Chapter 1

Model problems

Free boundary problems of obstacle type appear naturally in numerous applications, and the purpose of this chapter is to list some of the most interesting ones, from our point of view (§1.1). Not all of these problems will be treated in detail in this book, but the methods developed here will be applicable to all of them at least with a certain degree of success. For the detailed treatment we have selected three model problems (which we call **A**, **B**, and **C**) that can be put into a more general framework of obstacle-type problems $\mathbf{OT}_1$–$\mathbf{OT}_2$ (§1.2). At the end of this chapter (§1.3) we discuss the almost optimal $W^{2,p}_{\mathrm{loc}} \cap C^{1,\alpha}_{\mathrm{loc}}$ regularity of solutions for any $1 < p < \infty$, $0 < \alpha < 1$.

1.1. Catalog of problems

1.1.1. The classical obstacle problem.

1.1.1.1. *The Dirichlet principle.* A well-known variational principle of Dirichlet says that the solution of the boundary value problem

$$\Delta u = 0 \quad \text{in } D, \qquad u = g \quad \text{on } \partial D,$$

can be found as the minimizer of the (Dirichlet) functional

$$J_0(u) = \int_D |\nabla u|^2 dx,$$

among all u such that $u = g$ on ∂D. More precisely (and slightly more generally), if D is a bounded open set in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and $f \in L^\infty(D)$, then the minimizer of

$$J(u) = \int_D (|\nabla u|^2 + 2fu)dx \tag{1.1}$$

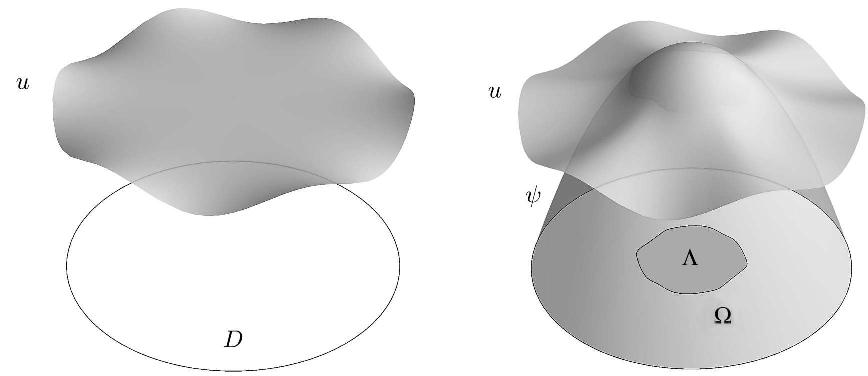

Figure 1.1. Free membrane and the solution of the obstacle problem

over the set

$$\mathfrak{K}_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\}$$

solves the Poisson equation

$$-\Delta u + f = 0 \quad \text{in } D, \qquad u = g \quad \text{on } \partial D,$$

in the sense of distributions, i.e.

$$\int_D (\nabla u \nabla \eta + f\eta)dx = 0,$$

for all test functions $\eta \in C_0^\infty(D)$ (and more generally for all $\eta \in W_0^{1,2}(D)$). One can think of the graph of u as a membrane attached to a thin wire (the graph of g over ∂D).

1.1.1.2. *The classical obstacle problem.* Suppose now that we are given a certain function $\psi \in C^2(D)$, known as the *obstacle*, satisfying the compatibility condition $\psi \leq g$ on ∂D in the sense that $(\psi - g)_+ \in W_0^{1,2}(D)$. Consider then the problem of minimizing the functional (1.1), but now over the constrained set

$$\mathfrak{K}_{g,\psi} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D), u \geq \psi \text{ a.e. in } D\}.$$

Since J is continuous and strictly convex on a convex subset $\mathfrak{K}_{g,\psi}$ of the Hilbert space $W^{1,2}(D)$, it has a unique minimizer on $\mathfrak{K}_{g,\psi}$.

As before, we may think of the graph of u as a membrane attached to a fixed wire, which is now forced to stay above the graph of ψ. A new feature in this problem is that the membrane can actually touch the obstacle; i.e. the set

$$\Lambda = \{u = \psi\},$$

known as the *coincidence set*, may be nonempty (see Fig. 1.1). We also denote

$$\Omega = D \setminus \Lambda.$$

The boundary

$$\Gamma = \partial\Lambda \cap D = \partial\Omega \cap D$$

is called the *free boundary*, since it is not known a priori. The study of the free boundary in this and related problems is the main objective in this book.

To obtain the conditions satisfied by the minimizer u, we observe that using the method of regularization, which we discuss in §1.3.2 (or, alternatively, the method of penalization; see Exercise 1.10 or Friedman [**Fri88**, Chapter 1]) one can show that the minimizer is not only in $W^{1,2}(D)$, but actually is in $W^{2,p}_{\mathrm{loc}}(D)$ for any $1 < p < \infty$ and consequently (by the Sobolev embedding theorem) is in $C^{1,\alpha}_{\mathrm{loc}}(D)$ for any $0 < \alpha < 1$. Then it is straightforward to show (see Exercise 1.1) that

$$\begin{aligned} \Delta u &= f && \text{in } \Omega = \{u > \psi\}, \\ \Delta u &= \Delta\psi && \text{a.e. on } \Lambda = \{u = \psi\}. \end{aligned}$$

Besides,

$$\Delta u \le f \quad \text{in } D$$

in the sense of distributions. Combining the properties above, we obtain that the solution of the obstacle problem is a function $u \in W^{2,p}_{\mathrm{loc}}(D)$ for any $1 < p < \infty$, which satisfies

$$-\Delta u + f \ge 0, \quad u \ge \psi, \quad (-\Delta u + f)(u - \psi) = 0 \quad \text{a.e. in } D, \tag{1.2}$$

$$u - g \in W^{1,2}_0(D). \tag{1.3}$$

This is known as the *complementarity problem* and uniquely characterizes the minimizers of J over $\mathfrak{K}_{g,\psi}$. Complementarity condition (1.2) is also written quite often as

$$\min\{-\Delta u + f, u - \psi\} = 0.$$

1.1.1.3. *Reduction to the case of zero obstacle.* Since the governing operator (Δ) is linear, it is possible to reduce the problem to the case when the obstacle is identically 0. Indeed, for a solution u of the obstacle problem, consider the difference $v = u - \psi$. Then it is straightforward to see that v is the minimizer of the functional

$$J_1(v) = \int_D (|\nabla v|^2 + 2f_1 v)dx$$

over the set $\mathfrak{K}_{g_1,0}$, where

$$f_1 = f - \Delta\psi, \quad g_1 = g - \psi.$$

Moreover, if one knows $v \in W^{2,p}_{\mathrm{loc}}(D)$ for some $p > n$, then it is also possible to show that

$$\Delta v = f_1 \chi_{\{v>0\}} \quad \text{in } D,$$

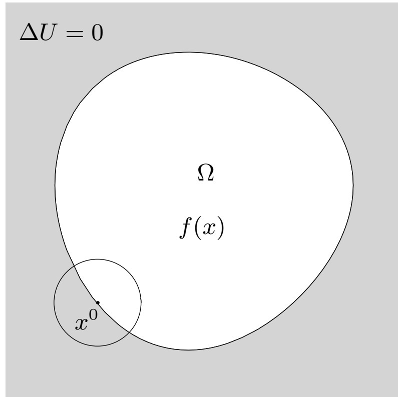

Figure 1.2. Harmonic continuation of Newtonian potentials

in the sense of distributions. This constitutes Exercise 1.2.

