_i\|_{s-1}^2\,\mathrm{d}t + \int_0^T \|c\|_{s-1}^2\,\mathrm{d}t\,. \tag{2.2.6}$$

We also note that $a_{ij} \in L^\infty(Q) \hookrightarrow L^1(0,T;L^\infty)$, and, by (2.1.6), the first of (2.2.2) implies that $\partial_t\, a_{ij} \in L^1(0,T;H^{s-1}) \hookrightarrow L^1(0,T;L^\infty)$; hence, the third claim of Proposition 1.7.6, with $X = L^\infty$ (which is not reflexive), implies that

$$a_{ij} \in AC([0,T];L^\infty)\,. \tag{2.2.7}$$

Likewise, (2.2.2) implies that

$$D\,a_{ij}\,,\; b_i\,,\; c \;\in\; L^1(0,T;H^{s-1}) \hookrightarrow L^1(0,T;L^\infty)\,. \tag{2.2.8}$$

This observation will be essential in the application of Theorem 2.2.1 below to the quasi-linear equations we study in Chapter 3.

2. For $m \in \mathbb{N}$ and $T > 0$, we define

$$\mathcal{X}_m(T) \;:=\; C([0,T];H^{m+1}) \cap C^1([0,T];H^m)\,, \tag{2.2.9}$$

$$\mathcal{Y}_m(T) \;:=\; \{u \in \mathcal{X}_m(T) \;|\; u_{tt} \in L^2(0,T;H^{m-1})\}\,; \tag{2.2.10}$$

these are Banach spaces, with respect to their natural norms

$$\|u\|_{\mathcal{X}_m(T)} \;=\; \max_{0\le t\le T}\left(\|u(t)\|_{m+1}^2 + \|u_t(t)\|_m^2\right)^{1/2}\,, \tag{2.2.11}$$

$$\|u\|_{\mathcal{Y}_m(T)} \;=\; \left(\|u\|_{\mathcal{X}_m(T)}^2 + \int_0^T \|u_{tt}\|_{m-1}^2\,\mathrm{d}t\right)^{1/2}\,. \tag{2.2.12}$$

Note that, with the notation of (1.7.31), $\mathcal{X}_m(T) = C^1([0,T];H^{m+1},H^m)$.

We are ready to state the main result on strong solutions of linear hyperbolic equations.

Theorem 2.2.1. *Let s and $m \in \mathbb{N}$, with $s > \frac{N}{2} + 1$ and $1 \leq m \leq s$. Let $u_0 \in H^{m+1}$, $u_1 \in H^m$, and $f \in L^2(0,T;H^m)$. Under the above stated assumptions on the coefficients of* (2.1.1)*, the following holds:*

(1) Existence: *There exists a unique $u \in \mathcal{Y}_m(T)$, which is a strong solution of the Cauchy problem* (2.1.1)+(2.1.2)*, in the sense that equation* (2.1.1) *holds in H^{m-1} for a.a. $t \in \,]0,T[$, as well as a.e. in Q.*

(2) A Priori Estimates: *u satisfies the estimates*

$$\|u\|_{\mathcal{X}_m(T)} \leq 2\, I_0\, \psi_0 \,, \tag{2.2.13}$$

$$\|u_{tt}\|_{L^2(0,T;H^{m-1})} \leq I_0\, \psi_1 \,, \tag{2.2.14}$$

where I_0 depends on the data u_0, u_1, f, and ψ_0 depends exponentially on T and the coefficients (via α_0, a bound a_1 on $a_{ij}(0)$, defined in (2.3.18) *below, and on M_1), and ψ_1 depends on ψ_0, T, and M_2. (The constants I_0, ψ_0 and ψ_1 are defined in* (2.3.21), (2.3.24) *and* (2.3.28) *below).*

(3) Well-Posedness: *As a consequence of* (2.2.13)*, the Cauchy problem* (2.1.1)+(2.1.2) *is well-posed; that is, the map*

$$(u_0, u_1, f) \mapsto u =: \Phi(u_0, u_1, f) \tag{2.2.15}$$

is continuous from $H^{m+1} \times H^m \times L^2(0,T;H^m)$ into $\mathcal{Y}_m(T)$.

(4) Regularity: *If, for some $k \in \mathbb{N}_{\leq m-1}$, and all $r \in \mathbb{N}_{\leq k}$,*

$$\partial_t^r f \in L^2(0,T;H^{m-r})\,, \quad \partial_t^r a_{ij} \in L^2(0,T;H^{s-r})\,, \tag{2.2.16}$$

$$\partial_t^r b_i\,, \quad \partial_t^r c \in L^2(0,T;H^{s-r})\,, \tag{2.2.17}$$

then

$$u \in \bigcap_{\ell=0}^{k+1} C^\ell([0,T];H^{m+1-\ell})\,, \quad \partial_t^{k+2} u \in L^2(0,T;H^{m-k-1})\,. \tag{2.2.18}$$

Remarks. 1) Estimates (2.2.13) and (2.2.14) are a priori, in the sense that they are automatically satisfied by any solution $u \in \mathcal{Y}_m(T)$ that problem (2.1.1)+(2.1.2) may have.

2) As a consequence of the Sobolev product estimates, we can see that the right side of (2.1.1) is in $L^2(0,T;H^m)$. Thus, while each of the two terms at the left side of (2.1.1) is in $L^2(0,T;H^{m-1})$, their sum is in $L^2(0,T;H^m)$. It is then natural to ask whether, separately, $\partial_i\partial_j u$ and $u_{tt} \in L^2(0,T;H^m)$. However, if this were the case, it would follow that $u \in L^2(0,T;H^{m+2})$; hence, by the trace theorem ((1.7.59) of Theorem 1.7.4),

$$u \in C([0,T];H^{m+3/2}) \cap C^1([0,T];H^{m+1/2})\,. \tag{2.2.19}$$

In turn, this would imply that the conditions $u_0 \in H^{m+3/2}$ and $u_1 \in H^{m+1/2}$, as opposed to $u_0 \in H^{m+1}$ and $u_1 \in H^m$ only, would be necessary for the solvability of (2.1.1)+(2.1.2).

3) In the special case that f and the coefficients a_{ij}, b_i, and c are independent of t, the solution to the Cauchy problem for (2.1.1) is in $\mathcal{Y}_m(T)$ for all $T > 0$; in fact, equation (2.1.1) generates a semiflow on the phase space $X_m := H^{m+1} \times H^m$ (see, e.g., Milani and Koksch [**119**, ch. 2, sct. 2.2]). This means that the family $\mathcal{S} = (S(t))_{t\geq 0}$ of solution operators, with $S(t) : X_m \to X_m$ defined by

$$S(t)(u_0, u_1) := (u(t), u_t(t)) \,, \tag{2.2.20}$$

u being the solution to the Cauchy problem (2.1.1)+(2.1.2), satisfies the following four properties:

1) $S(0) = I_m$, the identity in X_m.

2) For all t, $\theta \geq 0$, $S(t)\,S(\theta) = S(\theta)\,S(t) = S(t+\theta)$ (a semigroup property).

3) For all $t \geq 0$, the map $X_m \ni (u_0, u_1) \mapsto S(t)(u_0, u_1) \in X_m$ is continuous (a pointwise property of continuous dependence on the data).

4) For all $(u_0, u_1) \in X_m$, the map $\mathbb{R}_{\geq 0} \ni t \mapsto S(t)(u_0, u_1) \in X_m$ is continuous (a pointwise property of continuity with respect to the family parameter).

(A trivial example of semiflow in $\mathbb{R}^N$ is the family $(\mathrm{e}^{tA})_{t\geq 0}$ of the exponentials of a $N \times N$ matrix A.) ◇

2.3. Proof of Theorem 2.2.1

We prove Theorem 2.2.1 in various steps. First, we establish the a priori estimates (2.2.13) and (2.2.14) for solutions of (2.1.1) in $\mathcal{Y}_m(T)$, and use these estimates to prove the well-posedness of the Cauchy problem (2.1.1)+(2.1.2) in $\mathcal{Y}_m(T)$. Next, we construct sequences of approximate solutions to (2.1.1), and use estimates analogous to the a priori ones, to show that these approximations converge to a limit, which we verify to be the desired solution of (2.1.1)+(2.1.2) in $\mathcal{Y}_m(T)$. Finally, we prove the additional regularity result (2.2.18).

As agreed in part 4 of section 1.1, we denote by C, or K, a universal positive constant, which may vary from estimate to estimate, and even within the same estimate. Without loss of generality, we can take C, $K \geq 1$.

2.3.1. A Priori Estimates and Well-Posedness.

1. We prove that any solution in $\mathcal{Y}_m(T)$ of the Cauchy problem (2.1.1)+(2.1.2) satisfies the estimates (2.2.13) and (2.2.14). To this end, we first assume that u satisfies the additional regularity condition $u \in \mathcal{Y}_{m+1}(T)$; we then remove this assumption, resorting to a regularization process involving the Friedrichs' mollifiers of part 5 of section 1.5.1. The starting point of our procedure is the identity

$$\langle u_{tt} - a_{ij}\,\partial_i\partial_j u, u_t\rangle_m = \langle f + b_j\,\partial_j u + c\,u, u_t\rangle_m\,, \tag{2.3.1}$$

obtained by multiplying (2.1.1) in H^m by u_t, which is legitimate if $u \in \mathcal{Y}_m(T)$. However, in order to proceed we need to split the left side of (2.3.1) into the sum

$$\langle u_{tt}, u_t\rangle_m - \langle a_{ij}\,\partial_i\partial_j u, u_t\rangle_m\,, \tag{2.3.2}$$

and this is no longer legitimate if $u \in \mathcal{Y}_m(T)$ only, since, as we have remarked earlier, neither of the terms u_{tt} and $a_{ij}\,\partial_i\partial_j u$ is guaranteed to be in H^m, even a.e. in t. This explains why we first establish the estimates under the additional regularity assumption that $u \in \mathcal{Y}_{m+1}(T)$, in which case the two terms in (2.3.2) are indeed defined, at least for a.a. $t \in \,]0,T[$.

2. Let $u \in \mathcal{Y}_{m+1}(T)$ be a solution of the Cauchy problem (2.1.1)+(2.1.2). By the additional regularity of u, each term of the left side of (2.1.1) is in H^m for a.a. $t \in \,]0,T[$ (as opposed to only their difference); indeed, this is clear for u_{tt}, while for $a_{ij}\,\partial_i\partial_j u$ it is a consequence of Theorem 1.5.5, because $a_{ij}(t,\cdot) \in \tilde{H}^s$ for a.a. t, and $\partial_i\partial_j\,u(t,\cdot) \in H^m$ for all t. Thus, we can differentiate (2.1.1) with respect to x up to m times; fixing $\alpha \in \mathbb{N}^N$, with $|\alpha| \le m$, we obtain

$$(\partial_x^\alpha u)_{tt} - a_{ij}\partial_i\partial_j(\partial_x^\alpha u) = \partial_x^\alpha(f + b_i\partial_i u + c\,u) + G_\alpha(a_{ij}, \partial_i\partial_j u) =: R_\alpha\,, \tag{2.3.3}$$

where G_α is the commutator defined in (1.5.204); that is, $G_0 = 0$ and, if $\alpha > 0$,

$$G_\alpha(a_{ij}, \partial_i\partial_j u) = \sum_{0<\beta\le\alpha} \tbinom{\alpha}{\beta}\partial_x^\beta a_{ij}\;\partial_x^{\alpha-\beta}\partial_i\partial_j u\,. \tag{2.3.4}$$

The key point of this argument is that $R_\alpha(t) \in H^{m-|\alpha|}$ for a.a. $t \in \,]0,T[$, and its norm in this space can be estimated in terms of the norm of $u(t)$ in at most H^{m+1} (that is, explicitly, *not* requiring the additional regularity $u \in \mathcal{X}_{m+1}(T)$). This is a consequence of Proposition 1.5.10, with $\zeta = a_{ij}(t)$ and $w = \partial_i\partial_j u(t)$, which are, respectively, in $\tilde{H}^s$ and H^m for a.a. $t \in [0,T]$. Consequently, from (1.5.205) we deduce that

$$\|G_\alpha(a_{ij}, \partial_i\partial_j u)\|_{m-|\alpha|} \le C\,\|\nabla a_{ij}\|_{s-1}\,\|\nabla u\|_m\,. \tag{2.3.5}$$

From the analogous estimates for the terms $\partial_x^\alpha(b_i\partial_i u + cu)$, the definition of μ_1 in (2.2.3), and keeping in mind that $\partial_x^\alpha f(t) \in H^{m-|\alpha|}$ for a.a. t, (2.3.3) and (2.3.5) yield

$$\|R_\alpha\|_{m-|\alpha|} \le \|f\|_m + C\,\mu_1(t)\,(\|\nabla u\|_m + \|u\|_m)\ . \tag{2.3.6}$$

Since $H^{m-|\alpha|} \subseteq L^2$, we can multiply (2.3.3) in L^2 by $2\partial_x^\alpha u_t$ (which is in $H^{m+1-|\alpha|}$ for all t). At the left side of the resulting identity, we note that, again by the additional regularity of u, the function $t \mapsto \langle u_{tt}(t), u_t(t)\rangle_m$ is in $L^1(0,T)$; thus, by (1.7.63) of Proposition 1.7.5, with $m=0$,

$$2\,\langle \partial_x^\alpha u_{tt}, \partial_x^\alpha u_t\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\,\|\partial_x^\alpha u_t\|^2\ . \tag{2.3.7}$$

Likewise, the functions

$$t \;\mapsto\; 2\langle a_{ij}(t)\partial_i\partial_x^\alpha u(t), \partial_j\partial_x^\alpha u_t(t)\rangle\ , \tag{2.3.8}$$

$$t \;\mapsto\; \langle \partial_t\, a_{ij}(t)\,\partial_i\partial_x^\alpha u(t), \partial_j\partial_x^\alpha u(t)\rangle\ , \tag{2.3.9}$$

are integrable on $[0,T]$: the first, since $a_{ij}\in L^\infty(Q)$ and $D\,u\in C([0,T];H^m)$, and the second because, in addition, $\partial_t\, a_{ij} \in L^1(0,T;H^{s-1})$, by the first of (2.2.2). As a consequence, the function

$$t \;\mapsto\; \langle a_{ij}(t)\partial_i\partial_x^\alpha u(t), \partial_j\partial_x^\alpha u(t)\rangle \tag{2.3.10}$$

is absolutely continuous on $[0,T]$, since its distributional derivative is the sum of the two functions in (2.3.8) and (2.3.9), as can be seen by adapting the proof of Theorem 1.5.4. Thus, by the symmetry of the coefficients a_{ij}, we can compute that

$$\begin{aligned} &-2\,\langle a_{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u_t\rangle \\ &\quad= 2\,\langle a_{ij}\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u_t\rangle + 2\,\langle \partial_j a_{ij}\,\partial_i\partial_x^\alpha u, \partial_x^\alpha u_t\rangle \\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}\,\langle a_{ij}\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u\rangle + 2\,\langle \partial_j a_{ij}\,\partial_i\partial_x^\alpha u, \partial_x^\alpha u_t\rangle \\ &\qquad - \langle \partial_t a_{ij}\,\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u\rangle\ . \end{aligned} \tag{2.3.11}$$

Consequently, from (2.3.3),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,\big(\|\partial_x^\alpha u_t\|^2 + \langle a_{ij}\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u\rangle\big) \;&=\; \Lambda_\alpha \\ &:= 2\,\langle R_\alpha - \partial_j a_{ij}\,\partial_i\partial_x^\alpha u, \partial_x^\alpha u_t\rangle + \langle \partial_t a_{ij}\,\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u\rangle\ . \end{aligned} \tag{2.3.12}$$

We now set, for $k \in \mathbb{N}_{\ge 0}$ and $u \in H^{k+1}$,

$$Q_k(a,\nabla u) \;:=\; \sum_{|\alpha|\le k} \langle a_{ij}\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u\rangle\ ; \tag{2.3.13}$$

note that, by (2.2.1),

$$\alpha_0\,\|\nabla u\|_k^2 \le Q_k(a,\nabla u) \le \alpha_1\,\|\nabla u\|_k^2\ . \tag{2.3.14}$$

Summing all identities (2.3.12) for $|\alpha| \le m$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|_m^2 + Q_m(a,\nabla u)\right) = \sum_{|\alpha|\le m} \Lambda_\alpha \; =: \Lambda\,. \tag{2.3.15}$$

Recalling (2.3.6), by Schwarz' inequality (1.4.16) we deduce from (2.3.15) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|_m^2 + Q_m(a,\nabla u)\right) &\le 2\,\|f\|_m\,\|u_t\|_m \\ &\quad + 2\,C\mu_1(t)\,\left(\|\nabla u\|_m + \|u\|_m\right)\,\|u_t\|_m + \mu_1(t)\|\nabla u\|_m^2 \\ &\le \|f\|_m^2 + C\,(1+\mu_1(t))\left(\|u\|_m^2 + \|D\,u\|_m^2\right)\,,\end{aligned} \tag{2.3.16}$$

from which, integrating in $[0,t]$, and recalling (2.3.14), $0 < t \le T$,

$$\begin{aligned}\|u_t(t)\|_m^2 + \alpha_0\,\|\nabla u(t)\|_m^2 \; &\le \|u_t(t)\|_m^2 + Q_m(a(t),\nabla u(t)) \\ &\le \; \|u_1\|_m^2 + Q_m(a(0),\nabla u_0) + \int_0^t \|f\|_m^2\,\mathrm{d}\theta \\ &\quad + C\int_0^t (1+\mu_1(\theta))\left(\|u_t\|_m^2 + \|\nabla u\|_m^2 + \|u\|_m^2\right)\,\mathrm{d}\theta \\ &\le \; \|u_1\|_m^2 + \alpha_1\,\|\nabla u_0\|_m^2 + \int_0^T \|f\|_m^2\,\mathrm{d}\theta \\ &\quad + C\int_0^t (1+\mu_1(\theta))\left(\|u_t\|_m^2 + \|\nabla u\|_m^2 + \|u\|_m^2\right)\,\mathrm{d}\theta\,.\end{aligned} \tag{2.3.17}$$

For future reference, we note that estimate (2.3.17) remains true if the second of (2.3.14) holds only at $t = 0$; that is, if we replace α_1 by

$$a_1 := \max_{i,\,j} |a_{ij}(0,\cdot)|_\infty\,. \tag{2.3.18}$$

Integrating the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|^2 = 2\langle u, u_t\rangle \le \|u\|^2 + \|u_t\|^2\,, \tag{2.3.19}$$

we obtain

$$\|u(t)\|^2 \le \|u_0\|^2 + \int_0^t \left(\|u\|^2 + \|u_t\|^2\right)\,\mathrm{d}\theta\,. \tag{2.3.20}$$

We now set

$$I_0^2 \;:=\; \|u_1\|_m^2 + \|u_0\|_{m+1}^2 + \int_0^T \|f\|_m^2\,\mathrm{d}\theta\,, \tag{2.3.21}$$

$$a_2^2 \;:=\; \frac{\max\{1,a_1\}}{\min\{1,\alpha_0\}}\,. \tag{2.3.22}$$

Adding (2.3.20) to (2.3.17), we deduce that

$$\|u(t)\|^2 + \|D\,u(t)\|_m^2 \le a_2^2\, I_0^2 + C\, a_2^2 \int_0^t (1+\mu_1(\theta))\,\big(\|u\|^2 + \|D\,u\|_m^2\big)\,\mathrm{d}\theta\,. \tag{2.3.23}$$

Consequently, recalling (2.2.4) and setting further

$$\psi_0^2 := a_2^2\, \exp\big(C\, a_2^2\,(T + M_1)\big)\,, \tag{2.3.24}$$

by Gronwall's inequality we obtain from (2.3.23) that, for all $t \in [0,T]$,

$$\|u(t)\|^2 + \|D\,u(t)\|_m^2 \le I_0^2\,\psi_0^2\,. \tag{2.3.25}$$

Thus, we conclude that if $u \in \mathcal{Y}_{m+1}(T)$, u does satisfies estimate (2.2.13).

