Introduction

In this book we treat free boundary problems of the type

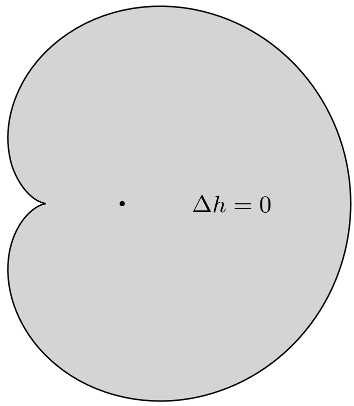
$$\Delta u = f(x, u, \nabla u) \quad \text{in } D \subset \mathbb{R}^n,$$

where the right-hand side f exhibits a jump discontinuity in its second and/or third variables. The discontinuity set is a priori unknown and therefore is said to be *free*. The prototypical example is the so-called *classical obstacle problem*, which minimizes the energy of a stretched membrane over a given obstacle (see more in §1.1) and in its simplest form can be reformulated as

$$\Delta u = \chi_{\{u>0\}}, \quad u \geq 0, \quad \text{in } D. \tag{0.1}$$

In this case $f(x, u, \nabla u) = \chi_{\{u>0\}}$ is the Heaviside function of u. The *free boundary* here is $\Gamma = \partial\{u > 0\} \cap D$. Also note that the *sign condition* $u \geq 0$ in (0.1) appears naturally in this problem. The classical obstacle problem and its variations have been the subject of intense studies in the past few decades. Today, there is a more or less complete and comprehensive theory for this problem, both from theoretical and numerical points of view.

The motivation for studying free boundary problems in general, and obstacle-type problems in particular, has roots in many applications. Classical applications of these problems originate (predominantly) in engineering sciences, where many problems (sometimes after a major simplification) could be formulated as variational inequalities or more general free boundary problems. In many cases variational inequalities can be viewed as obstacle-type problems, with an additional sign condition, as in the case of the classical obstacle problem. This particular feature significantly simplifies the problem, and most methods, up to the early 1990s, relied heavily on this strong property of solutions.



$$\int_\Omega h = \frac{3\pi}{2} h(0) + \frac{\pi}{2} \partial_x h(0)$$

Figure 0.1. Cardioid $\Omega = \{z = w + (1/2)w^2 : |w| < 1\}$ is a quadrature domain

Obstacle-type problems also arise in applications in potential theory, complex analysis, and geophysics. In contrast to variational inequalities, these problems typically lack the sign condition. To manifest, in the simplest form, one such application, we start with the mean-value property for harmonic functions in the unit ball B_1:

$$\int_{B_1} h(y)dy = c_n h(0),$$

where h is any harmonic function, integrable over B_1. More generally, given a distribution μ with compact support (say, a finite set), let Ω be a bounded domain containing $\operatorname{supp}\mu$ such that a generalized mean-value property (or a quadrature formula) holds:

$$\int_\Omega h(y)dy = \int h d\mu,$$

for any integrable harmonic function h in Ω. Such Ω are known as *quadrature domains*. There is an abundance of such domains; see [**Dav74, Sak82**]. An example of a quadrature domain is the so-called cardioid, which can be represented via the conformal mapping $z = w + (1/2)w^2$ of the unit circle (see Fig. 0.1). The distribution μ in this case equals $(\pi/2)(3\delta_0 + \partial_x \delta_0)$, where δ_0 is the Dirac delta function centered at the origin.