1.1.2. Problem from potential theory. Let Ω be a bounded open set in $\mathbb{R}^n$ and f a certain bounded measurable function on Ω. Consider then the Newtonian potential of the distribution of the mass $f\chi_\Omega$, i.e.

$$U(x) = \Phi_n * (f\chi_\Omega)(x) = \int_\Omega \Phi_n(x-y)f(y)dy,$$

where Φ_n is the fundamental solution of the Laplacian in $\mathbb{R}^n$, i.e. $\Delta\Phi_n = \delta$ in the sense of distributions. It can be shown (see e.g. Gilbarg-Trudinger [**GT01**, Theorem 9.9]) that the potential U is in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $1 < p < \infty$ and satisfies

$$\Delta U = f\chi_\Omega \quad \text{in } \mathbb{R}^n,$$

in the sense of distributions (or a.e., which amounts to the same in this case). In particular, U is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$ (see Fig. 1.2).

Let $x^0 \in \partial\Omega$ and suppose for some small $r > 0$ there is a harmonic function h in the ball $B_r(x^0)$ such that $h = U$ on $B_r \setminus \Omega$. We say in this case that h is a harmonic continuation of U into Ω at x^0. If such a continuation exists, the difference $u = U - h$ satisfies

$$\begin{aligned} \Delta u &= f\chi_\Omega \quad \text{in } B_r(x^0), \\ u = |\nabla u| &= 0 \quad \text{on } B_r(x^0) \setminus \Omega. \end{aligned} \tag{1.4}$$

Using the Cauchy-Kovalevskaya theorem, it is straightforward to show that the harmonic continuation exists if $\partial\Omega$ and f are real-analytic in a neighborhood of x^0. One may ask the converse to this question: if U admits a harmonic continuation in a neighborhood of x^0, then what can be said about the regularity of $\partial\Omega$?

Note that when $U \geq h$, or equivalently $u \geq 0$, u solves the obstacle problem in $B_r(x^0)$ with zero obstacle. However, the sign condition on u is rather unnatural in this setting, which leads to many complications as compared to the obstacle problem. The problem (1.4) has also been called the *no-sign* obstacle problem in the literature.

We also note that the study of quadrature domains described in the Introduction (see p. 2) leads to the same problem (1.4) with $f \equiv 1$ locally near points on $\partial\Omega$.

1.1.3. Pompeiu problem. A nonempty bounded open set $\Omega \subset \mathbb{R}^n$ is said to have the Pompeiu property if the only continuous function such that

$$\int_{\sigma(\Omega)} f(x)dx = 0$$

for all rigid motions $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is the identically zero function. A ball of any radius R fails this property: take $f(x) = \sin(ax_1)$ for $a > 0$ satisfying $J_{n/2}(aR) = 0$, where J_ν is the Bessel function of order ν. Furthermore, any finite disjoint union of balls of the same radius again fail the Pompeiu property, with the same function f.

A long standing conjecture in integral geometry says that if Ω fails the Pompeiu property and has a sufficiently regular (Lipschitz) boundary $\partial\Omega$ homeomorphic to the unit sphere, then Ω must be a ball. It is known (see Williams [**Wil76**]) that for such Ω there exists a solution to the problem

$$\Delta u + \lambda u = \chi_\Omega \quad \text{in } \mathbb{R}^n, \qquad u = |\nabla u| = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

for some $\lambda > 0$. This is a special case of an open conjecture of Schiffer which says that any Ω admitting solutions of the overdetermined problem above must be a ball.

1.1.4. A problem from superconductivity. In analyzing the evolution of vortices arising in the mean-field model of penetration of the magnetic field into superconducting bodies, one ends up with a degenerate parabolic-elliptic system. A simplified stationary model of this problem (in a local setting), where the scalar stream function admits a functional dependence on the scalar magnetic potential, reduces to finding u such that

$$\Delta u = f(x,u)\chi_{\{|\nabla u|>0\}} \quad \text{in } B_r(x^0),$$

with appropriate boundary conditions, where $f > 0$ and $f \in C^{0,1}(\mathbb{R}^n \times \mathbb{R})$.

This problem is more general than the problem in potential theory described in §1.1.2. For instance, the set $\Lambda = \{|\nabla u| = 0\}$ may consist of different components ("patches") Λ_j with u taking different constant values c_j on Λ_j (see Fig. 1.3). Exercise 1.4 at the end of this chapter gives an example of a solution with two different patches.

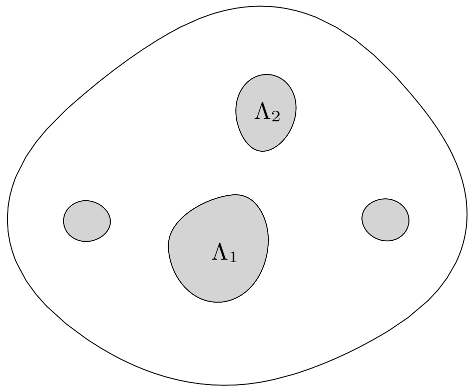

Figure 1.3. A problem from superconductivity: a solution with several "patches" $u \equiv c_j$ on Λ_j

1.1.5. Two-phase membrane problem. Given a bounded open set D in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and bounded measurable functions f_+ and f_- in D consider the problem of minimization of the functional

$$J_{f_+,f_-}(u) = \int_D (|\nabla u|^2 + 2f_+(x)u^+ + 2f_-(x)u^-)dx \tag{1.5}$$

over the set

$$\mathfrak{K}_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\}.$$

Here $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. When $f_- = 0$ and $g \geq 0$, the problem is equivalent to the obstacle problem with zero obstacle discussed at the end of §1.1.1.

To give a physical interpretation of this problem, consider a thin membrane (film), which is fixed on the boundary of a given domain, and some part of the boundary data of this film is below the surface of a thick liquid, heavier than the film itself (see Fig. 1.4). Now the weight of the film produces a force downwards (call it f_+) on the part of the film which is above the liquid surface. On the other hand, the part in the liquid is pushed upwards by a force f_-, since the liquid is heavier than the film. The equilibrium state of the film is given by a minimization of the above-mentioned functional.