Finally, from equation (2.1.1) itself, using the Sobolev product estimates of Theorem 1.5.4 we deduce that, for all $t \in [0,T]$,

$$\begin{aligned} \|u_{tt}(t)\|_{m-1} \;\le\; & \|f(t)\|_{m-1} + \|a_{ij}(t)\,\partial_i\partial_j u(t)\|_{m-1} \\ & + \|b_i(t)\|_{s-1}\,\|\partial_i u(t)\|_{m-1} \\ & + \|c(t)\|_{s-1}\,\|u(t)\|_{m-1}\,. \end{aligned} \tag{2.3.26}$$

As we have already remarked, $a_{ij}(t) \in \tilde{H}^s$ for a.a. $t \in [0,T]$; hence, by (1.5.94) of Theorem 1.5.5,

$$\|a_{ij}(t)\,\partial_i\partial_j u(t)\|_{m-1} \le C\,\left(|a_{ij}(t)|_\infty + \|\nabla a_{ij}(t)\|_{s-2}\right)\,\|\nabla u(t)\|_m\,. \tag{2.3.27}$$

Consequently, recalling (2.2.5) and (2.2.6), we deduce from (2.3.26) and (2.3.27) that

$$\begin{aligned} \int_0^T \|u_{tt}\|_{m-1}^2\,\mathrm{d}t \;&\le\; C \int_0^T \|f\|_{m-1}^2\,\mathrm{d}t + C\,M_2^2\,\|u\|_{\mathcal{X}_m(T)}^2 \\ &\le\; C\,I_0^2(1 + M_2^2\,\psi_0^2) =: I_0^2\,\psi_1^2\,, \end{aligned} \tag{2.3.28}$$

from which (2.2.14) follows. Note that the proof of (2.2.14) does not require the extra regularity $u \in \mathcal{Y}_{m+1}(T)$; in fact, (2.3.28) shows that the condition $u \in \mathcal{X}_m(T)$ is sufficient to deduce (2.2.14). For future reference, we record the fact that the constant ψ_0 and, therefore, ψ_1, depend of the coefficients only through the quantities M_1 and M_2 of (2.2.4) and (2.2.6); in particular, they depend on the a_{ij}'s through α_0, a_1, and the norms of a_{ij} in $L^\infty(Q)$, and $D\,a_{ij}$ in $L^1(0,T;H^{s-1})$.

Remark. An estimate similar to (2.2.13) can also be obtained under the less restrictive assumption that $f \in L^1(0,T;H^m)$. To see this, it is sufficient to modify estimate (2.3.17) into

$$\begin{aligned}\|u_t(t)\|_m^2 + Q_m(a(t),\nabla u(t)) &\le \|u_1\|_m^2 + Q_m(a(0),\nabla u_0)\\ &+ 2\int_0^t \|f\|_m \left(\|u_t\|_m^2 + Q_m(a,\nabla u) + \|u\|_m^2\right)^{1/2} \mathrm{d}\theta \\ &+ C\int_0^t (1+\mu_1(\theta))\left(\|u_t\|_m^2 + \|\nabla u\|_m^2 + \|u\|_m^2\right) \mathrm{d}\theta\,, \end{aligned} \tag{2.3.29}$$

add this to (2.3.20), and then use the modified version of Gronwall's inequality given in the following Proposition. ◇

Proposition 2.3.1. *Let $c > 0$, $[a, b] \subseteq \mathbb{R}$, $u \in C([a, b]; \mathbb{R}_{\ge 0})$, and v, $w \in L^1(a, b; \mathbb{R}_{\ge 0})$. Assume that, for all $t \in [a, b]$,*

$$u(t) \le c^2 + 2\int_a^t v(\theta)\sqrt{u(\theta)}\,\mathrm{d}\theta + \int_a^t w(\theta)\,u(\theta)\,\mathrm{d}\theta\,. \tag{2.3.30}$$

Then, u satisfies, in $[a, b]$, the inequality

$$u(t) \le \left(c + \int_a^t v(\theta)\,\mathrm{d}\theta\right)^2 \exp\left(\int_a^t w(\theta)\,\mathrm{d}\theta\right)\,. \tag{2.3.31}$$

Proof. Call $R(t)$ the right side of (2.3.30). Then, $R(t) \ge u(t)$, $R(t) \ge c^2 > 0$ for all $t \in [a, b]$, and the function $t \mapsto \sqrt{R(t)}$ satisfies the differential inequality

$$2\,\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{R} - w\sqrt{R} \le 2\,v\,. \tag{2.3.32}$$

Integrating (2.3.32), we find that

$$\sqrt{R(t)} \le \left(c + \int_a^t v(\theta)\,\mathrm{d}\theta\right) \exp\left(\frac{1}{2}\int_a^t w(\theta)\,\mathrm{d}\theta\right)\,. \tag{2.3.33}$$

Squaring this, and keeping in mind that $u(t) \le R(t)$, (2.3.31) follows. □

Remark. As in the standard Gronwall's inequality, Proposition 2.3.1 can be generalized to the case in which the constant c^2 in (2.3.30) is replaced by a variable $c^2(t)$, the map $t \mapsto c^2(t)$ being positive, differentiable and increasing in $[a, b]$. ◇

3. We now proceed to obtain (2.2.13), from which, as we have seen, (2.2.14) follows, assuming that $u \in \mathcal{Y}_m(T)$ only. To this end, we regularize u by means of the Friedrichs' mollifiers $\left(\rho^\delta\right)_{\delta>0}$ in the space variables, introduced in part 2 of section 1.4; that is, we set

$$u^\delta(t,x) := [\rho^\delta * u(t,\cdot)](x) = \frac{1}{\delta^N}\int \rho\left(\tfrac{x-y}{\delta}\right) u(t,y)\,\mathrm{d}y\,. \tag{2.3.34}$$

From (2.1.1), we derive that for each $\delta > 0$, u^δ solves the equation

$$u^\delta_{tt} - a_{ij}\partial_i\partial_j u^\delta = f^\delta + b_i\partial_i u^\delta + c\,u^\delta + F^\delta + L^\delta\,, \tag{2.3.35}$$

where, omitting the variables t and x, which are as in (2.3.34), $f^\delta := \rho^\delta * f$, and

$$F^\delta := \rho^\delta * (a_{ij}\partial_i\partial_j u) - a_{ij}\,\partial_i\partial_j u^\delta\,, \tag{2.3.36}$$
$$L^\delta := \rho^\delta * (b_i\partial_i u + cu) - (b_i\partial_i u^\delta + c\,u^\delta)\,. \tag{2.3.37}$$

Since $u^\delta \in \mathcal{Y}_{m+1}(T)$, it satisfies estimate (2.3.25), with I_0^2 replaced by

$$\begin{aligned}(I_0^\delta)^2 &:= \|u_1^\delta\|_m^2 + \|u_0^\delta\|_{m+1}^2 \\ &\qquad + 3\int_0^T \Big(\|f^\delta\|_m^2 + \|F^\delta\|_m^2 + \|L^\delta\|_m^2\Big)\,\mathrm{d}\theta\,;\end{aligned} \tag{2.3.38}$$

that is,

$$\|u^\delta(t)\|_m^2 + \|D\,u^\delta(t)\|_m^2 \le 3(I_0^\delta\,\psi_0)^2\,, \qquad t\in[0,T]\,. \tag{2.3.39}$$

Since $u_0 \in H^{m+1}$, $u_1 \in H^m$, and $f(t) \in H^m$ for a.a. $t \in [0,T]$, by Theorems 1.5.1 and 1.5.6, and Corollary 1.5.4, it follows that

$$\|u_1^\delta\|_m \le \|u_1\|_m\,, \quad \|u_0^\delta\|_{m+1} \le \|u_0\|_{m+1} \quad \|f^\delta(t)\|_m \le \|f(t)\|_m\,, \tag{2.3.40}$$

as well as

$$\|F^\delta(t)\|_m \le C\,\|\nabla a_{ij}(t)\|_{s-1}\|\nabla u(t)\|_m\,, \tag{2.3.41}$$

$$\|L^\delta(t)\|_m \le C\,\left(\|b_j(t)\|_s + \|c(t)\|_s\right)\,\|u(t)\|_m\,. \tag{2.3.42}$$

Moreover, as $\delta \to 0$, $u_0^\delta \to u_0$ in H^{m+1}, $u_1^\delta \to u_1$ in H^m, and, for a.a. $t \in [0,T]$,

$$\|f^\delta(t)\|_m \to \|f(t)\|_m\,, \quad \|F^\delta(t)\|_m \to 0\,, \quad \|L^\delta(t)\|_m \to 0\,. \tag{2.3.43}$$

This allows us to pass to the limit as $\delta \to 0$ in (2.3.38), by Lebesgue's dominated convergence Theorem. Since also

$$\|u^\delta(t)\|_m^2 + \|D\,u^\delta(t)\|_m^2 \to \|u(t)\|_m^2 + \|D\,u(t)\|_m^2\,, \tag{2.3.44}$$

we deduce from (2.3.39) that, for all $t \in [0,T]$,

$$\|u(t)\|_m^2 + \|D\,u(t)\|_m^2 \le 3\,I_0^2\,\psi_0^2\,; \tag{2.3.45}$$

that is, u satisfies (2.2.13), as claimed. □

4. As the proof of (2.2.14) shows, the regularity of u_{tt} is automatically determined by the regularity of f, u and the coefficients, via (2.1.1). We shall exploit this fact in the proof of the second regularity estimate in (2.2.18). In particular, we note that if, in addition to (2.2.2), a_{ij}, b_j, $c \in C([0,T];H^{s-1})$, and $f \in L^2(0,T;H^m) \cap C([0,T];H^{m-1})$, then $u_{tt} \in C([0,T];H^{m-1})$. For future reference, we summarize this conclusion, introducing, for $m \in \mathbb{N}$, the spaces

$$\mathcal{V}_m(T) := L^2(0,T;H^{m+1}) \cap C([0,T];H^m)\,, \tag{2.3.46}$$

$$\mathcal{Z}_m(T) := \bigcap_{j=0}^{2} C^j([0,T];H^{m+1-j})\,, \tag{2.3.47}$$

which are Banach spaces with respect to their natural norms, defined by

$$\|u\|^2_{\mathcal{V}_m(T)} := \int_0^T \|u\|^2_{m+1}\,\mathrm{d}t + \max_{0\le t\le T} \|u(t)\|^2_m\,, \tag{2.3.48}$$

$$\|u\|^2_{\mathcal{Z}_m(T)} := \sum_{j=0}^{2} \|\partial_t^j u\|^2_{C([0,T];H^{m+1-j})}\,. \tag{2.3.49}$$

Note that $\mathcal{Z}_m(T) \hookrightarrow \mathcal{Y}_m(T) \hookrightarrow \mathcal{X}_m(T) \hookrightarrow \mathcal{V}_m(T)$. Theorem 2.2.1 can then be supplemented by

Theorem 2.3.1. *Let $m \in \mathbb{N}$, with $1 \le m \le s$. If, in addition to the assumptions of Theorem* 2.2.1, *a_{ij}, b_j, $c \in C([0,T];\tilde{H}^{s-1})$, and $f \in \mathcal{V}_{m-1}(T)$, then the solution $u \in \mathcal{Y}_m(T)$ of the Cauchy problem* (2.1.1)+(2.1.2) *is in $\mathcal{Z}_m(T)$, and*

$$\|u\|_{\mathcal{Z}_m(T)} \le I_1\,\psi_1\,, \tag{2.3.50}$$

where $I_1^2 := (2I_0)^2 + \|f\|^2_{C([0,T];H^{m-1})}$.

Remark. In subsequent parts of these lectures, we will often establish a priori estimates similar to (2.2.13) and (2.2.14). We shall most often do so only formally, in the sense that we agree to have verified that it is possible to resort to a regularization process like the one followed in parts 2 and 3 above. ◇

5. As a consequence of the a priori estimates (2.2.13) and (2.2.14), we can show that problem (2.1.1)+(2.1.2) is well-posed in $\mathcal{Y}_m(T)$. Indeed, (2.2.13) and (2.2.14) imply that, for any solution $u \in \mathcal{Y}_m(T)$ of (2.1.1)+(2.1.2),

$$\|u\|^2_{\mathcal{Y}_m(T)} \le C\,(\psi_0 + \psi_1)^2 \left(\|u_0\|^2_{m+1} + \|u_1\|^2_m + \int_0^T \|f\|^2_m\,\mathrm{d}t \right). \tag{2.3.51}$$

Let u and $\tilde{u}$ be solutions of (2.1.1), corresponding to data $\{u_0, u_1, f\}$ and $\{\tilde{u}_0, \tilde{u}_1, \tilde{f}\}$. Then, their difference $z := u - \tilde{u}$ satisfies the Cauchy problem

$$\begin{cases} z_{tt} - a_{ij}\,\partial_i\partial_j z = (f - \tilde{f}) + b_j\,\partial_j z + c\,z\,, \\ z(0) = u_0 - \tilde{u}_0\,, \qquad z_t(0) = u_1 - \tilde{u}_1\,; \end{cases} \tag{2.3.52}$$

hence, applying (2.3.51) to z, we deduce that

(2.3.53)
$$\|u-\tilde u\|^2_{\mathcal{Y}_m(T)} \le C\left(\|u_0-\tilde u_0\|^2_{m+1}+\|u_1-\tilde u_1\|^2_m+\int_0^T \|f-\tilde f\|^2_m\,\mathrm{d}t\right),$$

where C depends only on the coefficients and T. This means that the norm of u in $\mathcal{Y}_m(T)$ depends continuously on that of the data. In particular, (2.3.53) implies that strong solutions of the Cauchy problem (2.1.1)+(2.1.2) are unique. □

2.3.2. Existence of Strong Solutions.

We prove the existence of a solution to problem (2.1.1)+(2.1.2), which we construct by means of a double approximation argument, in four steps. In the first, we restrict (2.1.1)+(2.1.2) to a sequence of expanding balls in $\mathbb{R}^N$, at the boundary of which we impose homogeneous Dirichlet boundary conditions; in the second, we solve the corresponding restricted initial-boundary value problems by means of a Galerkin sequence of approximations. Next, we extend each of these solutions to the whole space $\mathbb{R}^N$, thereby producing a sequence of approximated solutions on $\mathbb{R}^N$; finally, we show that these solutions converge to a limit, which we identify as the desired solution of (2.1.1)+(2.1.2).