By letting $h(y) = |x-y|^{2-n}$, $x \in \Omega^c$, in dimensions $n \geq 3$, and the corresponding logarithmic potential in dimension $n = 2$, one will obtain

$$\int_\Omega |x-y|^{2-n} dy = \int |x-y|^{2-n} d\mu(y), \quad x \in \Omega^c.$$

Now letting

$$u(x) := c_n \int_\Omega |x-y|^{2-n} dy - c_n \int |x-y|^{2-n} d\mu(y),$$

for an appropriately chosen constant c_n, we see that u satisfies

$$\Delta u = \chi_\Omega - \mu, \quad u = |\nabla u| = 0 \text{ in } \Omega^c \quad \text{in } \mathbb{R}^n.$$

In particular, in a small neighborhood $B_r(x^0)$ of any point $x^0 \in \partial\Omega$ we have

$$\Delta u = \chi_\Omega \quad \text{in } B_r(x^0), \quad u = |\nabla u| = 0 \quad \text{in } B_r(x^0) \setminus \Omega. \tag{0.2}$$

This problem bears resemblance to the classical obstacle problem; however, the important difference is that there is no sign condition imposed on u. More specifically, the solutions of (0.1) are precisely the nonnegative solutions of (0.2).

The boundary of Ω, $\Gamma = \partial\Omega$, is the free boundary in the problem, as we do not know beforehand the location, shape and regularity properties of Γ. Taking Ω to be the cardioid (see Fig. 0.1), we see that Γ may naturally exhibit cusp singularities, but otherwise is smooth (real analytic). It turns out that this is a typical regularity property of free boundaries for obstacle-type problems that we treat in this book.

The main approach to the study of local properties of the free boundary is the so-called method of *blowup*. This method originated in the work of Caffarelli [**Caf80**], motivated by a similar method in geometric measure theory, in the study of minimal surfaces. We will demonstrate this method by the example of the solution u of (0.2). For any $\lambda > 0$ consider the rescalings

$$u_\lambda(x) = u_{x^0,\lambda}(x) = \frac{u(x^0 + \lambda x)}{\lambda^2},$$

which will be defined now in $B_{r/\lambda}$. The factor of λ^2 is chosen so that u_λ still satisfy conditions similar to (0.2):

$$\Delta u_\lambda = \chi_{\Omega_\lambda} \quad \text{in } B_{r/\lambda}, \quad u_\lambda = |\nabla u_\lambda| = 0 \quad \text{in } B_{r/\lambda} \setminus \Omega_\lambda,$$

where

$$\Omega_\lambda = \{x : x^0 + \lambda x \in \Omega\} = \frac{1}{\lambda}(\Omega - x^0).$$

Heuristically, this corresponds to "zooming" with factor $1/\lambda$ near x^0. The idea is now to study the limit as $\lambda \to 0+$ (which would correspond to the idea of "infinite zoom" at x^0). To this end, we need to have some uniform estimates for the family of rescalings $\{u_\lambda\}$. It turns out, in fact, that one can prove the *quadratic growth* of the original solution u near x^0, i.e.,

$$|u(x)| \le M|x - x^0|^2, \quad x \in B_{r/2}(x^0), \tag{0.3}$$

which readily implies the uniform estimates

$$|u_\lambda(x)| \le M|x|^2, \quad x \in B_{r/(2\lambda)},$$

for any $\lambda > 0$, and then the a priori $C^{1,\alpha}$ estimates imply the convergence of u_λ to a certain u_0, locally in the entire space $\mathbb{R}^n$, over a subsequence

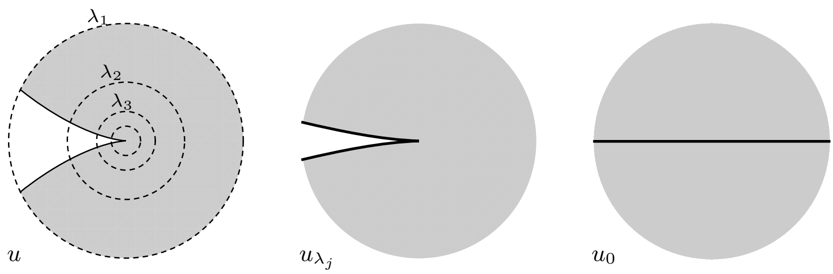


Figure 0.2. Blowup near the cusp point of the cardioid

$\lambda = \lambda_j \to 0+$ (see Fig. 0.2). Such u_0 will be called a *blowup* of u at x^0. There are a few difficulties that one encounters in this process:

1) The quadratic growth estimate is far from being obvious. In fact, if one treats u as being a solution of the equation $\Delta u = \chi_\Omega$ with a bounded right-hand side, then the best estimate one could hope for is

$$|u(x)| \leq M|x - x^0|^2 \log(1/|x - x^0|),$$

and this estimate does not scale well to provide uniform estimates for u_λ. However, using the additional structure of the problem in (0.2), it is possible to remove the logarithmic term. This is relatively easy in the presence of the sign condition $u \geq 0$. However, it requires the application of a powerful monotonicity formula of Alt-Caffarelli-Friedman in the general case (for details see Chapter 2).

2) In principle, the limits over different subsequences $\lambda = \lambda_j \to 0+$ may lead to different blowups. One of the main difficulties is to show that in fact the blowup u_0 is unique.

It is relatively easy to show that the blowups u_0 themselves satisfy a condition similar to (0.2), but in the entire space:

$$\Delta u_0 = \chi_{\Omega_0} \quad \text{in } \mathbb{R}^n, \quad u_0 = |\nabla u_0| = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega_0,$$

or we say that u_0 is a *global solution*. Similarly to the elliptic problems, there is a Liouville-type theorem for global solutions as above, which says for instance that u_0 must either be a quadratic polynomial or must be nonnegative and convex. Different types of blowups lead to a natural classification of free boundary points.

The next step in the study is to transfer some of the properties of the blowups to the rescalings and then to the original solution. For instance, if the blowup u_0 is convex and nonpolynomial, then there is a cone of directions $\mathcal{C}$ such that

$$\partial_e u_0 \geq 0 \quad \text{in } \mathbb{R}^n, \quad e \in \mathcal{C}.$$

It turns out that a slightly modified version of this property can be concluded also for the rescalings u_λ with small $\lambda > 0$, and in particular that for a small $\rho > 0$,

$$\partial_e u \geq 0 \quad \text{in } B_\rho(x^0), \quad e \in \mathcal{C}.$$

The monotonicity in a cone of directions then implies the Lipschitz regularity of Γ. After this initial regularity, one can push the regularity of the free boundary to $C^{1,\alpha}$ by an application of the so-called boundary Harnack principle and then further to $C^{k,\alpha}$ and real analyticity by the so-called method of partial hodograph-Legendre transform. A rigorous, yet readable, implementation of the steps above is the main purpose of this book.

Suggestions for reading/teaching

Anyone who intends to read this book, should have a basic knowledge in elliptic PDEs (maximum/comparison principle, the Harnack inequality, interior/boundary estimates, compactness arguments, etc). We also encourage the reader to look at the book by Caffarelli-Salsa [**CS02**], especially Part 3, for many technical tools that are extensively used in free boundary problems.

The book treats three different but methodologically close problems, here called Problems **A**, **B**, and **C**, and in addition, the so-called thin obstacle problem (Problem **S**). Problem **A** is essentially the same as problem (0.2) described above. For a beginner, we would suggest a study of this problem only, at least in the first reading.

Anyone who wants to give a course in the topic can choose Problem **A**, throughout Chapters 1–6, for a full-time 1/4 semester course. It can also be run over a semester with 2h/week lecture. An extended course can include model Problems **B** and **C** as well. Technically they are not very different, but sometime require finer analysis and new ideas to achieve results similar to those for Problem **A**.

Chapters 7–8 treat singular sets and the touch with fixed boundaries, and can be used as part of an examination where students can make presentations of various parts of these two chapters and do the exercises in these chapters.

In Chapter 9, we treat the so-called *thin obstacle problem* (Problem **S**). This chapter can be taught as a separate course, with some use of existing articles in the field. It can also be used as presentation material for students taking the above format of a course.

Each chapter closes with bibliographical notes for the results in that chapter and some of their generalizations.

At the very end of each chapter, we have tried to gather as many exercises as possible. Proofs of some of the theorems/lemmas are also put in exercises; in many cases hints (or even brief solutions) are provided.