One of the difficulties one confronts in this problem is that the interface $\{u = 0\}$ consists in general of two parts: one where the gradient of u is nonzero and one where the gradient of u vanishes. Close to points of the latter part we expect the gradient of u to have linear growth. However, because of the decomposition into two different types of growth, it is not possible to derive a growth estimate by classical techniques.

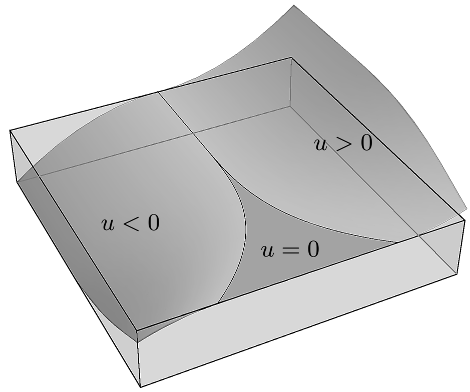

Figure 1.4. Two-phase membrane problem

Similarly to the classical obstacle problem, if one knows the $W^{2,p}_{\mathrm{loc}}$ regularity of the minimizer of J_{f_+,f_-} in D, then it will satisfy

$$\Delta u = f_+\chi_{\{u>0\}} - f_-\chi_{\{u<0\}} \quad \text{in } D.$$

This problem will be one of our model problems (Problem **B**) that we study in detail in this book, including the above-mentioned $W^{2,p}_{\mathrm{loc}}$ regularity, $1 < p < \infty$ (see §1.3.3).

1.1.6. Interior temperature control problem. Yet another application of the same equation as in the two-phase membrane problem arises in the temperature control through the interior, regulated by the temperature in the interior.

Suppose that we want to keep the temperature $u(x)$ in the domain D as close as possible to the range (θ_-, θ_+), where $\theta_-(x) \le \theta_+(x)$ are two prescribed functions in D. We want to achieve this with the help of cooling/heating devices that are distributed evenly over the domain D. The devices are assumed to be of limited power, so the heat flux $-f$ generated by them will be in the range $[-f_-, f_+]$, $f_\pm \ge 0$. There will be no heat generated when $u(x) \in [\theta_-(x), \theta_+(x)]$; however, when u is not in that range, the corrective heat flux $-f$ will be injected according to the following rule:

$$-f = \Phi(u) = \begin{cases} \min\{k_+(u-\theta_+), f_+\}, & u > \theta_+, \\ \max\{k_-(u-\theta_-), -f_-\}, & u < \theta_-, \end{cases}$$

where $k_\pm \ge 0$. In the equilibrium state, the temperature distribution will satisfy

$$\Delta u = \Phi(u) \quad \text{in } D.$$

Assuming $\theta_- = \theta_+ = 0$ and $k_\pm = +\infty$, the equation becomes

$$\Delta u = f_+ \chi_{\{u>0\}} - f_- \chi_{\{u<0\}} \quad \text{in } D.$$

The model described above is from the book by Duvaut-Lions [**DL76**, Chapter I, §2.3.2].

1.1.7. Composite membrane. A free boundary problem of obstacle type appears also in the construction of composite membranes. One wants to build a body of prescribed shape consisting of given materials (of varying densities) in such a way that the body has a prescribed mass and that the basic frequency of the resulting membrane (with fixed boundary) is as small as possible. Let us consider a more general problem. Suppose we are given a domain $D \subset \mathbb{R}^n$ (bounded, connected, with Lipschitz boundary) and numbers $\alpha > 0$, $A \in [0, |D|]$. For any measurable subset $\Omega \subset D$ let $\lambda_D(\alpha, \Omega)$ denote the lowest eigenvalue λ of the problem

$$-\Delta v + \alpha \chi_\Omega v = \lambda v \quad \text{in } D, \qquad v = 0 \quad \text{on } \partial D, \tag{1.6}$$

and set

$$\Lambda_D(\alpha, A) := \inf_{\Omega \subset D,\ |\Omega| = A} \lambda_D(\alpha, \Omega).$$

Any minimizer Ω in the latter equation is called an optimal configuration for the data. If Ω is an optimal configuration and v satisfies (1.6), then (v, Ω) is called an optimal pair (or solution). It is known that $\Omega = \{v \le t\}$ for some t such that $A = |\{v \le t\}|$; see [**CGI$^+$00**]. Now upon rewriting $u = v - t$ we can rephrase the above equation as

$$\Delta u = (\alpha \chi_{\{u \le 0\}} - \lambda)(u + t),$$

and with yet another rewriting we arrive at

$$\Delta u = \big((\alpha - \lambda)\chi_{\{u \le 0\}} - \lambda \chi_{\{u>0\}}\big)\,(u + t).$$

The particular case $\alpha < \lambda$ is of special interest, since the problem then does not fall into the category of obstacle-type problems, which we treat in detail in this book, but is rather akin to the *unstable obstacle problem*

$$\Delta u = -\chi_{\{u>0\}},$$

solutions of which fail in general to be $C^{1,1}$ (see the counterexample in §2.5).

1.1.8. Optimal stopping. In control theory one needs to find an optimal choice of strategy so that a cost/profit functional becomes minimal/maximal. Consider a domain $D \subset \mathbb{R}^n$, $x \in D$, and let $\mathbf{W}_x$ be a Brownian motion, with $\mathbf{W}_x(0) = x$. Further, let $\tau = \tau_x$ be a hitting time of ∂D and θ a stopping time (with respect to an underlying filtration for the Brownian motion).

For given functions f and g in $\overline{D}$, define an expected profit of stopping the Brownian motion at time $\theta \wedge \tau := \min(\theta, \tau)$ by

$$J_x(\theta) := \mathbb{E}\left[\int_0^{\theta\wedge\tau} \frac{1}{2} f(\mathbf{W}_x(s))ds + g(\mathbf{W}_x(\theta \wedge \tau))\right],$$

where $\mathbb{E}$ denotes the expected value. Now maximizing this over all possible stopping times and defining

$$u(x) := \sup_{\theta \text{ stopping time}} J_x(\theta),$$

it is relatively easy to show that u satisfies

$$u \geq g, \quad -\Delta u + f \geq 0, \quad (\Delta u - f)(u - g) = 0 \quad \text{in } D$$
$$u = g \quad \text{on } \partial D.$$

In other words, u solves the classical obstacle problem (1.2)–(1.3) with both the obstacle and boundary values given by g. For more explanation and details we refer to lecture notes of Evans [**Eva11**, pp. 103–107]. See also §9.1.2 for a version of this problem for jump processes.

1.1.9. Problems with lower-dimensional free boundaries. Over the past decade, there has been a renewed interest in problems that exhibit free boundaries of higher codimension, such as the thin obstacle problem (also known as the scalar Signorini problem), motivated by the new techniques as well as the increased range of applications.

On one hand, the methods for the study of such problems are quite similar to the ones for the problems described above, but on the other hand, there are substantial differences. For that reason, we decided to put the material related to the thin obstacle problem into a separate chapter, Chapter 9.