1. For $\ell\in\mathbb{N}$, we set $\Omega_\ell := \{x\in\mathbb{R}^N \mid |x|<\ell+1\}$ (the open ball of $\mathbb{R}^N$ of center 0 and radius $\ell+1$), and denote by $\|\cdot\|_{m,\ell}$ and $\langle\cdot,\cdot\rangle_{m,\ell}$ the norm and scalar product on $H^m(\Omega_\ell)$. We also denote by $[\,\cdot\,]_{m,\ell}$ the norm N_1 on $H^m_\Delta(\Omega_\ell)$ defined by (1.5.137). For $r\in\,]-1,1[$, we let

(2.3.54)
$$\varphi(r) := \exp\left(\frac{-r^2}{1-r^2}\right),$$

and define $\zeta_\ell\in C_0^\infty(\mathbb{R}^N)$ by

(2.3.55)
$$\zeta_\ell(x) := \begin{cases} 1 & \text{if} \quad |x|\le \ell\,, \\ \varphi(|x|-\ell) & \text{if} \quad \ell<|x|<\ell+1\,, \\ 0 & \text{if} \quad |x|\ge \ell+1\,. \end{cases}$$

Then, each ζ_ℓ is a cut-off function in $\mathbb{R}^N$, since $0\le\zeta_\ell(x)\le 1$ for all $x\in\mathbb{R}^N$, $\zeta_\ell(x)\equiv 1$ on $\Omega_{\ell-1}$ (we agree that $\Omega_{-1}=\{0\}$), and $\zeta_\ell\equiv 0$ off Ω_ℓ. In addition, for each $\alpha\in\mathbb{N}^N$ there is $C_\alpha>0$ such that, for all $\ell\in\mathbb{N}$,

(2.3.56)
$$\sup_{x\in\mathbb{R}^N}|\partial_x^\alpha\zeta_\ell(x)|\le C_\alpha\,,$$

and, since $\varphi^{(j)}(r) \to 0$ as $|r| \to 1^-$ for all $j \in \mathbb{N}$,

$$\frac{\partial^j}{\partial \nu_\ell^j}\, \zeta_\ell = 0 \qquad \text{on} \quad \partial\Omega_\ell\,, \tag{2.3.57}$$

where ν_ℓ denotes the outward unit normal to $\partial\Omega_\ell$. Finally, we abbreviate $-\Delta_\ell := -\Delta_{\Omega_\ell}$ (the Laplace operator with Dirichlet boundary conditions on $\partial\Omega_\ell$, introduced in section 1.5.2), denote by E_ℓ and R_ℓ the extension and restriction operators defined in part 4 of section 1.5.1, with $\Omega = \Omega_\ell$, and, given a function h defined on $\mathbb{R}^N$ (respectively, on Q), set

$$h^\ell := R_\ell(\zeta_\ell\, h) \qquad (\text{resp.,} \quad h^\ell(t,\cdot) := R_\ell(\zeta_\ell\, h(t,\cdot)\,)\,. \tag{2.3.58}$$

Note that (2.3.57) implies that $h^\ell \in H_0^m(\Omega_\ell)$ if $h \in H^m(\mathbb{R}^N)$, and, by (2.3.56),

$$\|h^\ell\|_{H^m(\Omega_\ell)} \le C_m\, \|h\|_{H^m(\mathbb{R}^N)}\,, \tag{2.3.59}$$

with $C_m := \max_{|\alpha|\le m} \big(\{\max_{\beta\le\alpha} \binom{\alpha}{\beta}\}\, C_\alpha\big)$.

2. For each fixed $\ell \in \mathbb{N}$, we consider the initial-boundary value problem

$$\begin{cases} u_{tt} - a_{ij}^\ell\, \partial_i\partial_j u = f^\ell + b_i^\ell\, \partial_i u + c^\ell\, u & \text{in} \quad]0,T[\,\times\Omega_\ell\,, \\ u(0,\cdot) = u_0^\ell\,, \qquad u_t(0,\cdot) = u_1^\ell & \text{on} \quad \{0\}\times\Omega_\ell\,, \\ u(t,\cdot)\big|_{\partial\Omega_\ell} = 0 & \text{on} \quad]0,T[\,\times\partial\Omega_\ell\,. \end{cases} \tag{2.3.60}$$

Our goal is to solve (2.3.60) by means of a Galerkin approximation method, with the choice, as a total basis of $L^2(\Omega_\ell)$, of the sequence $\mathcal{W} = (w_j)_{j\ge1}$ of the eigenfunctions of $-\Delta_\ell$. As we noted in part 3 of section 1.5.2, $\mathcal{W}$ is orthogonal in $H_0^m(\Omega_\ell)$ (endowed with the scalar product induced by $[\,\cdot\,]_{m,\ell}$), and $H_0^{m+k}(\Omega_\ell)$-regular, for all $m \in \mathbb{N}$ and $k \in \mathbb{N}_{\ge1}$. For each $n \in \mathbb{N}_{\ge1}$, we set

$$\mathcal{W}_n \;:=\; \operatorname{span}\{w_j \;\mid\; j \le n\}\,, \tag{2.3.61}$$

and project problem (2.3.60) into each finite-dimensional space $\mathcal{W}_n$. Heuristically, this means that, instead of (2.3.60), we consider the infinite set of scalar equations

$$\langle u_{tt} - a_{ij}^\ell\, \partial_i\partial_j u, w_r\rangle_{0,\ell} = \langle f^\ell + b_i^\ell\, \partial_i u + c^\ell\, u, w_r\rangle_{0,\ell}\,, \qquad r \ge 1\,, \tag{2.3.62}$$

each of which is to hold in $\mathbb{R}$, for a.a. $t \in\,]0,T[$.

To implement Galerkin's method, we proceed as follows. Since the restricted initial values u_0^ℓ and u_1^ℓ are in $H_0^{m+1}(\Omega_\ell)$ and $H_0^m(\Omega_\ell)$, respectively, by (1.5.147) of section 1.5.2 the sequences $(u_0^{n,\ell})_{n\ge0}$ and $(u_1^{n,\ell})_{n\ge0}$, defined respectively by

$$u_0^{n,\ell} \;:=\; \sum_{k=1}^n \langle u_0^\ell, w_k\rangle_{0,\ell}\, w_k\,, \qquad u_1^{n,\ell} \;:=\; \sum_{k=1}^n \langle u_1^\ell, w_k\rangle_{0,\ell}\, w_k\,, \tag{2.3.63}$$

are such that $u_0^{n,\ell}$, $u_1^{n,\ell} \in \mathcal{W}_n$ for all $n \geq 1$, and, as $n \to +\infty$,

$$u_0^{n,\ell} \to u_0^\ell \quad \text{in } H^{m+1}(\Omega_\ell)\,, \qquad u_1^{n,\ell} \to u_1^\ell \quad \text{in } H^m(\Omega_\ell)\,. \tag{2.3.64}$$

In fact, recalling that P_n is an orthogonal projection in $H_0^{m+1}(\Omega_\ell)$, by the first of (1.5.139) and by (1.5.25) and (2.3.56), it follows that

$$[u_0^{n,\ell}]_{m+1,\ell} \leq \|u_0^\ell\|_{m+1,\ell} = \|\zeta_\ell\, u_0\|_{m+1} \leq C_m\, \|u_0\|_{m+1}\,; \tag{2.3.65}$$

analogously,

$$[u_1^{n,\ell}]_{m,\ell} \leq C_m\, \|u_1\|_m\,. \tag{2.3.66}$$

Since also $[f^\ell(t)]_{m,\ell} \leq C_m\, \|f(t)\|_m$, we conclude that there exists $M > 0$, independent of n and of ℓ, such that for all $n \geq 1$ and $\ell \geq 0$,

$$\|u_0^{n,\ell}\|_{m+1,\ell}^2 + \|u_1^{n,\ell}\|_{m,\ell}^2 + \int_0^T \|f^\ell(t)\|_{m,\ell}^2\, \mathrm{d}t \leq M^2\,. \tag{2.3.67}$$

In analogy to the technique of separation of variables, we wish to find functions $u^{n,\ell} : [0,T] \to \mathcal{W}_n$, that is, of the form

$$u^{n,\ell}(t,x) = \sum_{k=1}^n \gamma_{kn}(t) w_k(x)\,, \tag{2.3.68}$$

which are approximate solutions of (2.3.62), in the sense that they should solve the n equations

$$\langle u_{tt}^{n,\ell} - a_{ij}^\ell\, \partial_i \partial_j u^{n,\ell}, w_r\rangle_{0,\ell} = \langle f^\ell + b_i^\ell\, \partial_i u^{n,\ell} + c^\ell\, u^{n,\ell}, w_r\rangle_{0,\ell}\,, \tag{2.3.69}$$

for $1 \leq r \leq n$. In addition, and consistently with (2.3.63), $u^{n,\ell}$ should satisfy the initial conditions

$$u^{n,\ell}(0,\cdot) = u_0^{n,\ell}\,, \qquad u_t^{n,\ell}(0,\cdot) = u_1^{n,\ell}\,. \tag{2.3.70}$$

We rewrite (2.3.69) in the form

$$\langle u_{tt}^{n,\ell} - \partial_j(a_{ij}^\ell\, \partial_i u^{n,\ell}), w_r\rangle_{0,\ell} = \langle f^\ell + \tilde{b}_i^\ell\, \partial_i u^{n,\ell} + c^\ell\, u^{n,\ell}, w_r\rangle_{0,\ell}\,, \tag{2.3.71}$$

where $\tilde{b}_i := b_i - \partial_j a_{ij}$. Since the system $\mathcal{W}$ is orthonormal in $L^2(\Omega_\ell)$ and contained in $H_0^1(\Omega_\ell)$, (2.3.71) is equivalent to the finite system of n scalar second order ordinary differential equations

$$\begin{cases} \gamma''_{rn}(t) + \displaystyle\sum_{k=1}^n \gamma_{kn}(t) \langle a_{ij}^\ell\, \partial_i w_k, \partial_j w_r\rangle_{0,\ell} \\ \qquad = \langle f^\ell, w_r\rangle_{0,\ell} + \displaystyle\sum_{k=1}^n \gamma_{kn}(t) \langle \tilde{b}_i^\ell\, \partial_i w_k + c^\ell\, w_k, w_r\rangle_{0,\ell}\,, \\ 1 \leq r \leq n\,, \end{cases} \tag{2.3.72}$$

in the unknowns $(\gamma_{rn}(t))_{1\le r\le n} =: \gamma_n(t) \in \mathbb{R}^n$. Comparing (2.3.68) for $t=0$ with (2.3.70) and (2.3.63), we attach to system (2.3.72) the initial conditions

$$\text{(2.3.73)} \qquad \begin{cases} \gamma_n(0) &= \gamma_0^n := \Big(\langle u_0^\ell, w_r\rangle_{0,\ell}\Big)_{1\le r\le n} \in \mathbb{R}^n\,, \\ \gamma_n{}'(0) &= \gamma_1^n := \Big(\langle u_1^\ell, w_r\rangle_{0,\ell}\Big)_{1\le r\le n} \in \mathbb{R}^n\,. \end{cases}$$

We can solve the Cauchy problem (2.3.72)+(2.3.73) by means of Carathéodory's theorem (see, e.g., Coddington and Levinson [**37**, ch. 1, § 2]), which yields a solution $\gamma_n \in C^1([0,T];\mathbb{R}^n)$ of (2.3.72) (for a.a. $t\in[0,T]$) and (2.3.73), with $\gamma'_n \in AC([0,T];\mathbb{R}^n)$. This solution is global on $[0,T]$, because (2.3.72) is linear (see the remark at the end of the proof of Lemma 2.3.1 below). We then *define* the function $u^{n,\ell}$ by (2.3.68); clearly,

$$\text{(2.3.74)} \qquad u^{n,\ell} \in C^1([0,T];\mathcal{W}_n)\,, \qquad u_t^{n,\ell} \in AC([0,T];\mathcal{W}_n)\,.$$

By construction, $u^{n,\ell}$ solves (2.3.71), a.e. in $[0,T]$; in fact, it also solves (2.3.69). Moreover, since Ω_ℓ is a ball, $\mathcal{W}_n \subset C^\infty(\overline{\Omega_\ell})$, so that (2.3.74) implies that

$$\text{(2.3.75)} \qquad u^{n,\ell}(t) \in H^2(\Omega_\ell)\,, \qquad u_{tt}^{n,\ell}(t) \in L^2(\Omega_\ell)$$

for all (respectively, almost all) $t\in[0,T]$, and (2.3.69) makes sense. Moreover, by (2.3.73), (2.3.68), and (2.3.63), $u^{n,\ell}$ also satisfies the initial conditions (2.3.70).

3. We now introduce, for $k\in\mathbb{N}$, the spaces

$$\text{(2.3.76)} \qquad W_0^k(\Omega_\ell) := \begin{cases} L^2(\Omega_\ell)\,, & \text{if} \quad k=0\,, \\ H^k(\Omega_\ell)\cap H_0^1(\Omega_\ell)\,, & \text{if} \quad k\ge 1\,, \end{cases}$$

and

$$\text{(2.3.77)} \qquad \mathcal{X}_{k,\ell}(T) := C([0,T];W_0^{k+1}(\Omega_\ell)) \cap C^1([0,T];W_0^k(\Omega_\ell))\,.$$

We claim:

Lemma 2.3.1. *The sequence $\big(u^{n,\ell}\big)_{n\ge1}$ is in a bounded set of $\mathcal{X}_{k,\ell}(T)$, $0\le k\le m$. The bounds on this sequence depend only on the data and the coefficients of* (2.1.1)+(2.1.2)*, via the constants M of* (2.3.67) *and M_1 of* (2.2.4)*, respectively. In particular, these bounds are independent of ℓ.*

Proof. We proceed by induction on k. For $k=0$, we multiply equation (2.3.71) by $\gamma_{rn}{}'(t)$, and then sum the resulting identities for $1\le r\le n$, to obtain

(2.3.78)

$$\langle u_{tt}^{n,\ell}, u_t^{n,\ell}\rangle_{0,\ell} + \langle a_{ij}^\ell \partial_i u^{n,\ell}, \partial_j u_t^{n,\ell}\rangle_{0,\ell} = \langle f^\ell + \tilde b_i^\ell\,\partial_i u^{n,\ell} + c^\ell\,u^{n,\ell}, u_t^{n,\ell}\rangle_{0,\ell}\,.$$

From this we obtain, as in (2.3.15),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_t^{n,\ell}\|_{0,\ell}^2 + Q_0(a^\ell, \nabla u^{n,\ell})\Big) = \langle \partial_t(a_{ij}^\ell)\,\partial_i u^{n,\ell}, \partial_j u^{n,\ell}\rangle_{0,\ell} + 2\langle f^\ell + \tilde b_i^\ell\,\partial_i u^{n,\ell} + c^\ell\,u^{n,\ell}, u_t^{n,\ell}\rangle_{0,\ell} =: R_{01} + R_{02}\,. \tag{2.3.79}$$

Since $0 \le \zeta_\ell \le 1$, recalling (2.2.3) we can estimate

$$|\partial_t a_{ij}^\ell(t)|_{L^\infty(\Omega_\ell)} \le |\partial_t a_{ij}(t)|_\infty \le \|\partial_t a_{ij}(t)\|_{s-1} \le \mu_1(t)\,; \tag{2.3.80}$$

hence,

$$R_{01} \le \mu_1(t)\,\|\nabla u^{n,\ell}\|_{0,\ell}^2\,. \tag{2.3.81}$$

Acting likewise for $\tilde b_i^\ell$ and c^ℓ, we obtain from (2.3.79) that, as in (2.3.16),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_t^{n,\ell}\|_{0,\ell}^2 + Q_0(a^\ell, \nabla u^{n,\ell})\Big) \le \|f^\ell\|_{0,\ell}^2 + C\,(1+\mu_1(t))\Big(\|u_t^{n,\ell}\|_{0,\ell}^2 + \|u^{n,\ell}\|_{1,\ell}^2\Big)\,. \tag{2.3.82}$$

We add to (2.3.82) the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u^{n,\ell}\|_{0,\ell}^2 \le \|u_t^{n,\ell}\|_{0,\ell}^2 + \|u^{n,\ell}\|_{0,\ell}^2 \tag{2.3.83}$$

(see (2.3.19)), and then integrate; recalling (2.3.22), by Gronwall's inequality we deduce that, for all $t \in [0,T]$,

$$\|u_t^{n,\ell}(t)\|_{0,\ell}^2 + \|u^{n,\ell}(t)\|_{1,\ell}^2 \le a_2^2\left(\|u_1^{n,\ell}\|_{0,\ell}^2 + \|u_0^{n,\ell}\|_{1,\ell}^2 + \int_0^T \|f^\ell\|_{0,\ell}^2\,\mathrm{d}t\right) \cdot \exp\left(C\int_0^T (1+\mu_1(t))\,\mathrm{d}t\right)\,. \tag{2.3.84}$$

Recalling (2.3.67) and (2.2.4), we finally conclude that, for all $t \in [0,T]$ and all $n \ge 1$,

$$\|u_t^{n,\ell}(t)\|_{0,\ell}^2 + \|u^{n,\ell}(t)\|_{1,\ell}^2 \le a_2^2\,M^2\,\mathrm{e}^{C(T+M_1)} =: \Lambda_0^2\,. \tag{2.3.85}$$

This proves the claim of Lemma 2.3.1 for $k = 0$. We assume then that the claim is true for $k-1$, $1 \le k \le m$, and proceed to prove it for k. To this end, we note that, by the first of (2.3.75), $u^{n,\ell}$, which satisfies (2.3.71), also solves (2.3.69). In this equation, we replace $w_r = \frac{1}{\lambda_r^k}\,(-\Delta_\ell)^k\,w_r$, to obtain that, for $1 \le r \le n$,

$$\langle u_{tt}^{n,\ell} - a_{ij}^\ell\,\partial_i\partial_j u^{n,\ell}, (-\Delta_\ell)^k\,w_r\rangle_{0,\ell} = \langle f^\ell + b_i^\ell\,\partial_i u^{n,\ell} + c^\ell\,u^{n,\ell}, (-\Delta_\ell)^k\,w_r\rangle_{0,\ell}\,. \tag{2.3.86}$$

We can integrate by parts in each of the terms of (2.3.86), as per (1.5.110), if k is even, or (1.5.111), if k is odd, because condition (1.5.112) is satisfied.

