# Preface

Free boundary problems (FBPs) are considered today as one of the most important directions in the mainstream of the analysis of partial differential equations (PDEs), with an abundance of applications in various sciences and real world problems. In the past two decades, various new ideas, techniques, and methods have been developed, and new important, challenging problems in physics, industry, finance, biology, and other areas have arisen.

The study of free boundaries is an extremely broad topic not only due to the diversity of applications but also because of the variety of the questions one may be interested in, ranging from modeling and numerics to the purely theoretical questions. This breadth presents challenges and opportunities!

A particular direction in free boundary problems has been the study of the regularity properties of the solutions and those of the free boundaries. Such questions are usually considered very hard, as the free boundary is not known a priori (it is part of the problem!) so the classical techniques in elliptic/parabolic PDEs do not apply. In many cases the success is achieved by combining the ideas from PDEs with the ones from geometric measure theory, the calculus of variations, harmonic analysis, etc.

Today there are several excellent books on free boundaries, treating various issues and questions: e.g. [**DL76**], [**KS80**], [**Cra84**], [**Rod87**], [**Fri88**], [**CS05**]. These books are great assets for anyone who wants to learn FBPs and related techniques; however, with the exception of [**CS05**], they date back two decades. We believe that there is an urge for a book where some of the most recent developments and new methods in the regularity of free boundaries can be introduced to the nonexperts and particularly to the graduate students starting their research in the field. This gap in the literature has been partially filled by the aforementioned book of Caffarelli and

Salsa [**CS05**], which treats the Stefan-type free boundary problems (with the Bernoulli gradient condition). Part 3 in [**CS05**], in particular, covers several technical tools that should be known to anyone working in the field of PDEs/FBPs.

Our intention, in this book, was to give a coherent presentation of the study of the regularity properties of the free boundary for a particular type of problems, known as *obstacle-type problems*. The book grew out of the lecture notes for the courses and mini-courses given by the authors at various locations, and hence we believe that the format of the book is most suitable for a graduate course (see the end of the Introduction for suggestions). Notwithstanding this, we have to warn the reader that this book is far from being a complete reference for the regularity theory. We hope that it gives a reasonably good introduction to techniques developed in the past two decades, including those due to the authors and their collaborators.

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