1.2. Model Problems A, B, C

Out of the examples described in the previous section, we have chosen three model problems (to be called Problems **A**, **B**, **C**) that we will study in detail throughout Chapters 2–8, as we believe they demonstrate the most typical techniques and difficulties associated with the obstacle-type problems. The forth model problem, Problem **S**, which is a version of the thin obstacle problem, will be treated in Chapter 9.

The model problems below have the following structure. We are given an open set D in $\mathbb{R}^n$ and a function $u \in L^\infty_{\text{loc}}(D)$ that satisfies

$$\Delta u = f_u \quad \text{in} \quad D,$$

where f_u is a certain function in $L^\infty(D)$, having a jump discontinuity along a certain set $\Gamma(u)$, which is the *free boundary* in the problem. The last equation is understood in the sense of distributions, so that

$$\int_D u\Delta\eta\,dx = \int_D f_u\eta\,dx,$$

for all test functions $\eta \in C_0^\infty(D)$.

1.2.1. Problem A. *No-sign obstacle problem.* Our first model problem is the particular case of the problem from potential theory described in §1.1.2 with $f(x) \equiv 1$:

$$(\mathbf{A}) \qquad \begin{aligned} \Delta u &= \chi_{\Omega(u)} \quad \text{in } D,\\ \Omega(u) &:= D \setminus \{u = |\nabla u| = 0\}. \end{aligned}$$

The free boundary in this case is $\Gamma(u) = \partial\Omega(u) \cap D$. As we noted earlier, if the solution u of this problem is nonnegative, then $\Omega = \{u > 0\}$ and u becomes a solution of the classical obstacle problem.

1.2.2. Problem B. *Superconductivity problem.* Our second model problem is the particular version of the problem from superconductivity in §1.1.4 with $f(x, u) \equiv 1$:

$$(\mathbf{B}) \qquad \begin{aligned} \Delta u &= \chi_{\Omega(u)} \quad \text{in } D,\\ \Omega(u) &:= \{|\nabla u| > 0\}. \end{aligned}$$

The free boundary here is again $\Gamma(u) = \partial\Omega(u) \cap D$.

1.2.3. Problem C. *Two-phase membrane problem.* Our third model problem is the particular form of the problem in §1.1.5 with $f_\pm(x) \equiv \lambda_\pm$ positive constants:

$$(\mathbf{C}) \qquad \begin{aligned} \Delta u &= \lambda_+\chi_{\Omega_+(u)} - \lambda_-\chi_{\Omega_-(u)} \quad \text{in } D,\\ \Omega_\pm(u) &:= \{\pm u > 0\}. \end{aligned}$$

The free boundary here is the union $\Gamma(u) = \Gamma_+(u) \cup \Gamma_-(u)$, where $\Gamma_\pm(u) = \partial\Omega_\pm(u) \cap D$. Since the behavior of u near the free boundary is going to be different depending on whether $|\nabla u|$ vanishes or not, we naturally subdivide Γ into the union of

$$\begin{aligned} \Gamma^0(u) &:= \Gamma(u) \cap \{|\nabla u| = 0\},\\ \Gamma^*(u) &:= \Gamma(u) \cap \{|\nabla u| \neq 0\}. \end{aligned}$$

Note that by the implicit function theorem Γ^* is locally a $C^{1,\alpha}$ graph (in fact, it can be shown to be real-analytic; see §4.5), so most of the difficulties lie in the study of the free boundary near the points in Γ^0.

1.2.4. Obstacle-type problems. Some of our results will be proved in a more general framework of what we will call *obstacle-type problems*, which is a slight abuse of the terminology, since there are still many interesting problems of obstacle type that do not fit into this framework.

We consider functions $u \in L^\infty_{\mathrm{loc}}(D)$ that satisfy

$$(\mathbf{OT}_1)\qquad \begin{aligned} \Delta u &= f(x,u)\chi_{G(u)} \quad \text{in } D,\\ |\nabla u| &= 0 \quad \text{on } D\setminus G(u), \end{aligned}$$

where $G(u)$ is an open subset of D and $f : D\times\mathbb{R}\to\mathbb{R}$ satisfies the following structural conditions: there exist $M_1, M_2 \geq 0$ such that

$$(\mathbf{OT}_2)\qquad \begin{aligned} |f(x,t)-f(y,t)| &\leq M_1|x-y|, \quad x,y\in D,\ t\in\mathbb{R},\\ f(x,s)-f(x,t) &\geq -M_2(s-t), \quad x\in D,\ s,t\in\mathbb{R},\ s\geq t. \end{aligned}$$

Locally, these conditions are equivalent to

$$|\nabla_x f(x,t)| \leq M_1, \quad \partial_t f(x,t) \geq -M_2$$

in the sense of distributions. The free boundary is going to be $\partial G(u)\cap D$ and/or the set of discontinuity of $f(x,u)$, depending on the problem.

Problems **A**, **B**, and **C** fit into the framework $\mathbf{OT}_1$–$\mathbf{OT}_2$ as follows:

- Problems **A**, **B**: $G = \Omega(u)$, $f(x,u) = 1$ in G. More generally, the problems $\Delta u = f(x)\chi_{\Omega(u)}$ with $\Omega(u)$ as in Problem **A** or **B** will fit in our framework if $f \in C^{0,1}(D)$.
- Problem **C**: $G = D$, $f(x,u) = \lambda_+\chi_{\Omega_+(u)} - \lambda_-\chi_{\Omega_-(u)}$. More generally, the equation $\Delta u = f_+(x)\chi_{\Omega_+(u)} - f_-(x)\chi_{\Omega_-(u)}$ fits in our framework if $f_\pm \in C^{0,1}(D)$.

1.3. $W^{2,p}$ regularity of solutions

1.3.1. Calderón-Zygmund estimates. In Chapter 2 we will see that the solutions u of obstacle-type problems $\mathbf{OT}_1$-$\mathbf{OT}_2$ enjoy the $C^{1,1}$ regularity, which is the optimal regularity of the solutions, since Δu is generally a discontinuous function. However, that will require the use of a rather powerful Alt-Caffarelli-Friedman monotonicity formula and its generalizations (see §2.2). Here, we remark that the almost optimal $W^{2,p}_{\mathrm{loc}} \cap C^{1,\alpha}_{\mathrm{loc}}$ regularity for any $1<p<\infty$ and $0<\alpha<1$ comes "for free" from the standard L^p-theory of elliptic equations (see Gilbarg-Trudinger [**GT01**, Chapter 9]).