Indeed, for the first term of (2.3.86) this is true because, by (2.3.68), $u^{n,\ell}_{tt}$ is a linear combination of $w_1, \ldots, w_n$, each of which does satisfy (1.5.112). For the other terms of (2.3.86), it is sufficient to recall that, by (2.3.57), both $a^\ell_{ij}\,\partial_i\partial_j u^{n,\ell}$ and $f^\ell + b^\ell_i\,\partial_i u^{n,\ell} + c^\ell\, u^{n,\ell} \in H^k_0(\Omega_\ell)$. Hence, assuming, e.g., that k is even (the other case is analogous), we obtain from (2.3.86) that

$$\begin{aligned}
&\langle(-\Delta_\ell)^{k/2}\, u^{n,\ell}_{tt}, (-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell} \\
&\qquad - \langle(-\Delta_\ell)^{k/2}\,(a^\ell_{ij}\,\partial_i\partial_j u^{n,\ell}), (-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell} \\
&\quad = \langle(-\Delta_\ell)^{k/2}\,(f^\ell + b^\ell_i\,\partial_i u^{n,\ell} + c^\ell\, u^{n,\ell}), (-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell}\,.
\end{aligned} \tag{2.3.87}$$

By Leibniz' formula, and recalling further that $(-\Delta_\ell)^{k/2}\, w_r$ vanishes on $\partial\Omega_\ell$, so that we can integrate by parts, we can rewrite the second line of (2.3.87) as

$$\begin{aligned}
&- \langle a^\ell_{ij}\,\partial_i\partial_j(-\Delta_\ell)^{k/2}u^{n,\ell} + \sum_{|\alpha|=k}\Gamma_\alpha(a^\ell_{ij}, u^{n,\ell}), (-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell} \\
&\quad = \langle a^\ell_{ij}\,\partial_i(-\Delta_\ell)^{k/2}u^{n,\ell}, \partial_j(-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell} \\
&\qquad + \langle(\partial_j a^\ell_{ij})\partial_i(-\Delta_\ell)^{k/2}u^{n,\ell} - \sum_{|\alpha|=k}\Gamma_\alpha(a^\ell_{ij}, u^{n,\ell}), (-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell}\,,
\end{aligned} \tag{2.3.88}$$

where $\Gamma_\alpha(a^\ell_{ij}, u^{n,\ell})$ has the form

$$\Gamma_\alpha(a^\ell_{ij}, u^{n,\ell}) := \sum_{0<\gamma\le\alpha} C_{\alpha\gamma}\,\partial^\gamma_x a^\ell_{ij}\,\partial^{\alpha-\gamma}_x\partial_i\partial_j u^{n,\ell}\,, \tag{2.3.89}$$

for suitable coefficients $C_{\alpha\gamma}$. Inserting (2.3.88) into (2.3.87), we obtain

$$\begin{aligned}
&\langle(-\Delta_\ell)^{k/2}\, u^{n,\ell}_{tt}, (-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell} \\
&\qquad + \langle a^\ell_{ij}\,(-\Delta_\ell)^{k/2}\,\partial_i u^{n,\ell}, (-\Delta_\ell)^{k/2}\partial_j\, w_r\rangle_{0,\ell} \\
&\quad = \sum_{|\alpha|=k}\langle\Gamma_\alpha(a^\ell_{ij}, u^{n,\ell}) - (\partial_j a^\ell_{ij})\partial_i(-\Delta_\ell)^{k/2}u^{n,\ell}, (-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell} \\
&\qquad + \langle(-\Delta_\ell)^{k/2}\,(f^\ell + b^\ell_i\,\partial_i u^{n,\ell} + c^\ell\, u^{n,\ell}), (-\Delta_\ell)^{k/2}\, w_r\rangle_{0,\ell}\,.
\end{aligned} \tag{2.3.90}$$

As in (2.3.78), we multiply (2.3.90) by ${\gamma_{rn}}'(t)$, and then sum the resulting identities for $1 \le r \le n$, to obtain

(2.3.91)
$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|(-\Delta_\ell)^{k/2}\,u_t^{n,\ell}\|_{0,\ell}^2 + \langle a_{ij}^\ell\,(-\Delta_\ell)^{k/2}\,\partial_i u^{n,\ell}, (-\Delta_\ell)^{k/2}\partial_j\,u^{n,\ell}\rangle_{0,\ell}\Big)\\
&\quad= \langle(\partial_t a_{ij}^\ell)\,\partial_i(-\Delta_\ell)^{k/2}\,u^{n,\ell}, \partial_j(-\Delta_\ell)^{k/2}\,u^{n,\ell}\rangle_{0,\ell}\\
&\qquad+ 2\langle\sum_{|\alpha|=k}\Gamma_\alpha(a_{ij}^\ell, u^{n,\ell}) - (\partial_j a_{ij}^\ell)\partial_i(-\Delta_\ell)^{k/2}u^{n,\ell}, (-\Delta_\ell)^{k/2}\,u_t^{n,\ell}\rangle_{0,\ell}\\
&\qquad+ 2\langle(-\Delta_\ell)^{k/2}\,(f^\ell + b_i^\ell\,\partial_i u^{n,\ell} + c^\ell\,u^{n,\ell}), (-\Delta_\ell)^{k/2}\,u_t^{n,\ell}\rangle_{0,\ell}\\
&\quad=: R_{k1} + R_{k2} + R_{k3}\,.
\end{aligned}
$$

As in (2.3.81),

$$
R_{k1} \le \|\partial_t a_{ij}(t)\|_{s-1}\,\|\nabla u^{n,\ell}(t)\|_{k,\ell}^2 \le \mu_1(t)\,\|Du^{n,\ell}(t)\|_{k,\ell}^2\,. \tag{2.3.92}
$$

Next, recalling (2.3.89), and that $0 < |\gamma| \le |\alpha| = k \le m \le s$, so that $s - |\gamma| \ge 0$ and $|\gamma| - 1 \ge 0$,

$$
\begin{aligned}
R_{k2} &\le C\sum_{0<\gamma\le\alpha}\|\partial_x^\gamma a_{ij}^\ell\|_{s-|\gamma|}\,\|\partial_x^{\alpha-\gamma}\partial_i\partial_j u^{n,\ell}\|_{|\gamma|-1}\,\|u_t^{n,\ell}\|_{k,\ell}\\
&\qquad + |\nabla a_{ij}^\ell|_\infty\,\|\nabla u^{n,\ell}\|_{k,\ell}\,\|u_t^{n,\ell}\|_{k,\ell}\\
&\le C\,\|\nabla a_{ij}^\ell\|_{s-1,\ell}\,\|\nabla u^{n,\ell}\|_{k,\ell}\,\|u_t^{n,\ell}\|_{k,\ell}\\
&\le C_m\,\|\nabla a_{ij}\|_{s-1}\,\|Du^{n,\ell}\|_{k,\ell}^2\,.
\end{aligned} \tag{2.3.93}
$$

Finally, recalling (1.5.25), (2.3.56), and that, by the induction assumption, $\|u^{n,\ell}\|_{k,\ell} \le \Lambda_{k-1}$ for suitable Λ_{k-1} independent of n and ℓ, by the Sobolev product estimates,

(2.3.94)
$$
\begin{aligned}
R_{k3} &\le 2\Big(\|f^\ell\|_{k,\ell} + \|b_i^\ell\|_{s,\ell}\,\|\nabla u^{n,\ell}\|_{k,\ell} + \|c^\ell\|_{s,\ell}\,\|u^{n,\ell}\|_{k,\ell}\Big)\,\|u_t^{n,\ell}\|_{k,\ell}\\
&\le 2\,C_m\Big(\|f\|_k + \|b_i\|_s\,\|\nabla u^{n,\ell}\|_{k,\ell} + \|c\|_s\,\Lambda_{k-1}\Big)\,\|u_t^{n,\ell}\|_{k,\ell}\\
&\le C\,(1+\mu_1(t))\Big(\|u_t^{n,\ell}\|_{k,\ell}^2 + \|\nabla u^{n,\ell}\|_{k,\ell}^2\Big) + C\,\|f\|_k^2 + \mu_1(t)\,\Lambda_{k-1}^2\,,
\end{aligned}
$$

with C independent of n and ℓ. Putting (2.3.92), (2.3.94), and (2.3.93) into (2.3.91), and then integrating, recalling (2.3.22) we deduce that, for all

$t \in [0,T]$,

$$\begin{aligned} &\|(-\Delta_\ell)^{k/2}\, u_t^{n,\ell}(t)\|_{0,\ell}^2 + \|(-\Delta_\ell)^{k/2}\, \nabla u^{n,\ell}(t)\|_{0,\ell}^2 \\ &\quad \le\ a_2^2\, \left(I_0^2(k,\ell) + M_1\, \Lambda_{k-1}^2\right) \\ &\qquad + C\, a_2^2 \int_0^t (1+\mu_1(\theta)) \left(\|u_t^{n,\ell}\|_{k,\ell}^2 + \|\nabla u^{n,\ell}\|_{k,\ell}^2\right) \mathrm{d}\theta\,, \end{aligned} \tag{2.3.95}$$

where, by (2.3.67),

(2.3.96)

$$\begin{aligned} I_0^2(k,\ell) &:= \|(-\Delta_\ell)^{k/2}\, u_1^{n,\ell}\|_{0,\ell}^2 + \|(-\Delta_\ell)^{k/2}\, \nabla u_0^{n,\ell}\|_{0,\ell}^2 + \int_0^T \|f\|_k^2\, \mathrm{d}t \\ &\le C \left(\|u_1^{n,\ell}\|_{k,\ell}^2 + \|\nabla u_0^{n,\ell}\|_{k,\ell}^2 + \int_0^T \|f\|_k^2\, \mathrm{d}t \right) \ \le\ C\, M^2\,. \end{aligned}$$

By the elliptic estimates (1.5.130) and (1.5.131) of Corollary 1.5.3, with m replaced by, respectively, $k-2$ (which is even) and $k-1$ (which is odd),

$$\|u_t^{n,\ell}(t)\|_{k,\ell}^2 \le C \left(\|(-\Delta_\ell)^{k/2}\, u_t^{n,\ell}(t)\|_{0,\ell}^2 + \|u_t^{n,\ell}(t)\|_{0,\ell}^2 \right)\,, \tag{2.3.97}$$

$$\|\nabla u^{n,\ell}(t)\|_{k,\ell}^2 \le C \left(\|\nabla(-\Delta_\ell)^{k/2}\, u^{n,\ell}(t)\|_{0,\ell}^2 + \|u^{n,\ell}(t)\|_{0,\ell}^2 \right)\,, \tag{2.3.98}$$

with C independent of ℓ. Replacing (2.3.85) and (2.3.95) into the sum of (2.3.97) and (2.3.98), we deduce, via Gronwall's inequality, that

$$\|u_t^{n,\ell}(t)\|_{k,\ell}^2 + \|\nabla u^{n,\ell}(t)\|_{k,\ell}^2 \le \Lambda_k^2\,, \tag{2.3.99}$$

where

(2.3.100)

$$\Lambda_k^2 := C\, \left(\Lambda_0^2 + a_2^2\, M^2 + a_2^2\, M_1\, \Lambda_{k-1}^2\right)\, \exp\left(C \int_0^T (1+\mu_1(t))\, \mathrm{d}t \right)\,.$$

Thus, the claim of Lemma 2.3.1 also holds for even k. The proof when k is odd follows analogous lines; hence, we can conclude that $(u^{n,\ell})_{n\ge 0}$ is bounded in $\mathcal{X}_{m,\ell}(T)$, as claimed. □

Remark. The fact that the bounds obtained in (2.3.85) and (2.3.99) are independent of t allows us to show that each solution $u^{n,\ell}$ of (2.3.69), which a priori may be defined only locally, i.e. on a maximal interval $[0,t_n] \subseteq [0,T]$, can indeed be extended to the whole of $[0,T]$. ◇

4. By Lemma 2.3.1, for each $\ell \in \mathbb{N}$ there is $u^\ell \in L^\infty(0,T;W_0^{m+1}(\Omega_\ell))$, with $u_t^\ell \in L^\infty(0,T;W_0^m(\Omega_\ell))$, as well as a subsequence of $\left(u^{n,\ell}\right)_{n\ge 0}$, which we still denote by $\left(u^{n,\ell}\right)_{n\ge 0}$ (not only for convenience, since, by uniqueness, any such subsequence will converge to the same solution), such that, as $n \to +\infty$,

$$u^{n,\ell} \to u^\ell \quad \text{in } L^\infty(0,T;W_0^{m+1}(\Omega_\ell)) \quad \text{weak}^* \,, \tag{2.3.101}$$

$$u^{n,\ell} \to u^\ell \quad \text{in } L^2(0,T;W_0^{m+1}(\Omega_\ell)) \quad \text{weak} \,, \tag{2.3.102}$$

$$u_t^{n,\ell} \to u_t^\ell \quad \text{in } L^\infty(0,T;W_0^{m}(\Omega_\ell)) \quad \text{weak}^* \,, \tag{2.3.103}$$

$$u_t^{n,\ell} \to u_t^\ell \quad \text{in } L^2(0,T;W_0^{m}(\Omega_\ell)) \quad \text{weak} \,. \tag{2.3.104}$$

We wish to pass to the limit as $n \to \infty$ in (2.3.71). To this end, we multiply (2.3.71) by an arbitrary $\psi \in \mathcal{D}(]0,T[)$, and integrate by parts also in t, as allowed by the second of (2.3.74): we obtain, for $1 \le r \le n$,

$$\begin{aligned} &\int_0^T \Big(- \langle u_t^{n,\ell}, \psi' \, w_r \rangle_{0,\ell} + \langle a_{ij}^\ell \partial_i u^{n,\ell}, \psi \, \partial_j w_r \rangle_{0,\ell} \Big) \, \mathrm{d}t \\ &\qquad = \int_0^T \langle f^\ell + \tilde{b}_i^\ell \partial_i u^{n,\ell} + c^\ell u^{n,\ell}, \psi \, w_r \rangle_{0,\ell} \, \mathrm{d}t \,. \end{aligned} \tag{2.3.105}$$

We can now let $n \to \infty$ in (2.3.105) (with r fixed, and $n \ge r$). Indeed, considering each term separately, we first see that the weak convergence (2.3.104) implies that

$$\int_0^T \langle - u_t^{n,\ell}, \psi' \, w_r \rangle_{0,\ell} \, \mathrm{d}t \;\to\; \int_0^T \langle - u_t^{\ell}, \psi' \, w_r \rangle_{0,\ell} \, \mathrm{d}t \,. \tag{2.3.106}$$

Next, to show that

$$A_n := \int_0^T \langle a_{ij}^\ell (\partial_i u^{n,\ell} - \partial_i u^\ell), \psi \, \partial_j w_r \rangle_{0,\ell} \, \mathrm{d}t \;\to\; 0 \,, \tag{2.3.107}$$

we note that $A_n = \Psi_1(u^{n,\ell} - u^\ell)$, where Ψ_1 is the map on $L^2(0,T;W_0^1(\Omega_\ell))$ defined by

$$v \;\mapsto\; \Psi_1(v) := \int_0^T \langle a_{ij}^\ell \, \partial_i v, \psi \, \partial_j w_r \rangle_{0,\ell} \, \mathrm{d}t \,. \tag{2.3.108}$$

This map is linear, and also continuous, as follows from the estimate

$$|\Psi_1(v)| \le C_1(a,\psi,w_r) \, \|v\|_{L^2(0,T;H^1)} \,, \tag{2.3.109}$$

where, recalling (2.2.7),

$$C_1(a,\psi,w_r) := C_m \, |a_{ij}|_{L^\infty(Q)} \, \|\psi\|_{L^2(0,T)} \, \|w_r\|_{W_0^1(\Omega_\ell)} \,. \tag{2.3.110}$$

Thus, $\Psi_1 \in (L^2(0,T;W_0^1(\Omega_\ell)))'$, and (2.3.102) implies that $A_n \to 0$, as desired. An analogous argument, based on (2.3.101) and the fact that, by (2.2.8), $\tilde{b}_i^\ell$ and $c^\ell \in L^1(0,T;L^\infty)$, shows that

$$\int_0^T \langle \tilde{b}_i^\ell (\partial_i u^{n,\ell} - \partial_i u^\ell) + c^\ell \, (u^{n,\ell} - u^\ell), \psi \, w_r \rangle_{0,\ell} \, \mathrm{d}t \;\to\; 0 \,. \tag{2.3.111}$$

Indeed, consider, e.g., the term

$$B_n := \int_0^T \langle \partial_j a_{ij}^\ell \, (\partial_i u^{n,\ell} - \partial_i u^\ell), \psi \, w_r \rangle_{0,\ell} \, \mathrm{d}t \, . \tag{2.3.112}$$

Then, as above, we can write $B_n = \langle\langle \partial_i u^{n,\ell} - \partial_i u^\ell, \Psi_2 \rangle\rangle$, where the duality is between the spaces $L^\infty(0,T;L^2(\Omega_\ell))$ and $L^1(0,T;L^2(\Omega_\ell))$, and $\Psi_2 := (\partial_j a_{ij}^\ell) \, \psi \, w_r \in L^1(0,T;L^2(\Omega_\ell))$, because

$$\begin{aligned} \int_0^T \|\Psi_2(t)\| \, \mathrm{d}t &\le C_2(a,\psi,w_r) \\ &=: \; C_m \left(\max_{0\le t\le T} |\psi(t)| \right) \|w_r\| \int_0^T |\partial_j a_{ij}(t)|_\infty \, \mathrm{d}t \, . \end{aligned} \tag{2.3.113}$$