Theorem 1.1. *Let $u \in L^1(D)$, $f\in L^p(D)$, $1<p<\infty$, be such that $\Delta u = f$ in D in the sense of distributions. Then $u \in W^{2,p}_{\mathrm{loc}}(D)$ and*

$$\|u\|_{W^{2,p}(K)} \leq C\left(\|u\|_{L^1(D)} + \|f\|_{L^p(D)}\right),$$

for any $K \Subset D$ with $C = C(p,n,K,D)$. □

Remark 1.2. It should be noted that in the typical versions of Theorem 1.1 in the literature one has $\|u\|_{L^p(D)}$ instead of $\|u\|_{L^1(D)}$ on the right-hand side and a priori assumes that $u \in W^{2,p}_{\mathrm{loc}}(D) \cap L^p(D)$; see e.g. Gilbarg-Trudinger [**GT01**, Theorem 9.11]. However, that is a superfluous assumption. The proof is outlined in Exercise 1.6, based on the $W^{2,p}$ estimates for the Newtonian potentials.

Thus, for solutions of $\Delta u = f$ in D with $u \in L^1(D)$ and $f \in L^\infty(D)$ we have

$$u \in W^{2,p}_{\mathrm{loc}}(D), \quad \text{for all} \quad 1 < p < \infty. \tag{1.7}$$

Consequently, we also have

$$u \in C^{1,\alpha}_{\mathrm{loc}}(D), \quad \text{for all} \quad 0 < \alpha < 1, \tag{1.8}$$

by the Sobolev embedding $W^{2,p} \hookrightarrow C^{1,\alpha}$ with $\alpha = 1 - n/p$ for $p > n$.

An easy counterexample (see Exercise 1.7) shows that in general we cannot have $p = \infty$ in (1.7) and $\alpha = 1$ in (1.8). Instead, we have the following estimate.

Theorem 1.3. *Let $u \in L^1(D)$, $f \in L^\infty(D)$ be such that $\Delta u = f$ in the sense of distributions. Then $u \in W^{2,p}_{\mathrm{loc}}(D) \cap C^{1,\alpha}_{\mathrm{loc}}(D)$ for all $1 < p < \infty$, $0 < \alpha < 1$ and*

$$|\nabla u(x) - \nabla u(y)| \leq C(\|u\|_{L^1(D)} + \|f\|_{L^\infty(D)})|x - y| \log \frac{1}{|x - y|},$$

for any $x, y \in K \Subset D$ with $|x - y| \leq 1/e$ and $C = C(n, K, D)$. □

We refer again to Exercise 1.6 for the outline of the proof.

As we will see later, the logarithmic term in this theorem can be dropped if one assumes the additional structure on the right-hand side as in $\mathbf{OT}_1$–$\mathbf{OT}_2$. That would give us a starting point for the analysis of the free boundary in those problems.

1.3.2. Classical obstacle problem. The $W^{2,p}$ estimates come "for free" in Problems **A**, **B**, and **C**, since we state them in the form of equations with bounded right-hand sides. However, recall that to obtain such an equation for the solutions of the classical obstacle problem (§1.1.1) or the two-phase membrane problem (§1.1.5), we needed to have the $W^{2,p}$ regularity in the first place. So to avoid this circular reasoning, we establish the $W^{2,p}$ regularity here.

We start with the classical obstacle problem. Recall that by simply subtracting the obstacle (see §1.1.1.3) we reduce it to the case of the obstacle problem with zero obstacle, i.e. the problem of minimizing

$$J(u) = \int_D (|\nabla u|^2 + 2fu)dx$$

over the convex closed subset

$$\mathfrak{K}_{g,0} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D), u \geq 0 \text{ a.e. in } D\}.$$

Here we assume $g \in W^{1,2}(D)$, $g \geq 0$ on ∂D in the sense that $g^- \in W_0^{1,2}(D)$ and $f \in L^\infty(D)$.

It is easy to see that the functional J is strictly convex and bounded below on $\mathfrak{K}_{g,0}$. Hence, J has a unique minimizer on $\mathfrak{K}_{g,0}$. We show next that the minimizer is in $W^{2,p}_{\mathrm{loc}}(D)$ for any $1 < p < \infty$ and will consequently solve

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in } D.$$

The first step is getting rid of the obstacle at the expense of losing the regularity of the functional J.

Lemma 1.4. *A function $u \in W^{1,2}(D)$ is a minimizer of J over $\mathfrak{K}_{g,0}$ iff u is a minimizer of the functional*

$$\tilde{J}(u) = \int_D (|\nabla u|^2 + 2fu^+)dx$$

over $\mathfrak{K}_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\}$.

Proof. For any $u \in \mathfrak{K}_g$ we have that $u^+ \in \mathfrak{K}_{g,0}$ and

$$\nabla(u^+) = (\nabla u)\chi_{\{u>0\}}.$$

We then claim that

$$\tilde{J}(u^+) \leq \tilde{J}(u) \quad \text{for any } u \in \mathfrak{K}_g,$$

with equality iff $u = u^+$. Indeed,

$$\tilde{J}(u) = \int_D (|\nabla u|^2 + 2fu^+)dx \geq \int_D (|\nabla u|^2\chi_{\{u>0\}} + 2fu^+)dx = \tilde{J}(u^+).$$

The equality $\tilde{J}(u^+) = \tilde{J}(u)$ holds iff $\nabla u = 0$ a.e. on $\{u \leq 0\}$, which is equivalent to having $\nabla u^- = 0$ a.e. in D. The latter means that u^- is locally constant, and since $u^- \in W_0^{1,2}(D)$, it follows that $u^- = 0$. Hence, $\tilde{J}(u^+) = \tilde{J}(u)$ iff $u \geq 0$ a.e. in D, or equivalently $u \in \mathfrak{K}_{g,0}$. Thus, $\tilde{J}$ attains its minimum on $\mathfrak{K}_{g,0}$.

On the other hand

$$\tilde{J} = J \quad \text{on } \mathfrak{K}_{g,0}.$$

Hence these two functionals have the same set of minimizers. □

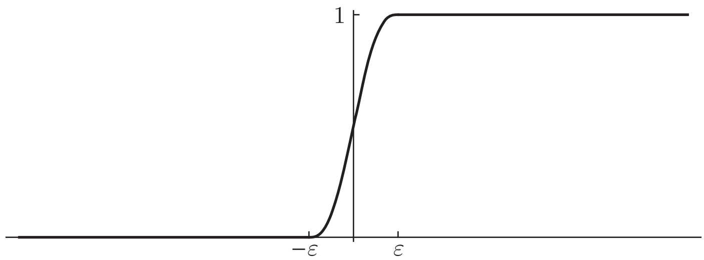

Figure 1.5. The function χ_ε

Thus, we reduced the problem to studying the minimizers of $\tilde{J}$ with given boundary values g on ∂D. In the next step, consider a family of regularized problems

$$\begin{aligned} \Delta u_\varepsilon &= f\chi_\varepsilon(u_\varepsilon) && \text{in } D,\\ u_\varepsilon &= g && \text{on } \partial D, \end{aligned}$$

for $0 < \varepsilon < 1$, where $\chi_\varepsilon(s)$ is a smooth approximation of the Heaviside function $\chi_{\{s>0\}}$ such that

$$\chi'_\varepsilon \geq 0, \quad \chi_\varepsilon(s) = 0 \quad \text{for } s \leq -\varepsilon, \quad \chi_\varepsilon(s) = 1 \quad \text{for } s \geq \varepsilon.$$

A solution u_ε to this problem can be obtained by minimizing the functional

$$J_\varepsilon(u) = \int_D (|\nabla u|^2 + 2f(x)\Phi_\varepsilon(u))dx$$

over $\mathfrak{K}_g$, where

$$\Phi_\varepsilon(s) = \int_{-\infty}^{s} \chi_\varepsilon(t)dt.$$

To be more precise, the minimizer u_ε will satisfy $\Delta u_\varepsilon = f(x)\chi_\varepsilon(u_\varepsilon)$ in D in the sense that

$$\int_D (\nabla u_\varepsilon \nabla \eta + f(x)\chi_\varepsilon(u_\varepsilon)\eta) = 0$$

for any $\eta \in W_0^{1,2}(D)$. We next want to show that the family $\{u_\varepsilon\}$ is uniformly bounded in $W^{1,2}(D)$ as well as in $W^{2,p}(K)$ for any $K \Subset D$.

To show the uniform $W^{1,2}(D)$ estimate, we take $\eta = u_\varepsilon - g$ in the above equation. Then an application of the Poincaré inequality gives

$$\int_D |\nabla(u_\varepsilon - g)|^2 dx \leq C(f, g),$$

uniformly for $0 < \varepsilon < 1$. This implies the claimed uniform boundedness of $\{u_\varepsilon\}$ in $W^{1,2}(D)$.

As a consequence, we obtain that there exists $u \in W^{1,2}(D)$ such that over a subsequence $\varepsilon = \varepsilon_k \to 0$,

$$u_\varepsilon \to u \quad \text{weakly in } W^{1,2}(D),$$
$$u_\varepsilon \to u \quad \text{strongly in } L^2(D).$$

Moreover, since $u_\varepsilon - g \in W_0^{1,2}(D)$ and $W_0^{1,2}(D)$ is a closed subspace (thus also closed with respect to weak convergence) of the Hilbert space $W^{1,2}(D)$, we conclude that $u \in \mathfrak{K}_g$.

Next, recalling that u_ε are weak solutions of the regularized problem $\Delta u_\varepsilon = f(x)\chi_\varepsilon(u_\varepsilon)$ and applying the Calderón-Zygmund estimate (Theorem 1.1), we will obtain that

$$\begin{aligned}\|u_\varepsilon\|_{W^{2,p}(K)} &\le C(K,D)(\|u_\varepsilon\|_{L^2(D)} + \|f\chi_\varepsilon(u_\varepsilon)\|_{L^\infty(D)})\\ &\le C(K,D,f,g),\end{aligned}$$

for any $K \Subset D$ and $1 < p < \infty$. Thus, we may further assume that over the same sequence $\varepsilon = \varepsilon_k \to 0$ as above,

$$u_\varepsilon \to u \quad \text{weakly in } W^{2,p}_{\mathrm{loc}}(D),$$

for any $1 < p < \infty$. Clearly, $u \in W^{2,p}_{\mathrm{loc}}(D)$ for any $1 < p < \infty$.

Now, we are ready to pass to the limit as $\varepsilon = \varepsilon_k \to 0$. Since

$$\int_D |\nabla u|^2 dx \le \liminf_{\varepsilon\to 0} \int_D |\nabla u_\varepsilon|^2 dx,$$
$$\int_D f u^+ dx = \lim_{\varepsilon\to 0} \int_D f\Phi_\varepsilon(u_\varepsilon) dx,$$

we see that

$$\tilde{J}(u) \le \liminf_{\varepsilon=\varepsilon_k\to 0} J_\varepsilon(u_\varepsilon) \le \liminf_{\varepsilon=\varepsilon_k\to 0} J_\varepsilon(v) = \tilde{J}(v),$$

for any $v \in \mathfrak{K}_g$. Thus, by Lemma 1.4, u is the solution of the obstacle problem. Finally, we verify that u satisfies

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in } D,$$

in the sense of distributions. Since $u \in W^{2,p}_{\mathrm{loc}}(D)$, we readily have that Δu is an L^p function locally in D, and thus we have to verify that

$$\Delta u = f\chi_{\{u>0\}} \quad \text{a.e. in } D.$$

To this end, we remark that since $u_\varepsilon \in W^{2,p}_{\mathrm{loc}}(D)$, the equation $\Delta u_\varepsilon = f(x)\chi_\varepsilon(u_\varepsilon)$ is now satisfied in the strong sense, i.e., for a.e. $x \in D$. Moreover, taking p large, by the Sobolev embedding theorem we may assume that over $\varepsilon = \varepsilon_k \to 0$ we have

$$u_\varepsilon \to u \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(D).$$

Then the locally uniform convergence implies that $\Delta u = f$ a.e. in the open set $\{u > 0\}$. Besides, using the fact that $u \in W^{2,p}_{\mathrm{loc}}(D)$ one more time, we get $\Delta u = 0$ a.e. on $\{u = 0\}$. Combining these two facts, we obtain that $\Delta u = f(x)\chi_{\{u>0\}}$ a.e. in D.

1.3.3. Two-phase membrane problem. Given any two nonnegative bounded functions f_+ and f_- we want to show here that the minimizer of the energy functional

$$J(v) = \int_D (|\nabla v|^2 + 2f_+u^+ + 2f_-u^-)dx$$

on the set $\mathfrak{K}_g$, $g \in W^{1,2}(D)$, is in $W^{2,p}_{\mathrm{loc}}(D)$ for any $1 < p < \infty$ and solves

$$\Delta u = f_+(x)\chi_{\{u>0\}} - f_-(x)\chi_{\{u<0\}} \quad \text{a.e. in } D.$$

To this end consider the approximating problems

$$\begin{aligned} \Delta u &= f_+\chi_\varepsilon(u) - f_-\chi_\varepsilon(-u) && \text{in } D, \\ u &= g && \text{on } \partial D, \end{aligned}$$

and the solutions u_ε obtained by minimizing the functional

$$J_\varepsilon(u) = \int_D (|\nabla u|^2 + 2f_+\Phi_\varepsilon(u) + 2f_-\Phi_\varepsilon(-u))dx,$$

where the approximations χ_ε and Φ_ε are as in the previous subsection. Then, following the arguments as before one can establish that for a subsequence $\varepsilon = \varepsilon_k \to 0$ the minimizers u_ε converge weakly in $W^{2,p}_{\mathrm{loc}}(D)$ for any $1 < p < \infty$ to a solution of the desired problem. We leave it to the reader to fill in the details (see Exercise 1.9).