Consequently, (2.3.101) and (2.2.8) imply that $B_n \to 0$, as claimed. In conclusion, we deduce from (2.3.105) that, for all $r \ge 1$,

$$\begin{aligned} \int_0^T &\left(- \langle u_t^\ell, \psi' \, w_r \rangle_{0,\ell} + \langle a_{ij}^\ell \partial_i u^\ell, \psi \, \partial_j w_r \rangle_{0,\ell} \right) \mathrm{d}t \\ &= \int_0^T \langle f^\ell + \tilde{b}_i^\ell \partial_i u^\ell + c^\ell u^\ell, \psi \, w_r \rangle_{0,\ell} \, \mathrm{d}t \, . \end{aligned} \tag{2.3.114}$$

The base $\mathcal{W}$ is total in $L^2(\Omega_\ell)$, and $H_0^1(\Omega_\ell)$-regular: this implies that for all $w \in H_0^1(\Omega_\ell)$, the identity

$$\nabla w = \sum_{r=0}^\infty \langle w, w_r \rangle_{0,\ell} \, \nabla w_r \tag{2.3.115}$$

holds in $L^2(\Omega_\ell)$. Therefore, from (2.3.114) it follows that

$$\begin{aligned} \int_0^T &\left(- \langle u_t^\ell, \psi' \, w \rangle_{0,\ell} + \langle a_{ij}^\ell \partial_i u^\ell, \psi \, \partial_j w \rangle_{0,\ell} \right) \mathrm{d}t \\ &= \int_0^T \langle f^\ell + \tilde{b}_i^\ell \partial_i u^\ell + c^\ell u^\ell, \psi \, w \rangle_{0,\ell} \, \mathrm{d}t \end{aligned} \tag{2.3.116}$$

for all $w \in H_0^1(\Omega_\ell)$; in particular, for all $w \in \mathcal{D}(\Omega_\ell)$. In turn, since the set of product functions $(t,x) \mapsto \psi(t) \, w(x)$, with $\psi \in \mathcal{D}(]0,T[)$ and $w \in \mathcal{D}(\Omega_\ell)$, is total in $\mathcal{D}(]0,T[\times\Omega_\ell)$ (as seen in Theorem 1.7.7), we conclude from (2.3.116) that, for all $\varphi \in \mathcal{D}(]0,T[\times\Omega_\ell)$,

$$\begin{aligned} \int_0^T &\left(- \langle u_t^\ell, \varphi_t \rangle_{0,\ell} + \langle a_{ij}^\ell \partial_i u^\ell, \partial_j \varphi \rangle_{0,\ell} \right) \mathrm{d}t \\ &= \int_0^T \langle f^\ell + \tilde{b}_i^\ell \partial_i u^\ell + c^\ell u^\ell, \varphi \rangle_{0,\ell} \, \mathrm{d}t \, . \end{aligned} \tag{2.3.117}$$

This means that the identity

$$u_{tt}^\ell = \partial_j (a_{ij}^\ell \partial_i u^\ell) + f^\ell + \tilde{b}_i^\ell \partial_i u^\ell + c^\ell u^\ell \tag{2.3.118}$$

holds in $\mathcal{D}'(]0,T[\times\Omega_\ell)$. But since $u^\ell \in L^2(0,T;W_0^{m+1}(\Omega_\ell))$, the right side of (2.3.118) is in $L^2(0,T;H^{m-1}(\Omega_\ell))$; hence, so is u^ℓ_{tt}, and (2.3.118) holds in $H^{m-1}(\Omega_\ell)$, for a.a. $t \in]0,T[$. Furthermore, since the identity

$$\partial_j(a^\ell_{ij}\,\partial_i u^\ell) = (\partial_j a^\ell_{ij})\partial_i u^\ell + a^\ell_{ij}\,\partial_i\partial_j u^\ell \tag{2.3.119}$$

holds, for a.a. t, in $H^{m-1}(\Omega_\ell)$, we can rewrite (2.3.118) in the original form

$$u^\ell_{tt} - a^\ell_{ij}n\partial_i\partial_j u^\ell = f^\ell + b^\ell_i\partial_i u^\ell + c^\ell u^\ell\,. \tag{2.3.120}$$

With an argument similar to the one we use in parts 2 and 3 of the proof of Lemma 2.3.2 below, we can show that, because of (2.3.64), u^ℓ also satisfies the initial conditions of (2.3.60); thus, u^ℓ solves the Cauchy problem (2.3.60).

5. By means of Proposition 1.5.1, we can extend each u^ℓ to a function $\tilde{u}^\ell \in L^\infty(0,T;H^{m+1})$. Since the extension operator does not involve the variable t, we can check that $\partial_t(\tilde{u}^\ell) = \widetilde{u^\ell_t}$ in $\mathcal{D}'(Q)$ (and, consequently, in $L^\infty(0,T;H^m)$); hence, we conclude that $\tilde{u}^\ell \in W^{1,\infty}(0,T;H^{m+1},H^m)$. As per (1.5.30), it follows from Lemma 2.3.1 that the sequence $\big(\tilde{u}^\ell\big)_{\ell\in\mathbb{N}}$ is bounded in $W^{1,\infty}(0,T;H^{m+1},H^m)$; thus, there exists a function $u \in W^{1,\infty}(0,T;H^{m+1},H^m)$, as well as a subsequence of $\big(\tilde{u}^\ell\big)_{\ell\geq 0}$, which we still denote by $\big(\tilde{u}^\ell\big)_{\ell\geq 0}$, such that, as $\ell \to +\infty$, as in (2.3.101), …, (2.3.104),

$$\tilde{u}^\ell \to u \quad \text{in} \quad L^\infty(0,T;H^{m+1}) \quad \text{weak}^*\,, \tag{2.3.121}$$

$$\tilde{u}^\ell \to u \quad \text{in} \quad L^2(0,T;H^{m+1}) \quad \text{weak}\,, \tag{2.3.122}$$

$$\tilde{u}^\ell_t \to u_t \quad \text{in} \quad L^\infty(0,T;H^m) \quad \text{weak}^*\,, \tag{2.3.123}$$

$$\tilde{u}^\ell_t \to u_t \quad \text{in} \quad L^2(0,T;H^m) \quad \text{weak}\,. \tag{2.3.124}$$

We claim that u solves equation (2.1.1). To this end, we fix an arbitrary $\varphi \in \mathcal{D}(Q)$, and a corresponding compact set $K \subset \mathbb{R}^N$, such that $\operatorname{supp}(\varphi) \subseteq]0,T[\,\times K$. There exists then $\ell_0 \in \mathbb{N}$ such that, for all $\ell \geq \ell_0$, $K \subset \Omega_{\ell-1}$. From (2.3.120), we deduce that, for $\ell \geq \ell_0$,

$$\begin{aligned} &\int_0^T \int_{\Omega_{\ell-1}} (u^\ell_{tt} - a^\ell_{ij}\partial_i\partial_j u^\ell)\,\varphi\,\mathrm{d}x\mathrm{d}t \\ &\qquad = \int_0^T \int_{\Omega_{\ell-1}} (f^\ell + b^\ell_i\partial_i u^\ell + c^\ell u^\ell)\,\varphi\,\mathrm{d}x\mathrm{d}t\,. \end{aligned} \tag{2.3.125}$$

Now, clearly, $u^\ell = \tilde{u}^\ell$ on $\Omega_{\ell-1}$; in addition, since $\zeta_\ell \equiv 1$ on $\Omega_{\ell-1}$ (by (2.3.55)), it follows that, on $\Omega_{\ell-1}$, also $a^\ell_{ij} = a_{ij}$, $b^\ell_i = b_i$, $c^\ell = c$, and $f^\ell = f$. Hence, (2.3.125) implies that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} (\tilde{u}^\ell_{tt} - a_{ij}\partial_i\partial_j \tilde{u}^\ell)\,\varphi\,\mathrm{d}x\mathrm{d}t \\ &\qquad = \int_0^T \int_{\mathbb{R}^N} (f + b_i\partial_i \tilde{u}^\ell + c\,\tilde{u}^\ell)\,\varphi\,\mathrm{d}x\mathrm{d}t\,. \end{aligned} \tag{2.3.126}$$

Integrating by parts in t in the first term of (2.3.126), and then letting $\ell \to +\infty$, by (2.3.122) and (2.3.124) we deduce that

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} (-\, u_t\, \varphi_t - (a_{ij}\partial_i\partial_j u)\, \varphi)\, \mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_{\mathbb{R}^N} (f + b_i\partial_i u + c\, u)\, \varphi\, \mathrm{d}x\mathrm{d}t\,.\end{aligned} \tag{2.3.127}$$

Since φ is arbitrary, (2.3.127) means that the identity

$$u_{tt} = a_{ij}\, \partial_i\partial_j u + f + b_i\, \partial_i u + c\, u \tag{2.3.128}$$

holds in $\mathcal{D}'(Q)$. But since $u \in L^2(0,T;H^{m+1})$, the right side of (2.3.128) is in $L^2(0,T;H^{m-1})$; hence, so is u_{tt}, and (2.3.128) holds in H^{m-1}, for a.a. $t \in \,]0,T[$. Thus, we conclude that u is, a.e. in t, a solution of the original equation (2.1.1), as claimed.

6. So far, the solution u we have constructed is such that, by (2.3.121), (2.3.123), and (2.3.128),

$$u \in L^\infty(0,T;H^{m+1}) \subseteq L^2(0,T;H^{m+1})\,, \tag{2.3.129}$$

$$u_t \in L^\infty(0,T;H^m) \subseteq L^2(0,T;H^m)\,, \tag{2.3.130}$$

$$u_{tt} \in L^2(0,T;H^{m-1})\,. \tag{2.3.131}$$

Thus, to conclude the proof of Theorem 2.2.1, we still have to prove that $u \in C([0,T];H^{m+1})$ and $u_t \in C([0,T];H^m)$. To this end, we first note that, by (1.7.59) of the trace theorem, (2.3.129), (2.3.130) and (2.3.131) allow us to modify u on a set of measure zero of $[0,T]$, in such a way that

$$u \in C([0,T];H^m)\,, \qquad u_t \in C([0,T];H^{m-1})\,. \tag{2.3.132}$$

By Proposition 1.7.1, it follows from (2.3.129) and the first of (2.3.132) (respectively, from (2.3.130) and the second of (2.3.132)), that

$$u \in C_{\mathrm{w}}([0,T];H^{m+1})\,, \qquad u_t \in C_{\mathrm{w}}([0,T];H^m)\,. \tag{2.3.133}$$

We now show that, as per the last claim of Proposition 1.7.1,

Lemma 2.3.2. *The function u, thus modified, is such that*

$$u \in C([0,T];H^{m+1})\,, \qquad u_t \in C([0,T];H^m)\,. \tag{2.3.134}$$

Hence, $u \in \mathcal{Y}_m(T)$. Moreover, u satisfies the initial conditions (2.1.2).

Proof. 1) Recalling (2.3.13), we set

$$\tilde{E}^u_m(t) := \|u_t(t)\|^2_m + Q_m(a(t), \nabla u(t))\,, \tag{2.3.135}$$

and proceed to show that the function $t \mapsto \tilde{E}^u_m(t)$ is continuous. To this end, we consider again the regularized functions u^δ, which satisfy (2.3.35) and, therefore, an identity analogous to (2.3.15), that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{E}_m^{u^\delta} = \Lambda^\delta = \sum_{|\alpha|\le m} \Lambda_\alpha^\delta\,, \tag{2.3.136}$$

where

$$\Lambda_\alpha^\delta := 2\langle R_\alpha^\delta - (\partial_j a_{ij})\partial_i\partial_x^\alpha u^\delta, \partial_x^\alpha u_t^\delta\rangle + \langle(\partial_t a_{ij})\partial_i\partial_x^\alpha u^\delta, \partial_j\partial_x^\alpha u^\delta\rangle \tag{2.3.137}$$

$$R_\alpha^\delta := \partial_x^\alpha(f^\delta + F^\delta + L^\delta + b_i\,\partial_i u^\delta + c\,u^\delta) + G_\alpha(a_{ij}, \partial_i\partial_j u^\delta)\,. \tag{2.3.138}$$

Since the right side of (2.3.136) is in $L^1(0,T)$, we can integrate (2.3.136) in any interval $[t_1,t_2]\subseteq[0,T]$ to obtain

$$\tilde{E}_m^{u^\delta}(t_2) - \tilde{E}_m^{u^\delta}(t_1) = \int_{t_1}^{t_2} \Lambda^\delta(t)\,\mathrm{d}t\,. \tag{2.3.139}$$

We estimate the right side of (2.3.139) as in (2.3.16). Since $u(t)$ is in a bounded set of H^{m+1}, independent of t, (1.7.35) (with m replaced by $m+1$) implies that $u^\delta(t)$ is in a bounded set of H^{m+1}, independent of t and δ. An analogous argument shows that $u_t^\delta(t)$ is in a bounded set of H^m, independent of t and δ. Thus, we obtain from (2.3.139) and (2.3.137), (2.3.138), that

(2.3.140)

$$\begin{aligned}|\tilde{E}_m^{u^\delta}(t_2) - \tilde{E}_m^{u^\delta}(t_1)| &\le 2\int_{t_1}^{t_2}\Big(\|f^\delta + F^\delta + L^\delta\|_m\,\|u_t\|_m \\ &\qquad\quad + \,C\,\mu_1\,\big(\|u\|_{m+1}^2 + \|u_t\|_m^2\big)\Big)\,\mathrm{d}t \\ &\le K\int_{t_1}^{t_2}(\|f^\delta\|_m + \|F^\delta\|_m + \|L^\delta\|_m + \mu_1)\,\mathrm{d}t\,,\end{aligned}$$

for some K depending on $\|u\|_{\mathcal{X}_m(T)}$ and, therefore, on the data and the coefficients. In particular, K does not depend on t, nor on δ. Because of (2.3.43) and (2.3.44), we can let $\delta\to 0$ in (2.3.140), and deduce that

(2.3.141)

$$\begin{aligned}|\tilde{E}_m^{u}(t_2) - \tilde{E}_m^{u}(t_1)| &\le K\int_{t_1}^{t_2}(\|f\|_m + \mu_1)\,\mathrm{d}t \\ &\le K\left(|t_2 - t_1|\int_0^T \|f\|_m^2\,\mathrm{d}t\right)^{1/2} + K\int_{t_1}^{t_2}\mu_1(t)\,\mathrm{d}t\,.\end{aligned}$$

Since $\mu_1\in L^1(0,T)$, (2.3.141) implies that $\tilde{E}$ is continuous on $[0,T]$. Fix now $t_0\in[0,T]$, and set

$$F(t,t_0) := \|u_t(t) - u_t(t_0)\|_m^2 + Q_m(a(t), \nabla(u(t) - u(t_0)))\,. \tag{2.3.142}$$

We compute that

$$\begin{aligned} F(t,t_0) &= \tilde{E}^u_m(t) + \|u_t(t_0)\|^2_m - 2\,\langle u_t(t), u_t(t_0)\rangle_m \\ &\qquad + \sum_{|\alpha|\le m} \langle a_{ij}(t)\partial_i\partial_x^\alpha u(t_0), \partial_j\partial_x^\alpha u(t_0)\rangle \\ &\qquad - 2\sum_{|\alpha|\le m} \langle a_{ij}(t)\partial_i\partial_x^\alpha u(t), \partial_j\partial_x^\alpha u(t_0)\rangle\,. \end{aligned} \tag{2.3.143}$$

We now show that, as $t \to t_0$,

$$\langle a_{ij}(t)\partial_i\partial_x^\alpha u(t), \partial_j\partial_x^\alpha u(t_0)\rangle \;\to\; \langle a_{ij}(t_0)\partial_i\partial_x^\alpha u(t_0), \partial_j\partial_x^\alpha u(t_0)\rangle\,. \tag{2.3.144}$$

Indeed, we decompose

$$\begin{aligned} &\langle a_{ij}(t)\,\partial_i\partial_x^\alpha u(t) - a_{ij}(t_0)\,\partial_i\partial_x^\alpha u(t_0), \partial_j\partial_x^\alpha u(t_0)\rangle \\ &\quad= \langle (a_{ij}(t) - a_{ij}(t_0))\,\partial_i\partial_x^\alpha u(t), \partial_j\partial_x^\alpha u(t_0)\rangle \\ &\qquad + \langle a_{ij}(t_0)\,(\partial_i\partial_x^\alpha u(t) - \partial_i\partial_x^\alpha u(t_0)), \partial_j\partial_x^\alpha u(t_0)\rangle\,. \end{aligned} \tag{2.3.145}$$

Since the function $t \mapsto \partial_i\partial_x^\alpha u(t)$ is bounded in L^2,

$$\begin{aligned} &|\langle (a_{ij}(t) - a_{ij}(t_0))\,\partial_i\partial_x^\alpha u(t), \partial_j\partial_x^\alpha u(t_0)\rangle| \\ &\qquad \le\; C\,|a_{ij}(t) - a_{ij}(t_0)|_\infty\,\|\partial_j\partial_x^\alpha u(t_0)\|\,; \end{aligned} \tag{2.3.146}$$

thus, recalling also (2.2.7), the first term at the right side of (2.3.145) vanishes as $t \to t_0$. Noting that $a_{ij}(t_0)\,\partial_j\partial_x^\alpha u(t_0) \in L^\infty \cdot L^2 \hookrightarrow L^2$, the weak continuity of $t \mapsto \partial_i\partial_x^\alpha u(t)$ implies that also the last term of (2.3.145) vanishes as $t \to t_0$. In conclusion, using the weak continuity of the function $t \mapsto u_t(t) \in H^m$, we deduce that, as $t \to t_0$,

$$\begin{aligned} F(t,t_0) &\to \tilde{E}^u_m(t_0) + \|u_t(t_0)\|^2_m - 2\,\|u_t(t_0)\|^2_m \\ &\qquad - 2\sum_{|\alpha|\le m} \langle a_{ij}(t_0)\partial_i\partial_x^\alpha u(t_0), \partial_j\partial_x^\alpha u(t_0)\rangle \\ &\qquad + \sum_{|\alpha|\le m} \langle a_{ij}(t_0)\partial_i\partial_x^\alpha u(t_0), \partial_j\partial_x^\alpha u(t_0)\rangle \\ &= \tilde{E}^u_m(t_0) - \|u_t(t_0)\|^2_m - Q_m(a(t_0)\nabla u(t_0)) \;=\; 0\,. \end{aligned} \tag{2.3.147}$$

The inequalities

$$\|u_t(t) - u_t(t_0)\|^2_m \le F(t,t_0) \tag{2.3.148}$$

and

$$\alpha_0\,\|\nabla u(t) - \nabla u(t_0)\|^2_m \le Q_m(a(t), \nabla(u(t) - u(t_0))) \le F(t,t_0) \tag{2.3.149}$$

show that $D\,u \in C([0,T]; H^m)$. Together with (2.3.132), this allows us to deduce that (2.3.134) holds.