Notes

The obstacle problem originated in the work of Stampacchia [**Sta64**], with the obstacle being a characteristic function of a set (in relation to the capacity of that set). The Signorini problem (akin to the thin obstacle problem; see Chapter 9) have appeared even earlier in Signorini [**Sig59**] and Fichera [**Fic64**]. Lions-Stampacchia [**LS67**] gave the first systematic treatment of variational inequalities (of which the obstacle and Signorini problems are particular examples).

Problems from potential theory similar to the one in §1.1.2 and Problem **A** can be found in Sakai [**Sak82, Sak91, Sak93**]. We also refer to the lectures of Gustafsson [**Gus04**] for related problems. This kind of problems are also of interest in inverse problems and geophysics; see e.g. Strakhov [**Str74**], Margulis [**Mar82**], and the book by Isakov [**Isa90**]. Our main reference for Problem **A** is Caffarelli-Karp-Shahgholian [**CKS00**].

The Pompeiu problem in §1.1.3 originated in the work of Pompeiu [**Pom29**]. The problem was reduced by Williams [**Wil76**] to a special case in Schiffer's conjecture (see e.g. Yau [**Yau82**, p. 688, Problem 80]). Analyticity of the boundary of the domains failing the Pompeiu property was proved by Williams [**Wil81**] in case the boundary is assumed to be Lipschitz. This was extended to domains satisfying a thickness condition by Caffarelli-Karp-Shahgholian [**CKS00**].

The problem from superconductivity in §1.1.4 (and also Problem **B**) is a simplified time-independent version of the parabolic-elliptic mean-field model of Chapman [**Cha95**]. There are related models by Berestycki-Bonnet-Chapman [**BBC94**] and Chapman-Rubinstein-Schatzman [**CRS96**], with a rigorous derivation from the Ginzburg-Landau model by Sandier-Serfaty [**SS00**]. Viscosity solutions were studied by Elliott-Schätzle-Stoth [**ESS98**] and Caffarelli-Salazar [**CS02**]. For particular configurations (with single patches) the free boundary was studied by Bonnet-Monneau [**BM00**] and Monneau [**Mon04**]. The free boundary in the general case was first studied by Caffarelli-Salazar [**CS02**], followed by Caffarelli-Salazar-Shahgholian [**CSS04**]. The latter paper is our main source for the results on Problem **B**.

The two-phase membrane problem in §1.1.5 (and Problem **C**) was known in the literature at least since the mid 1970s as a limiting case in the interior temperature control problem, as described in §1.1.6 (see also Duvaut-Lions [**DL76**, Chapter I, §2.3.2]). From the free boundary point of view it was first studied by Weiss [**Wei01**]. There is now more or less complete understanding of the problem, thanks to the works by Uraltseva [**Ura01**], Shahgholian-Weiss [**SW06**], and Shahgholian-Uraltseva-Weiss [**SUW07**].

The composite membrane problem in §1.1.7 can be found in Chanillo-Grieser-Imai-Kurata-Ohnishi [**CGI$^+$00**]. Partial regularity of the free boundary was proved in dimension two by Shahgholian [**Sha07**] (see also Blank [**Bla04**]) and Chanillo-Kenig [**CK08**] in higher dimensions. The full regularity (analyticity) in dimension two was proved by Chanillo-Kenig-To [**CKT08**]. The composite membrane problem is related to the unstable obstacle problem, first studied by Monneau-Weiss [**MW07**] in connection with a model in solid combustion. For some recent results on this interesting problem, we refer to Andersson-Shahgholian-Weiss [**ASW10**].

The optimal stopping problems in §1.1.8 can be viewed as a simplified version of the optimal pricing problem in mathematical finance for the so-called American (call/put) options, see e.g. Wilmott-Howison-Dewynne [**WHD95**, Chap. 7] and Evans [**Eva98**, pp. 107–111]. Optimal stopping problems for jump processes (stable Lévy processes) were also studied in the literature. They lead to obstacle-type problems for nonlocal integro-differential operators; see e.g. the thesis of Silvestre [**Sil07**]. In a particular case of the half-Laplacian, this has a direct relation to the thin obstacle problem; see §9.1.2 for more details.

The $W^{2,p}$ regularity for solutions of the obstacle problem is due to Lewy-Stampacchia [**LS69, LS70, LS71**]. The method of penalization (as in Exercise 1.10) was used in Brezis-Stampacchia [**BS68**], Lions [**Lio69**], Brezis [**Bre72**], and Brezis-Kinderlehrer [**BK74**].

In the recent literature, the obstacle problem was generalized to governing operators of various types. Below, we mention just a few of the available papers. Obstacle-type problems for p-Laplacian were studied by Choe-Lewis [**CL91**], Karp-Kilpeläinen-Petrosyan-Shahgholian [**KKPS00**], and Lee-Shahgholian [**LS03**]; see also the references therein. Obstacle-type problems for uniformly elliptic fully nonlinear equations were studied by Lee in [**Lee98**] and for Monge-Ampére equation in [**Lee01**]. A subelliptic obstacle problem was studied by Danielli-Garofalo-Salsa [**DGS03**] and Danielli-Garofalo-Petrosyan [**DGP07**].

Exercises

1.1. Let u be a solution of the obstacle problem as in §1.1.1.2. Under the additional assumption that $u \in W^{2,p}_{\mathrm{loc}}(D)$ for some $p > n$ prove that

$$\begin{aligned} \Delta u &= f && \text{in } \Omega = \{u > \psi\}, \\ \Delta u &= \Delta\psi && \text{a.e. on } \Lambda = \{u = \psi\}, \end{aligned}$$

and that

$$\Delta u \le f \qquad \text{in } D,$$

in the sense of distributions, or equivalently,

$$\int_D (\nabla u \nabla \eta + f\eta)dx \ge 0,$$

for any nonnegative $\eta \in W^{1,2}_0(D)$. Verify that this implies the complementarity condition (1.2).

Hint: To show that $\Delta u \le f$ let $\varepsilon \to 0+$ in

$$\frac{J(u+\varepsilon\eta) - J(u)}{\varepsilon} \ge 0.$$

1.2. Prove the assertions in §1.1.1.3, i.e. the reduction of the classical obstacle problem to the case with zero obstacle.

1.3. Show that in §1.1.2, if $\partial\Omega \cap B_r(x^0)$ is a real-analytic surface for some $r > 0$, then the Newtonian potential U admits a harmonic continuation into Ω at x^0. Use the Cauchy-Kovalevskaya theorem.

1.4. The following example constructs a solution to the problem in §1.1.4 with two different "patches" of the set $\{|\nabla u| = 0\}$.

Consider a dumbbell-shaped region $D \subset \mathbb{R}^2$,

$$D := B_1(x^1) \cup B_1(x^2) \cup \{x : |x_2| < \varepsilon,\ |x_1| < 2\}$$

with $x^1 = (2, 0)$, $x^2 = (-2, 0)$, and ε very small positive number. The solution to

$$\Delta v = f \quad \text{in } D, \qquad v = 0 \quad \text{on } \partial D,$$

forms a shape of hanging graphs over D, symmetric with respect to the x_1-axis.