2) By (2.3.122) and (2.3.124), $\tilde{u}^\ell \to u$ weakly in $W^{1,2}(0,T;H^{m+1},H^m)$. The trace operator $\Phi_0 : W^{1,2}(0,T;H^{m+1},H^m) \to H^m$, defined by

$$\Phi_0(u) := u(0)\,, \tag{2.3.150}$$

is linear and continuous. Hence, by Proposition 1.2.1,

$$\Phi_0(\tilde{u}^\ell) = \tilde{u}^\ell(0) \to \Phi_0(u) = u(0)\,, \tag{2.3.151}$$

weakly in H^m. On the other hand, $\Phi_0(\tilde{u}^\ell) = \tilde{u}^\ell_0 \to u_0$ in H^{m+1}; thus, we conclude that $u(0) = u_0$. That is, the first of the initial conditions (2.1.2) holds.

3) To proceed, we slightly modify the procedure that led to (2.3.127) from (2.3.120), by taking φ of the form $\varphi(t,x) = \psi(t)\,\chi(x)$, with arbitrary $\psi \in \mathcal{D}(]-T,T[)$ and $\chi \in \mathcal{D}(\mathbb{R}^N)$. Choosing ψ so that $\psi(0) = 1$, and recalling that $\tilde{u}^\ell_t(0) = \tilde{u}^\ell_1 \to u_1$ in H^m, we obtain that, in analogy to (2.3.127),

$$\begin{aligned}\int_0^T \big(-\langle u_t, \psi'\,\chi\rangle + \langle a_{ij}\partial_i u, \psi\,\partial_j\chi\rangle\big)\,\mathrm{d}\theta &= \langle u_1, \chi\rangle \\ &+ \int_0^T \langle f + (b_i - \partial_j a_{ij})\partial_i u + c\,u, \psi\,\chi\rangle\,\mathrm{d}\theta\,.\end{aligned} \tag{2.3.152}$$

At the same time, multiplying (2.3.128) by $\psi\,\chi$ in $L^2(0,T;L^2)$,

$$\begin{aligned}\int_0^T \big(-\langle u_t, \psi'\,\chi\rangle + \langle a_{ij}\partial_i u, \psi\,\partial_j\chi\rangle\big)\,\mathrm{d}\theta &= \langle u_t(0), \chi\rangle \\ &+ \int_0^T \langle f + (b_i - \partial_j a_{ij})\partial_i u + cu, \psi\,\chi\rangle\,\mathrm{d}\theta\,.\end{aligned} \tag{2.3.153}$$

Comparing (2.3.153) with (2.3.152) shows that $u_t(0) = u_1$; that is, the second of the initial conditions (2.1.2) holds. This concludes the proof of Lemma 2.3.2, and we can conclude that u is the desired solution to problem (2.1.1)+(2.1.2). □

2.3.3. Additional Regularity.

In this section we prove that if the additional regularity assumptions (2.2.16) and (2.2.17) hold, then the solution u, whose existence has been established above, satisfies the regularity described in (2.2.18). For simplicity, we assume that $b_i \equiv 0$ and $c \equiv 0$; the proof carries over to the general case without difficulty.

We proceed by induction on k. For $k = 0$, (2.2.18) just means that $u \in \mathcal{Y}_m(T)$. Assume then that, for $1 \le k \le m-1$,

$$u \in \bigcap_{r=0}^{k} C^r([0,T];H^{m+1-r})\,. \tag{2.3.154}$$

We can differentiate equation (2.1.1) $k-1$ times with respect to t, to deduce that

$$\partial_t^{k+1} u = \partial_t^{k-1} f + \sum_{r=0}^{k-1} \binom{k-1}{r} \partial_t^r a_{ij}\, \partial_t^{k-1-r} \partial_i \partial_j u\,. \tag{2.3.155}$$

By (2.2.16), $\partial_t^k f \in L^2(0,T;H^{m-k})$ and $\partial_t^{r+1} a_{ij} \in L^2(0,T;H^{s-r-1})$ (note that $s-r-1>0$, since $r \le k-1 \le m-2 < s-1$); thus, by the trace theorem (1.7.59), $\partial_t^{k-1} f \in C([0,T];H^{m-k})$ and $\partial_t^r a_{ij} \in C([0,T];H^{s-r-1})$. Since $\partial_t^{k-1-r}\partial_i\partial_j u \in C([0,T];H^{m-k+r})$ by (2.3.154), we conclude, via the Sobolev product properties, that $\partial_t^{k+1} u \in C([0,T];H^{m-k})$. Together with (2.3.154), this implies the first part of (2.2.18). The second part is proven similarly: indeed, (2.3.154) allows us to differentiate once more; in the sum at the right side of

$$\partial_t^{k+2} u = \partial_t^k f + \sum_{r=0}^{k} \binom{k}{r} \partial_t^r a_{ij}\, \partial_i \partial_j \partial_t^{k-r} u\,, \tag{2.3.156}$$

we have $\partial_t^r a_{ij} \in L^2(0,T;H^{s-r})$ and $\partial_t^{k-r}\partial_i\partial_j u \in C([0,T];H^{m-1-k+r})$, and we can deduce, as before, that $\partial_t^{k+2} u \in L^2(0,T;H^{m-k-1})$. The proof of Theorem 2.2.1 is now complete. □

Remark. As the proof of (2.2.18) shows, the additional regularity result can be improved, in the sense that if, for some $h \ge 0$,

$$u_0 \in H^{m+1+h}\,, \quad u_1 \in H^{m+h}\,, \quad \partial_t^r f \in L^2(0,T;H^{m-r+h}) \tag{2.3.157}$$

for $0 \le r \le m-1+h$, as well as

$$\partial_t^r a_{ij}\,,\ \partial_t^r b_i\,,\ \partial_t^r c\ \in L^2(0,T;H^{s-r+h})\,, \tag{2.3.158}$$

then

$$u \in \bigcap_{\ell=0}^{m+h} C^\ell([0,T];H^{m+1-\ell+h})\,, \qquad \partial_t^{m+1+h} u \in L^2(0,T;L^2)\,. \tag{2.3.159}$$

It follows that, if the data and the coefficients are C^∞, the solution of problem (2.1.1)+(2.1.2) is C^∞. ◇

2.4. Weak Solutions

In this section we briefly consider the Cauchy problem (2.1.3)+(2.1.2) for linear equations in divergence form; that is,

$$\begin{cases} u_{tt} - \partial_j(a_{ij}\,\partial_i u) = f + b_i\,\partial_i u + c\,u\,, \\ u(0) = u_0\,, \qquad u_t(0) = u_1\,, \end{cases} \tag{2.4.1}$$

which formally corresponds to the case $m = 0$ in the previous sections, and we show that (2.4.1) is well-posed in a suitable class of weak solutions. We assume that a_{ij}, b_i and $c \in L^\infty(Q)$, and that, in addition to (2.2.1),

$$\partial_t a_{ij} \in L^1(0,T;L^\infty)\,. \tag{2.4.2}$$

By (2.2.7), we can again define a_1 as in (2.3.18). In analogy to (2.2.3) and (2.2.4), we also set

$$\mu_0(t) := |\partial_t a_{ij}(t)|_\infty + |b_i(t)|_\infty + |c(t)|_\infty\,, \qquad M_0 := \int_0^T \mu(t)\,\mathrm{d}t\,. \tag{2.4.3}$$

We claim:

Theorem 2.4.1. *Let $u_0 \in H^1$, $u_1 \in L^2$, $f \in L^2(0,T;L^2)$. Under the above assumptions on the coefficients of* (2.1.3)*, there exists a unique $u \in \mathcal{Y}_0(T)$, which is a weak solution of the Cauchy problem* (2.4.1)*, in the sense that equation* (2.1.3) *holds in H^{-1} for a.a. $t \in \,]0,T[$; that is, for all $w \in H^1$* (*omitting the dependence on t*),

$$\langle u_{tt}, w\rangle_{H^{-1}\times H^1} + \langle a_{ij}\partial_i u, \partial_j w\rangle = \langle f + b_i\partial_i u + cu, w\rangle\,. \tag{2.4.4}$$

Any weak solution of (2.4.1) *satisfies the a priori estimate*

$$\|u\|_{\mathcal{Y}_0(T)} \le I_0\,\psi\,, \tag{2.4.5}$$

where I_0 depends on the data, and ψ depends exponentially on the coefficients, via M_0 of (2.4.3)*, α_0, a_1, and T. As a consequence, the Cauchy problem* (2.4.1) *is well-posed; that is, the map Φ defined in* (2.2.15) *is continuous from $H^1 \times L^2 \times L^2(0,T;L^2)$ into $\mathcal{Y}_0(T)$.*

Remarks. Analogous remarks to those we made for Theorem 2.2.1 hold. In particular, the underlying phase space for (2.1.3) is $H^1 \times L^2$; estimate (2.4.5) is a priori, and it need not be true that, separately, $\partial_i\partial_j u$ and $u_{tt} \in L^2(0,T;L^2)$, because this would require, as a necessary condition for solvability, that $u_0 \in H^{3/2}$ and $u_1 \in H^{1/2}$. ◇

Sketch of Proof. The proof of Theorem 2.4.1 is similar to that of Theorem 2.2.1, and is again based on the validity of the a priori estimate (2.4.5), which is proved as in section 2.3.1.

1) To prove (2.4.5), we only need to note that when we regularize equation (2.1.3), the convolution with the space differential operator has to be understood in the sense of the convolution of the distribution $\partial_j(a_{ij}\partial_i u)$ with the test functions $y \mapsto \rho^\delta(x-y)$ (with x fixed; see, e.g., Yosida [**165**, ch. VI, sct. 3]). More precisely, the term F^δ in (2.3.36) now reads

$$F^\delta := \rho^\delta * (\partial_j(a_{ij}\partial_i u)) - \partial_j(a_{ij}\partial_i(\rho^\delta * u))\,; \tag{2.4.6}$$

that is, explicitly, as in (1.5.245) of Theorem 1.5.7,

(2.4.7)
$$F^\delta(t,x) = \int_{\mathbb{R}^N} \frac{1}{\delta^N} \nabla\rho\left(\tfrac{x-y}{\delta}\right) \cdot (A(t,y)\,\nabla u(t,y))\,\mathrm{d}y$$
$$- \partial_j \left(a_{ij}(t,x)\partial_i \int_{\mathbb{R}^N} \frac{1}{\delta^N} \rho\left(\tfrac{x-y}{\delta}\right) u(t,y)\,\mathrm{d}y \right),$$

where $A(t,x)$ is the $N \times N$ matrix with entries $a_{ij}(t,x)$. By Theorem 1.5.7, we still have that $F^\delta(t,\cdot) \in L^2$ for a.a. $t \in \,]0,T[$, and, as in (2.3.41),

$$\|F^\delta(t)\| \le C\,\|\nabla a_{ij}(t)\|_{s-1}\,\|\nabla u(t)\|\,. \tag{2.4.8}$$

This is enough to show that the a priori estimate (2.4.5) holds; in turn, this implies the asserted continuity of Φ.

2) The existence claim of Theorem 2.4.1 is proven again by means of a Galerkin approximation scheme, which we can now implement directly on the whole space $\mathbb{R}^N$ (that is, without having to restrict the problem to the sequence of expanding balls Ω_ℓ). Indeed, we know from section 1.6 of Chapter 1 that L^2 admits a total basis $\mathcal{W} = (w_k)_{k\in\mathbb{N}^N}$, which is (H^1_*)-regular, in the sense that if $g \in L^2$, its Fourier series (1.2.9) with respect to $\mathcal{W}$ converges in L^2, and also in H^1 if, in addition, $g \in H^1_*$. For each $n \in \mathbb{N}$, we set, as in (2.3.61),

$$\mathcal{W}_n \,:=\, \operatorname{span}\{w_j \;\mid\; |j| \le n\}\,, \tag{2.4.9}$$

and we project problem (2.4.1) into each finite-dimensional space $\mathcal{W}_n$; that is, assuming for simplicity that $b_i \equiv 0$ and $c \equiv 0$, we consider, in analogy to (2.3.62), the infinite set of scalar equations

$$\langle u_{tt} - \partial_j(a_{ij}\,\partial_i u), w_r\rangle = \langle f, w_r\rangle\,, \qquad |r| \ge 0\,. \tag{2.4.10}$$

We temporarily assume that $u_0 \in C^\infty_0(\mathbb{R}^N)$, and later remove this restriction by means of an approximation argument. Then, the sequences $(u^n_0)_{n\ge0}$ and $(u^n_1)_{n\ge0}$, defined respectively by

$$u^n_0 \,:=\, \sum_{|k|=0}^{n} \langle u_0, w_k\rangle\, w_k\,, \qquad u^n_1 \,:=\, \sum_{|k|=0}^{n} \langle u_1, w_k\rangle\, w_k\,, \tag{2.4.11}$$

are such that u^n_0, $u^n_1 \in \mathcal{W}_n$ for all $n \ge 0$, and, as $n \to +\infty$,

$$u^n_0 \to u_0 \quad \text{in } H^1\,, \qquad u^n_1 \to u_1 \quad \text{in } L^2\,. \tag{2.4.12}$$

As in part 2 of section 2.3.2, we find functions $u^n : [0,T] \to \mathcal{W}_n$, which are approximate solutions of (2.4.10), in the sense that they solve the equations

$$\langle u^n_{tt}, w_r\rangle + \langle a_{ij}\,\partial_i u^n, \partial_j w_r\rangle = \langle f, w_r\rangle\,, \tag{2.4.13}$$

for $0 \le |r| \le n$, together with the initial conditions $u^n(0) = u_0^n$ and $u_t^n(0) = u_1^n$. As in Lemma 2.3.1, we show that the sequence $(u^n)_{n\ge 0}$ is bounded in $\mathcal{X}_0(T)$, with a bound independent of the additional regularity condition on u_0; as in part 4 of section 2.3.2, we show that the weak limit u of this sequence, which is in $W^{1,2}(0,T;H^1,L^2)$, satisfies equation (2.1.3). Then, we proceed as in Lemma 2.3.2, and show that $u \in \mathcal{Y}_0(T)$, and satisfies the initial conditions (2.1.2). In this step, in order to show that, as in (2.3.144),

$$\langle a_{ij}(t)\partial_i u(t), \partial_j u(t_0)\rangle \to \langle a_{ij}(t_0)\partial_i u(t_0), \partial_j u(t_0)\rangle \,, \tag{2.4.14}$$

we need, in addition to the weak continuity of $t \mapsto \partial_i u(t)$ into L^2, which we have, also the strong continuity of the function $t \mapsto a_{ij}(t)$ into L^∞, which follows from assumption (2.4.2). Finally, we resort to an approximation argument to remove the condition $u_0 \in C_0^\infty(\mathbb{R}^N)$. More precisely, let $(u_0^n)_{n\ge 0} \subset C_0^\infty(\mathbb{R}^N)$ be such that $u_0^n \to u_0$ in H^1. By our results so far, the corresponding Cauchy problem (2.4.1), with u_0 replaced by u_0^n, has a unique solution $u^n \in \mathcal{Y}_0(T)$, satisfying the bound (2.4.5) uniformly in n. Thus, the sequence $(u^n)_{n\ge 0}$ is bounded in $\mathcal{Y}_0(T)$, and we show that its weak limit u is the desired solution of (2.4.1), again with methods analogous to those used in parts 5 and 6 of section 2.3.2. □

2.5. The Parabolic Cauchy Problem

We briefly describe the type of results one can obtain, with the same techniques of the previous sections, for strong solutions of the parabolic initial-value problem consisting of equation (2.1.5), together with the initial condition

$$u(0,x) = u_0(x) \,. \tag{2.5.1}$$

For simplicity, we only consider strong solutions of equation (2.1.5); however, with methods similar to those of section 2.4, we could also consider weak solutions of the corresponding equations in the divergence form

$$u_t - \partial_j(a_{ij}\,\partial_i u) = f + b_i\,\partial_i u + c\,u \,. \tag{2.5.2}$$

2.5.1. Strong Solutions.

Given $m \in \mathbb{N}$ and $T > 0$, we define

$$\mathcal{P}_m(T) := \{u \in C([0,T];H^{m+1}) \mid Du \in L^2(0,T;H^m)\} \,, \tag{2.5.3}$$

which is a Banach space with respect to the norm defined by

$$\|u\|^2_{\mathcal{P}_m(T)} := \max_{0\le t\le T} \|u(t)\|^2_{m+1} + \int_0^T \|Du\|^2_m \,\mathrm{d}t \,. \tag{2.5.4}$$

In addition to (2.2.1), we assume that

$$a_{ij} \in L^\infty(Q)\,,\;\; D\,a_{ij} \in L^1(0,T;H^{s-1})\,,\quad b_i\,,\; c \,\in\, L^1(0,T;H^s)\,, \tag{2.5.5}$$

and claim:

Theorem 2.5.1. *Let s and $m \in \mathbb{N}$, with $s > \frac{N}{2} + 1$ and $1 \le m \le s$. Given $f \in L^2(0,T;H^m)$ and $u_0 \in H^{m+1}$, the following holds.*

(1) Existence: *There exists a unique solution $u \in \mathcal{P}_m(T)$ of problem (2.1.5)+(2.5.1), which satisfies (2.1.5) in H^m for a.a. $t \in [0,T]$, as well as a.e. in Q.*

(2) A Priori Estimates: *u satisfies the estimate*

$$\|u\|^2_{\mathcal{P}_m(T)} = \max_{0 \le t \le T} \|u(t)\|^2_{m+1} + \int_0^T \|Du\|^2_m \, \mathrm{d}t \le J_0^2 \, \tilde{\psi}_0^2 \,, \tag{2.5.6}$$

where J_0 depends on the data, and $\tilde{\psi}_0$ depends on the coefficients and T.

(3) Well-Posedness: *As a consequence of these estimates, u depends continuously on the data u_0 and f, in the sense that the map $(u_0, f) \mapsto u$ is continuous from $H^{m+1} \times L^2(0,T;H^m)$ into $\mathcal{P}_m(T)$. That is, problem (2.1.5)+(2.5.1) is well-posed.*

(4) Global Regularity: *If, for some k, $0 \le k \le \lfloor \frac{m}{2} \rfloor$, and all $r \in \{0, \ldots, k\}$,*

$$\partial_t^r f \in L^2(0,T;H^{m-2r}) \tag{2.5.7}$$

and

$$\partial_t^r a_{ij} \,,\; \partial_t^r b_i \,,\; \partial_t^r c \in L^2(0,T;H^{s-2r}) \,, \tag{2.5.8}$$

then

$$\partial_t^{r+1} u \in L^2(0,T;H^{m-2r}) \,. \tag{2.5.9}$$

Sketch of Proof. 1) The proof follows the same steps of the proof of Theorem 2.2.1; namely, we first prove the a priori estimate (2.5.6), from which the well-posedness of (2.1.5)+(2.5.1) follows. The existence part is again proven by a similar double approximation argument (that is, restriction to a sequence of initial-boundary value problems on expanding balls in $\mathbb{R}^N$, and Galerkin approximations for each of the latter problems). For an alternative approach, we refer to Krylov [**84**, ch. 2].

2) To establish (2.5.6), at least formally, we differentiate (2.1.5) α times with respect to the space variables, with $|\alpha| \le m$, and multiply the corresponding equations in L^2 by $2\,\partial_x^\alpha u$ and $2\,\partial_x^\alpha u_t$. Summing with respect to α, we obtain an estimate of

$$\max_{0 \le t \le T} \|u(t)\|^2_{m+1} + \int_0^T \|Du\|^2_m \, \mathrm{d}t \,. \tag{2.5.10}$$

As we have remarked in section 2.3.1, (2.5.10) can be established rigorously, by a regularization process involving Friedrichs' mollifiers.

3) The global regularity claim (2.5.9) is again proven by induction on k. For $k = 0$, (2.5.9) just reads $u_t \in L^2(0,T;H^m)$, which does hold if $u \in \mathcal{P}_m(T)$. Assuming in the sequel that b_i, $c \equiv 0$ for simplicity, to prove the claim for $k+1$, $k \geq 0$ and $2(k+1) \leq m$, it is sufficient to show that

$$\partial_t^{k+2} u \in L^2(0,T;H^{m-2(k+1)})\,. \tag{2.5.11}$$

This follows from the identity

$$\partial_t^{k+2} u = \partial_t^{k+1} f + \sum_{\ell=0}^{k+1} \tbinom{k+1}{\ell} \partial_t^\ell a_{ij}\, \partial_t^{k+1-\ell} \partial_i \partial_j u\,, \tag{2.5.12}$$

at the right side of which all the terms are in $L^2(0,T;H^{m-2(k+1)})$. This is clear for $\partial_t^{k+1} f$, by (2.5.7) with $r = k+1$. In the terms with $\ell \leq k$ of the sum, we note that $\partial_t^\ell a_{ij} \in C([0,T];H^{s-2\ell-1})$ by the trace theorem, and $\partial_t^{k+1-\ell}\partial_i\partial_j u \in L^2(0,T;H^{m-2k-2+2\ell})$ by the induction assumption; thus, by the Sobolev product properties, $\partial_t^\ell a_{ij} \quad \partial_t^{k+1-\ell}\partial_i\partial_j u \in L^2(0,T;H^{m-2k-2})$. When $\ell = k+1$, we reach the same conclusion, since $\partial_t^{k+1} a_{ij} \in L^2(0,T;H^{s-2k-2})$ and $\partial_i\partial_j u \in C([0,T];H^{m-1})$. In either case, (2.5.11) follows. □

From equation (2.1.5) and the existence part of Theorem 2.5.1, it follows that

$$-\,a_{ij}\,\partial_i\partial_j u = f + b_i\,\partial_i u + c\,u \;\in L^2(0,T;H^m)\,. \tag{2.5.13}$$

If the underlying space domain were bounded, and u were subject to suitable boundary conditions, (2.5.13) would imply, by standard results on elliptic regularity theory, similar to those in Proposition 1.5.5, that $u \in L^2(0,T;H^{m+2})$. When (2.1.5) is considered in all of $\mathbb{R}^N$, as in the present situation, we can still recover this result, at least under an additional regularity assumption on the coefficients.

Theorem 2.5.2. *In addition to* (2.5.5)*, assume that* $\nabla a_{ij} \in L^1(0,T;H^s)$. *Let* $u \in \mathcal{P}_m(T)$ *be the solution of* (2.1.5)+(2.5.1) *determined by Theorem* 2.5.1. *Then,* $u \in L^2(0,T;H^{m+2})$*, and its norm in this space depends continuously on the norms of* f *in* $L^2(0,T;H^m)$ *and of* u_0 *in* H^{m+1}.

Idea of the Proof. The additional regularity of the coefficients allows us to differentiate (2.1.5) one more time with respect to the space variables; that is, with $|\alpha| \leq m+1$. Multiplying the resulting identities in L^2 by $\partial_x^\alpha u$, we arrive at an estimate of

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|_{m+1}^2 + \alpha_0\,\|\nabla u(t)\|_{m+1}^2\,. \tag{2.5.14}$$

The only detail to note is that, in establishing (2.5.14), one has to deal with the terms $\partial_x^\alpha(f + b_i\,\partial_i u + c\,u)$, in which the total order of differentiation can be up to $m+1$, in contrast to the fact that $f(t) + b_i(t)\,\partial_i u(t) + c(t)\,u(t) \in H^m$

only, for a.a. t. However, such terms appear in a product, in which we can integrate by parts, thereby reducing the order of differentiation to exactly m. More precisely, when $|\alpha| = m+1$, i.e. $\partial_x^\alpha = \partial_k \partial_x^\gamma$ with $k \in \{1, \ldots, N\}$ and $|\gamma| = m$, we estimate

$$\begin{aligned} \langle \partial_k \, \partial_x^\gamma f, \partial_k \partial_x^\gamma u \rangle &= -\langle \partial_x^\gamma f, \partial_k^2 \partial_x^\gamma u \rangle \le \|\partial_x^\gamma f\| \, \|\nabla \partial_x^\alpha u\| \\ &\le \|f\|_m \, \|\nabla u\|_{m+1} \le C_\eta \, \|f\|_m^2 + \eta \, \|\nabla u\|_{m+1}^2 \,, \end{aligned} \tag{2.5.15}$$

and analogously for the other terms involving $b_i \, \partial_i u$ and $c\, u$. We can then choose $\eta > 0$ sufficiently small so as to absorb the last term of (2.5.15) into the corresponding term that appears at the left side of (2.5.14);, we can then integrate, and proceed as before, to obtain an estimate of u in $C([0,T]; H^{m+1}) \cap L^2(0,T; H^{m+2})$. □

2.5.2. Regularity for $t > 0$.

Let $u_0 \in L^2$. It is well known that the Cauchy problem for the linear, homogeneous heat equation, that is (see (0.0.6))

$$\begin{cases} u_t - \Delta\, u = 0 \,, \\ u(0) = u_0 \,, \end{cases} \tag{2.5.16}$$

has a solution $u \in L^2(0, +\infty; H^1) \cap C_{\mathrm{b}}([0, +\infty[; L^2)$, which is such that $u \in C^\infty(\mathbb{R}_{>0} \times \mathbb{R}^N)$. This improvement of the regularity of the solution from $t = 0$ to $t > 0$ is somewhat typical of parabolic equations, and is sometimes referred to as a consequence of the so-called SMOOTHING EFFECT of parabolic operators. In the same spirit, when $f \equiv 0$ (or is very smooth), we can establish an improved regularity result for equation (2.1.5). More precisely, we claim:

Theorem 2.5.3. *Assume that the coefficients of* (2.1.5) *satisfy the second of* (2.2.16), *and* (2.2.17)*; that is,*

$$\partial_t^r a_{ij} \,, \quad \partial_t^r b_i \,, \quad \partial_t^r c \, \in L^2(0,T; H^{s-r}) \qquad \text{for} \quad 0 \le r \le s \,. \tag{2.5.17}$$

For $0 \le m \le s$, let $u_0 \in H^{m+1}$ and $f \in W^{m,2}(0,T; H^m, L^2)$. Then, for all $\tau \in \,]0,T[$,

$$u \in \bigcap_{\ell=0}^{m} C^\ell([\tau, T]; H^{m+1-\ell}) \tag{2.5.18}$$

and

$$\sqrt{(\cdot\, - \tau)}\; \partial_t^{m+1} u \, \in \, L^2(\tau, T; L^2) \,. \tag{2.5.19}$$

That is, if u_0 and the coefficients satisfy the same regularity assumptions as for the hyperbolic equation (2.1.1)*, then the solution of the parabolic equation* (2.1.5) *enjoys, for $t > 0$, the same regularity as the solution to the hyperbolic equation.*

Proof. Again, we assume for simplicity that b_i, $c \equiv 0$. By (1.7.58) of Theorem 1.7.4,

$$\partial_t^r f \in L^2(0,T;H^{m-r})\,, \qquad \text{for} \quad 0 \le r \le m\,. \tag{2.5.20}$$

Let $\tau \in \,]0,T[$. We wish to prove that, for $0 \le \ell \le m$,

$$\partial_t^\ell u \;\in\; C([\tau,T];H^{m+1-\ell})\,, \tag{2.5.21}$$

$$\sqrt{(\cdot-\tau)}\,\partial_t^{\ell+1} u \;\in\; L^2(\tau,T;H^{m-\ell})\,. \tag{2.5.22}$$

We proceed by induction on ℓ. For $\ell = 0$, (2.5.21) and (2.5.22) follow from the fact that $u \in \mathcal{P}_m(T)$; in fact, when $\ell = 0$ we can take $\tau = 0$. Assume then that (2.5.21) and (2.5.22) hold for $0 \le \ell \le m-1$. To prove they hold also for $\ell+1$, we follow a formal procedure, which can, again, be justified by means of a regularization process. We set $\tau_1 := \frac{1}{3}\tau$, $\tau_2 := \frac{2}{3}\tau$. Differentiating equation (2.1.5) $k+1$ times with respect to t, $0 \le k \le m-1$, we obtain

$$\partial_t^{k+2} u - a_{ij}\,\partial_i\partial_j(\partial_t^{k+1}u) = \partial_t^{k+1} f + \sum_{r=1}^{k+1} \binom{k+1}{r}\,\partial_t^r a_{ij}\,\partial_t^{k+1-r}\partial_i\partial_j u\,. \tag{2.5.23}$$

We multiply this identity in H^{m-1-k} by $2(t-\tau_2)\partial_t^{k+2}u$, to obtain

(2.5.24)

$$2(t-\tau_2)\,\|\partial_t^{k+2}u\|^2_{m-1-k} + \frac{\mathrm{d}}{\mathrm{d}t}\,\Big((t-\tau_2)\,Q_{m-1-k}(a,\nabla\partial_t^{k+1}u)\Big)$$

$$= Q_{m-1-k}(a,\nabla\partial_t^{k+1}u) + 2(t-\tau_2)\,\left(\Lambda_f + \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4\right)\,,$$

where $Q_{m-1-k}(a,\cdot)$ is defined in (2.3.13), and

$$\Lambda_f \;:=\; \langle \partial_t^{k+1} f, \partial_t^{k+2}u\rangle_{m-1-k}\,, \tag{2.5.25}$$

$$\Lambda_1 \;:=\; -\sum_{|\alpha|\le m-1-k} \langle \partial_j a_{ij}\;\partial_i\partial_x^\alpha\partial_t^{k+1}u, \partial_x^\alpha\partial_t^{k+2}u\rangle\,, \tag{2.5.26}$$

$$\Lambda_2 \;:=\; \tfrac{1}{2}\sum_{|\alpha|\le m-1-k} \langle \partial_t a_{ij}\;\partial_i\partial_x^\alpha\partial_t^{k+1}u, \partial_j\partial_x^\alpha\partial_t^{k+1}u\rangle\,, \tag{2.5.27}$$

$$\Lambda_3 \;:=\; \sum_{r=1}^{k+1}\binom{k+1}{r}\langle \partial_t^r a_{ij}\;\partial_t^{k+1-r}\partial_i\partial_j u, \partial_t^{k+2}u\rangle_{m-1-k}\,, \tag{2.5.28}$$

$$\Lambda_4 \;:=\; \sum_{|\alpha|\le m-1-k}\;\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\langle A^k_{\alpha\beta}(u), \partial_x^\alpha\partial_t^{k+2}u\rangle\,, \tag{2.5.29}$$

with

$$A^k_{\alpha\beta}(u) := \partial_x^\beta a_{ij}\,\partial_x^{\alpha-\beta}\partial_i\partial_j\,\partial_t^{k+1}u\,. \tag{2.5.30}$$

We estimate these terms in the usual way. At first, for almost all $t \in [\tau_2, T]$,

(2.5.31)
$$\Lambda_f \leq C \, \|\partial_t^{k+1} f\|^2_{m-k-1} + \tfrac{1}{8} \, \|\partial_t^{k+2} u\|^2_{m-k-1} \, .$$

Then,

(2.5.32)
$$\begin{aligned} \Lambda_1 \ &\leq \ C \, |\nabla a_{ij}|_\infty \, \|\nabla \partial_t^{k+1} u\|_{m-1-k} \, \|\partial_t^{k+2} u\|_{m-1-k} \\ &\leq \ C \, \|\nabla a_{ij}\|^2_{s-1} \, \|\nabla \partial_t^{k+1} u\|^2_{m-1-k} + \tfrac{1}{8} \, \|\partial_t^{k+2} u\|^2_{m-1-k} \, , \end{aligned}$$

and, analogously,

(2.5.33)
$$\Lambda_2 \leq C \, |\partial_t a_{ij}|_\infty \, \|\nabla \partial_t^{k+1} u\|^2_{m-1-k} \leq C \, \|\partial_t a_{ij}\|_{s-1} \, \|\nabla \partial_t^{k+1} u\|^2_{m-1-k} \, .$$

Next, by the Sobolev product properties,

(2.5.34)
$$\Lambda_3 \leq C \sum_{r=1}^{k+1} \|\partial_t^r a_{ij}\|^2_{s-r} \, \|\partial_t^{k+1-r} \partial_i \partial_j u\|^2_{m-k-2+r} + \tfrac{1}{8} \, \|\partial_t^{k+2} u\|^2_{m-k-1} \, ,$$

and, recalling that $\beta > 0$,

(2.5.35)
$$\begin{aligned} \Lambda_4 \leq C &\sum_{0<\beta\leq\alpha} \|\partial_x^\beta a_{ij}\|^2_{s-|\beta|} \, \|\partial_x^{\alpha-\beta} \partial_i \partial_j \partial_t^{k+1} u\|^2_{m-k-2-|\alpha|+|\beta|} \\ &+ \ \tfrac{1}{8} \, \|\partial_t^{k+2} u\|^2_{m-k-1} \\ \leq C \, &\|\nabla a_{ij}\|^2_{s-1} \, \|\nabla \partial_t^{k+1} u\|^2_{m-1-k} + \tfrac{1}{8} \, \|\partial_t^{k+2} u\|^2_{m-k-1} \, . \end{aligned}$$

Replacing these estimates in (2.5.24), and integrating on $[\tau_2, t]$, $\tau_2 < t \leq T$, we obtain