Then solve the obstacle problem $\Delta u = f\chi_{\{u>\psi\}}$ in D with zero boundary values and the obstacle ψ which is smooth and equal to $\min v + \delta_i$ on each ball $B_{1/2}(x^i)$. Here $\delta_1 > \delta_2 > 0$ are small constants.

1.5. Let D be an open set with a Lipschitz boundary in $\mathbb{R}^n$. For a given function $h \in L^\infty(\partial D)$ and a control function $f \in L^\infty(D)$ from the class

$$U_{\text{ad}} = \left\{ \sup_D |f| \le 1, \int_D f = \int_{\partial D} h \right\},$$

consider the following problem:

$$\Delta u = f \quad \text{in } D, \quad \partial_\nu u = h \quad \text{on } \partial D.$$

Here ν is an outward normal to ∂D. Then minimize the functional

$$I(u) = \int_D |\nabla u|^2 + |u| - \int_{\partial D} hu$$

among all solutions with $f \in U_{\text{ad}}$. Show that the minimizer satisfies

$$\Delta u = \operatorname{sgn} u \quad \text{in } D,$$

in the weak sense. Note that this is a particular case of the two-membrane problem in §1.1.5 with $f_\pm \equiv 1$.

Hint: Show that for any $f \in U_{\text{ad}}$,

$$I(u) = \int_D |u|(1 - f \operatorname{sgn} u).$$

Consequently, $I(u) \ge 0$ for solutions u for any $f \in U_{\text{ad}}$, and the minimum $I(u) = 0$ is attained if $f = \operatorname{sgn} u$.

1.6. (i) Let D be a bounded open set in $\mathbb{R}^n$, $n \ge 3$, $f \in L^p(D)$, $1 < p < \infty$. Consider the Newtonian potential

$$w(x) = c_n \int_D f(y)|x - y|^{2-n} dy,$$

with the appropriately chosen c_n. Use the $W^{2,p}$ estimates for w (see [**GT01**, Lemma 7.12, Theorem 9.9]):

$$\|w\|_{L^p(D)} + \|D^2 w\|_{L^p(D)} \leq C(n, D)\|f\|_{L^p(D)}$$

to deduce Theorem 1.1.

Hint: Represent $u = v + w$, where w is the Newtonian potential as before and v a harmonic function in D. Then use the estimates above combined with the interior derivative estimates for harmonic functions (see [**Eva98**, §2.2, Theorem 7]),

$$\|D^k v\|_{L^\infty(K)} \leq C\|v\|_{L^1(D)},$$

to deduce Theorem 1.1. In $\mathbb{R}^2$, add a dummy variable to extend the functions to $\mathbb{R}^3$.

(ii) Assume now that $f \in L^\infty(D)$ and w is the Newtonian potential of f as above. Then use the estimate (see [**Mor08**, Theorem 2.5.1])

$$|\nabla w(x) - \nabla w(y)| \leq C(n, K, D)\|f\|_{L^\infty(D)}|x - y| \log \frac{1}{|x - y|}$$

for $x, y, \in K \Subset D$, $|x - y| < 1/e$, to deduce Theorem 1.3.

1.7. Show that the function $u(x_1, x_2) = (x_1^2 - x_2^2)\log(x_1^2 + x_2^2)$ defined in $\mathbb{R}^2$ is locally bounded and satisfies $\Delta u = f$ in the sense of distributions for a certain $f \in L^\infty(\mathbb{R}^2)$ but is not in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^2) = C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$.

1.8. Check that to prove the $W^{2,p}_{\mathrm{loc}}$ estimates for the minimizers of the functional J in §1.3.2 for a fixed $1 < p < \infty$, it is enough to assume that $f \in L^p(D)$. [Showing that the minimizer u solves $\Delta u = f\chi_{\{u>0\}}$ may not be easy if $p < n/2$, since we do not even know if the set $\{u > 0\}$ is open in that case.]

1.9. Complete the proof of the $W^{2,p}_{\mathrm{loc}}$ regularity of the solutions of the two-phase membrane problem in §1.3.3.

1.10. The purpose of this exercise is to give an alternative proof of the $W^{2,p}_{\mathrm{loc}}$ regularity of solutions of the obstacle problem. This is known as the *method of penalization*. We will assume that ∂D is sufficiently smooth and that the boundary data $g \in C^{1,\alpha}(\partial D)$ for some $\alpha > 0$.

For $\varepsilon > 0$, let $\beta_\varepsilon \in C^\infty(\mathbb{R})$ be such that

$$\beta_\varepsilon' \geq 0, \quad \beta_\varepsilon \leq 0, \quad \beta_\varepsilon(s) = 0 \text{ for } s \geq 0, \quad \beta_\varepsilon(s) = \frac{s}{\varepsilon} \text{ for } s < -\varepsilon.$$

Also let $\beta_{\varepsilon,N} = \max\{\beta_\varepsilon, -N\}$ for any $N > 0$. Now let $u^{\varepsilon,N}$ be a solution of the *penalized problem*

$$\begin{aligned} \Delta u = \beta_{\varepsilon,N}(u) + f(x) \quad &\text{in } D, \\ u = g \quad &\text{on } \partial D. \end{aligned}$$

(a) Prove that the function $\zeta^{\varepsilon,N}(x) = \beta_{\varepsilon,N}(u^{\varepsilon,N}(x))$ is bounded in D, uniformly in ε and N. In fact,

$$-\|f\|_{L^\infty(D)} \le \zeta^{\varepsilon,N} \le 0 \quad \text{in } D.$$

Hint: Let x^0 be a point where $\zeta^{\varepsilon,N}$ achieves its negative minimum. Then $x^0 \in D$ and $u^{\varepsilon,N}$ has a local minimum at that point, which implies that $\Delta u^{\varepsilon,N}(x^0) \le 0$. Use this fact to find a bound on $\zeta^{\varepsilon,N}(x^0)$ from below.

(b) Conclude from (a) that $u^\varepsilon = u^{\varepsilon,N}$ for large enough N solves

$$\Delta u = \beta_\varepsilon(u) + f(x) \quad \text{in } D,$$
$$u = g \quad \text{on } \partial D,$$

and that the u^ε are uniformly bounded in $W^{2,p}(K)$ for $K \Subset D$, and in $C^{1,\alpha}(\overline{D})$ for some $\alpha > 0$.

(c) Rigorously show that the limit u of any subsequence u^{ε_k} with $\varepsilon_k \to 0$ is a solution of the obstacle problem with zero obstacle and boundary data g.
Hint: Use (a) to deduce that $u \ge 0$ in D and show that the complementarity condition (1.2) holds with $\psi = 0$.