(2.5.36)
$$\begin{aligned} &\int_{2\tau/3}^t (\theta - \tau_2) \, \|\partial_t^{k+2} u\|^2_{m-1-k} \, \mathrm{d}\theta \ + \ (t - \tau_2) \, \alpha_0 \, \|\nabla \partial_t^{k+1} u(t)\|^2_{m-1-k} \\ &\qquad \leq \ C(R_1 + R_2 + R_3 + R_4 + R_5) \, , \end{aligned}$$

where

$$R_1 := \int_{2\tau/3}^{t} |a_{ij}|_\infty \, \|\nabla \partial_t^{k+1} u\|_{m-1-k}^2 \, \mathrm{d}\theta \,, \tag{2.5.37}$$

$$R_2 := \int_{2\tau/3}^{t} (\theta - \tau_2) \, \|\partial_t^{k+1} f\|_{m-1-k}^2 \, \mathrm{d}\theta \,, \tag{2.5.38}$$

$$R_3 := \int_{2\tau/3}^{t} (\theta - \tau_2) \, \|\partial_t a_{ij}\|_{s-1} \, \|\nabla \partial_t^{k+1} u\|_{m-1-k}^2 \, \mathrm{d}\theta \,, \tag{2.5.39}$$

$$R_4 := \sum_{r=1}^{k+1} \int_{2\tau/3}^{t} (\theta - \tau_2) \, \|\partial_t^r a_{ij}\|_{s-r}^2 \, \|\partial_t^{k+1-r} \partial_i \partial_j u\|_{m-2-k+r}^2 \, \mathrm{d}\theta \,, \tag{2.5.40}$$

$$R_5 := \int_{2\tau/3}^{t} (\theta - \tau_2) \, \|\nabla a_{ij}\|_{s-1}^2 \, \|\nabla \partial_t^{k+1} u\|_{m-1-k}^2 \, \mathrm{d}\theta \,. \tag{2.5.41}$$

We now choose $k = \ell$, and note that the corresponding terms R_1, R_2 and R_4 of (2.5.36) can be estimated in terms of known bounds, via assumptions (2.5.20), (2.5.21) and (2.5.22) (the latter, as the induction assumption). Indeed, in R_2 we know from (2.5.20) that $\partial_t^{\ell+1} f \in L^2(0,T;H^{m-\ell-1})$; for R_1, recalling that, by the trace theorem, the first of (2.5.17), with $r = 0$ and $r = 1$, imply that $a_{ij} \in C([0,T];H^{s-1}) \hookrightarrow L^\infty(Q)$, setting $C_a := \sup_Q |a_{ij}|$ we estimate

$$\begin{aligned} R_1 &\le C_a \int_{2\tau/3}^{t} \|\nabla \partial_t^{\ell+1} u\|_{m-\ell-1}^2 \, \mathrm{d}\theta \\ &\le \frac{3}{\tau} C_a \int_{2\tau/3}^{t} (\theta - \tau_1) \|\partial_t^{\ell+1} u\|_{m-\ell}^2 \, \mathrm{d}\theta \\ &\le \frac{3}{\tau} C_a \int_{\tau/3}^{T} (\theta - \tau_1) \|\partial_t^{\ell+1} u\|_{m-\ell}^2 \, \mathrm{d}\theta \,, \end{aligned} \tag{2.5.42}$$

and note that this term is finite, by the induction assumption (2.5.22), with τ replaced by τ_1. For R_4, we resort to the first of (2.5.17) and the induction assumption (2.5.21), with ℓ replaced by $\ell + 1 - r \le \ell \le m - 1$, and τ replaced by τ_2, so that $\partial_t^{\ell+1-r} \partial_i \partial_j u \in C([\tau_2,T];H^{m-2-\ell+r})$. In conclusion, there exists a constant $K_0 > 0$ such that for all $t \in [\tau_2, T]$,

(2.5.43)

$$\begin{aligned} \int_{2\tau/3}^{t} (\theta - \tau_2) \, \|\partial_t^{\ell+2} u\|_{m-1-\ell}^2 \, \mathrm{d}\theta + \underbrace{(t - \tau_2) \, \|\nabla \partial_t^{\ell+1} u(t)\|_{m-1-\ell}^2}_{=: \varphi(t)} \\ \le K_0^2 + C \int_{2\tau/3}^{t} (\|\partial_t a_{ij}\|_{s-1} + \|\nabla a_{ij}\|_{s-1}^2) \, \varphi(\theta) \, \mathrm{d}\theta \,, \end{aligned}$$

from which, by Gronwall's inequality,

(2.5.44)

$$\int_{2\tau/3}^{t} (\theta - \tau_2)\, \|\partial_t^{\ell+2} u\|_{m-1-\ell}^2 \,\mathrm{d}\theta + (t - \tau_2)\, \|\nabla \partial_t^{\ell+1} u(t)\|_{m-1-\ell}^2$$
$$\leq K_0^2 \exp\left(C \int_0^T (1 + \|Da_{ij}\|_{s-1}^2) \,\mathrm{d}\theta \right) =: K_1^2 .$$

From this, we first deduce that

(2.5.45)

$$\int_{\tau}^{T} (\theta - \tau)\, \|\partial_t^{\ell+2} u\|_{m-1-\ell}^2 \,\mathrm{d}\theta \leq \int_{\tau}^{T} (\theta - \tau_2)\, \|\partial_t^{\ell+2} u\|_{m-1-\ell}^2 \,\mathrm{d}\theta$$
$$\leq \int_{2\tau/3}^{T} (\theta - \tau_2)\, \|\partial_t^{\ell+2} u\|_{m-1-\ell}^2 \,\mathrm{d}\theta \leq K_1^2 ,$$

which shows that (2.5.22) holds for $\ell + 1$. Moreover, since $t - \tau_2 \geq \tau_1$ if $t \geq \tau$, (2.5.44) also implies that for all $t \in [\tau, T]$,

$$\|\nabla \partial_t^{\ell+1} u(t)\|_{m-\ell-1}^2 \leq \frac{3}{\tau} K_1^2 , \qquad (2.5.46)$$

from which

$$\nabla \partial_t^{\ell+1} u \in L^\infty(\tau, T; H^{m-\ell-1}) . \qquad (2.5.47)$$

Going back to (2.5.24), (2.5.45) and (2.5.47), together with (2.5.31), ..., (2.5.35), yield that

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big((\cdot - \tau_2)\, Q_{m-1-\ell}(a, \nabla \partial_t^{\ell+1} u) \Big) \in L^1(\tau, T) ; \qquad (2.5.48)$$

consequently, the function $t \mapsto Q_{m-1-\ell}(a(t), \nabla \partial_t^{\ell+1} u(t))$ is continuous on $[\tau, T]$. From this, we can deduce that $\nabla \partial_t^{\ell+1} u \in C([\tau, T]; H^{m-\ell-1})$, with an argument similar to the one used, on the function $F(t, t_0)$ of (2.3.142), to show (2.3.147). More precisely, we fix $t_0 \in [\tau, T]$, and compute that

$$\alpha_0\, \|\nabla \partial_t^{\ell+1} u(t) - \nabla \partial_t^{\ell+1} u(t_0)\|_{m-\ell-1}^2$$
$$\leq Q_{m-1-\ell}(a(t), \nabla \partial_t^{\ell+1} u(t) - \nabla \partial_t^{\ell+1} u(t_0))$$
$$= Q_{m-1-\ell}(a(t), \nabla \partial_t^{\ell+1} u(t)) + Q_{m-1-\ell}(a(t), \nabla \partial_t^{\ell+1} u(t_0)) \qquad (2.5.49)$$
$$- 2 \sum_{|\alpha| \leq m-1-\ell} \underbrace{\langle a_{ij}(t)\, \partial_i \partial_x^\alpha \partial_t^{\ell+1} u(t), \partial_i \partial_x^\alpha \partial_t^{\ell+1} u(t_0) \rangle}_{=:\, \psi_\alpha(t)}$$
$$=: V_1(t) + V_2(t) + V_3(t) .$$

Since, as previously shown, the functions V_1 and V_2 are continuous,

$$V_1(t) + V_2(t) \to V_1(t_0) + V_2(t_0) = 2\,V_1(t_0)\,. \tag{2.5.50}$$

We now show that, in $V_3(t)$, $\psi_\alpha(t) \to \psi_\alpha(t_0)$, for each multi-index α. Arguing as in (2.3.144), it is sufficient to show that the map $t \mapsto \nabla\partial_t^{\ell+1}u(t)$ is weakly continuous from $[\tau, T]$ into $H^{m-\ell-1}$. This follows from Proposition 1.7.1, recalling (2.5.47), and noting that $\nabla\partial_t^{\ell+1}u \in C([\tau,T]; H^{m-\ell-2})$. To see the latter, we refer to equation (2.5.23), with $k = \ell - 1$, which implies that

$$\partial_t^{\ell+1}u = \partial_t^\ell f + \sum_{r=0}^{\ell} \binom{\ell}{r}\, \partial_t^{\ell-r}a_{ij}\, \partial_t^r \partial_i\partial_j u\,. \tag{2.5.51}$$

By the trace theorem, it follows that $\partial_t^\ell f \in C([0,T]; H^{m-\ell-1})$ and $\partial_t^{\ell-r}a_{ij} \in C([0,T]; H^{s-\ell+r-1})$. The induction assumption (2.5.21), with ℓ replaced by r, $r \le \ell \le m-1$, implies that $\partial_t^r\partial_i\partial_j u \in C([\tau,T]; H^{m-r-1})$. Since

$$s - \ell + r - 1 \ge m - \ell - 1\,, \qquad m - r - 1 \ge m - \ell - 1 \tag{2.5.52}$$

and

$$(s-\ell+r-1)+(m-r-1) = (s-1)+(m-\ell-1) > \tfrac{N}{2}+(m-\ell-1)\,, \tag{2.5.53}$$

by the Sobolev product properties it follows that the right side of (2.5.51) is in $C([\tau,T]; H^{m-\ell-1})$. Thus, so is $\partial_t^{\ell+1}u$, and $\nabla\partial_t^{\ell+1}u \in C([\tau,T]; H^{m-\ell-2})$, as claimed. Consequently, $\psi_\alpha(t) \to \psi_\alpha(t_0)$. Then,

$$V_3(t) \to V_3(t_0) = -\,2\,V_2(t_0) = -\,2\,V_1(t_0)\,, \tag{2.5.54}$$

and we deduce from (2.5.49) that $\nabla\partial_t^{\ell+1}u \in C([\tau,T]; H^{m-\ell-1})$. Since, as we have shown from (2.5.51), $\partial_t^{\ell+1}u \in C([\tau,T]; H^{m-\ell-1})$, we conclude that $\partial_t^{\ell+1}u \in C([0,T]; H^{m-\ell})$; that is, (2.5.21) also holds for $\ell+1$, as claimed. This completes the proof of Theorem 2.5.3. □

Remarks. 1) As estimate (2.5.46) shows, it is not reasonable to expect that the regularity result (2.5.18) should hold on the whole interval $[0,T]$. Indeed, if this were the case, that is, if problem (2.1.5)+(2.5.1) did have a solution $u \in \bigcap_{\ell=0}^{m} C^\ell([0,T]; H^{m+1-\ell})$, then it would follow that, for $0 \le \ell \le m$,

$$v_\ell := \partial_t^\ell u(0) \;\in\; H^{m+1-\ell}\,. \tag{2.5.55}$$

On the other hand, the values of these derivatives can be recursively computed from the initial value u_0, by means of equation (2.1.5). More precisely, starting from u_0, we can generate the functions

$$u_{\ell+1} := \partial_t^\ell f(0) + \sum_{h=0}^{\ell} \binom{\ell}{h} \partial_t^h a_{ij}(0)\; \partial_i\partial_j u_{\ell-l}\,; \tag{2.5.56}$$

for example,

$$u_1 = f(0) + a_{ij}(0)\,\partial_i\partial_j u_0\,, \tag{2.5.57}$$

$$u_2 = f_t(0) + a_{ij}(0)\,\partial_i\partial_j u_1 + \partial_t a_{ij}(0)\,\partial_i\partial_j u_0\,. \tag{2.5.58}$$

Note that the functions u_ℓ can be defined independently of the existence of a solution of (2.1.5). Thus, if problem (2.1.5)+(2.5.1) had a solution $u \in \bigcap_{\ell=0}^{m} C^\ell([0,T]; H^{m+1-\ell})$, we would conclude that the identities $u_\ell = v_\ell$ would have to hold for all $\ell \in \{1, \ldots, k\}$. Therefore, by (2.5.55), the conditions

$$u_\ell \in H^{m+1-\ell}\,, \qquad 0 \le \ell \le k\,, \tag{2.5.59}$$

would be necessary in order for the regularity result (2.5.18) to hold on the whole interval $[0,T]$. For example, (2.5.59) would require that $u_1 \in H^m$; but from (2.5.57) we can only deduce, from the assumptions on u_0, f and the a_{ij} (i.e., $u_0 \in H^{m+1}$, $f \in C([0,T]; H^{m-1})$ and $a_{ij} \in C([0,T]; H^{s-1})$), that $u_1 \in H^{m-1}$.

2) In fact, conditions (2.5.59) are also sufficient for the regularity result (2.5.18) to hold on $[0,T]$. Indeed, in this case we do not need to consider the factor $t-\tau_2$ in (2.5.24), and, recalling that, by (2.5.55), $\partial_t^{k+1}u(0) = v_{k+1} = u_{k+1}$, estimate (2.5.36) can be replaced by

$$\begin{aligned}
&\int_0^t \|\partial_t^{k+2}u\|^2_{m-1-k}\,\mathrm{d}\theta + \alpha_0\,\|\nabla\partial_t^{k+1}u(t)\|^2_{m-1-k} \\
&\quad\le a_1\,\|\nabla u_{k+1}\|^2_{m-1-k} + \int_0^t \|\partial_t^{k+1}f\|^2_{m-1-k}\,\mathrm{d}\theta \\
&\qquad + C\sum_{r=1}^{k+1}\int_0^t \|\partial_t^r a_{ij}\|^2_{s-r}\,\|\partial_t^{k+1-r}\partial_i\partial_j u\|^2_{m-k+r-1}\,\mathrm{d}\theta \\
&\qquad + C\int_0^t \|\nabla a_{ij}\|^2_{s-1}\,\|\nabla\partial_t^{k+1}u\|^2_{m-1-k}\,\mathrm{d}\theta\,,
\end{aligned} \tag{2.5.60}$$

in the right side of which the first three terms are finite: the first, if (2.5.59) holds, the second, by (2.5.20), and the third is estimated as the term R_4 in (2.5.40).

3) As we remarked at the end of section 2.3 for the hyperbolic problem (2.1.1)+(2.1.2), the regularity results for (2.1.5)+(2.5.1), both global (i.e., in $[0,T]$) and for $t > 0$ (i.e., in $[\tau,T]$), can be improved in an analogous way. In particular, if the data and the coefficients are C^∞, the solution of problem (2.1.5)+(2.5.1) is C^∞ as well. ◇

2.5.3. Sobolev and Hölder Solutions.

We conclude by showing that Theorem 2.5.1 is consistent with the classical solvability results of the parabolic Cauchy problem (2.1.5)+(2.5.1) in the Hölder spaces $C^{1+\alpha/2,2+\alpha}(\overline{Q})$, $Q = \,]0,T[\,\times\mathbb{R}^N$. Indeed, assuming again for simplicity that b_j, $c \equiv 0$, take $m = s = \lfloor\frac{N}{2}\rfloor + 2$ in Theorem 2.5.1, and assume that

$$u_0 \in H^{s+1}\,, \qquad f \in L^2(0,T;H^s)\,, \quad f_t \in L^2(0,T;H^{s-2})\,, \tag{2.5.61}$$

as well as

$$a_{ij} \in L^2(0,T;H^{s+1})\,, \qquad \partial_t a_{ij} \in L^2(0,T;H^{s-2})\,. \tag{2.5.62}$$

Then, by the Sobolev imbedding (1.5.61) and Theorem 1.7.5 (with, respectively, $m = s+1$, $r = 2$, and $m = s-1$, $r = 0$ in their statements), it follows that

$$u_0 \in C^{2,\alpha}(\overline{\mathbb{R}^N})\,, \quad a_{ij}\,,\ f \in C^{\alpha/2,\alpha}(\overline{Q}) \qquad (0 < \alpha \le \tfrac{1}{2})\,. \tag{2.5.63}$$

As a consequence of theorem 8.10.1 of Krylov [**83**], conditions (2.5.63) are sufficient to yield a unique solution $\tilde{u} \in C^{1+\alpha/2,2+\alpha}(\overline{Q})$ to problem (2.1.5)+(2.5.1). On the other hand, by (2.5.62) we can apply Theorem 2.5.2 and part 4 of Theorem 2.5.1, with $k = 1$, to deduce that problem (2.1.5)+(2.5.1) also has a unique solution $u \in L^2(0,T;H^{s+2})$, with $u_{tt} \in L^2(0,T;H^{s-2})$. Again by Theorem 1.7.5 (with $m = s+1$ and $r = 1$), it follows that $u \in C^{1+\alpha/2,2+\alpha}(\overline{Q})$. By the uniqueness of both classical and Sobolev solutions, it follows that $u = \tilde{u}$, which shows the asserted consistency. Thus, we obtain the commutative diagram

$$\begin{array}{ccc} \text{Sobolev data} & \longrightarrow & \text{Hölder data} \\ \downarrow & & \downarrow \\ \text{Sobolev solution} & \longrightarrow & \text{Hölder solution} \end{array} \tag{2.5.64}$$

where the horizontal arrows refer to the Sobolev imbeddings. In the same way, we can show that, if f and the coefficients are more regular, in the sense of part 4 of Theorem 2.5.1 with $k > 1$, the Hölder and Sobolev additional regularity of u are also consistent (the former, as per theorem 8.12.1 of Krylov [**83**]).